Name: ....

Teacher: ....

SCEGGS Darlinghurst

HSC Assessment 1
Friday, 24th March, 2006

Extension 1 Mathematics

General Instructions

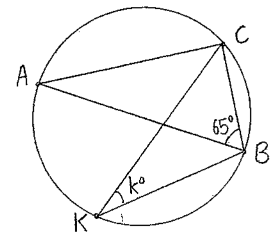
- Time allowed – 75 minutes
- Weighting 25%
- This paper has **four** questions
- Attempt **all** questions and show all necessary working
- Marks may be deducted for careless or badly arranged work
- Write using blue or black pen, diagrams in pencil
- Write your name and your teacher's name at the top of each page
- Approved calculators, mathematical templates and geometrical instruments may be used
- A table of standard integrals is provided at the back of this paper

Questions	Total	Comm.	Reas.	Calc.
1	/10		/3	/4
2	/12	/3	/7	
3	/10	/3	/7	
4	/11	/2	/6	/3
TOTAL	/43	/8	/23	/7

Question 1 (10 marks) **Marks**

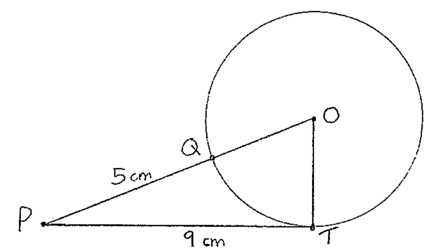
(a) Find: $\displaystyle\lim_{x \to 0} \frac{3x}{\sin 2x}$ 1

(b) 3



In the diagram above, AB is the diameter of a circle, $\angle ABC = 65°$, and $\angle BKC = k°$.
Find the value of k, giving full reasons for your answer.

(c) 2



In the diagram above, O is the centre of a circle with PT the tangent at T. If $PT = 9\,\text{cm}$ and $PQ = 5\,\text{cm}$, calculate the length of the radius of the circle. Give full reasons for your answer.

(d) i. Prove that the equation 4

$$2 \sin x - 10x + 5 = 0$$

has a solution between $x = 0$ and $x = 1$.

ii. Taking $x = 0.5$ as a first approximation, find a better approximation with one application of Newton's method (correct to 3 significant figures).

Question 2 begins on page 2 ...

START A NEW PAGE

Question 2 (12 marks) **Marks**

(a) By making the substitution $t = \tan(\frac{\theta}{2})$, or otherwise, show that 2

$$\cot\theta + \tan\frac{\theta}{2} = \text{cosec}\theta$$

(b) i. Express $\sqrt{3}\sin t + \cos t$ in the form $R\sin(t+\alpha)$, where α is in radians. 2

ii. Hence, or otherwise, find (in exact form) the general solution of the equation 2

$$\sqrt{3}\sin t + \cos t = 1$$

(c) Use mathematical induction to prove that, for any positive integer n, $5^n + 2(11)^n$ is a multiple of 3. 3

(d) An employer wishes to choose two people for a job. There are 8 applicants, 3 of whom are women and 5 of whom are men. 3

i. If each applicant is interviewed separately and all the women are interviewed before any of the men, find how many ways there are to carry out the interviews.

ii. How many ways can two applicants be chosen so that at least one of those chosen is a women.

Question 3 begins on page 3 ...

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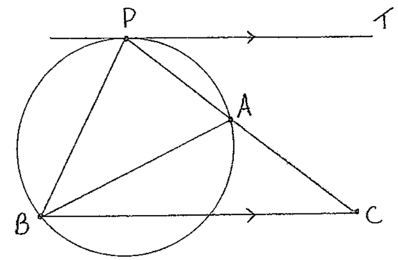
Question 3 (10 marks) **Marks**

(a)



In the diagram, A, P, and B, are points on the circle. The line PT is tangent to the circle at the point P, and PA is produced to C so that BC is parallel to PT.

i. Show that $\angle PBA = \angle PCB$ 2

ii. Deduce that $PB^2 = PA \times PC$ 2

(b) 5 males and 5 females are seated around a circular table. 3

i. How many seating arrangements are possible if there are no restrictions?

ii. If no two people of the same sex are to sit next to eachother, how many arrangements are possible?

iii. If Amy refuses to sit next to Benton, and there are no other restrictions, how many seating arrangements are possible?

(c) Use mathematical induction to prove that, for all positive integers n, 3

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n+1)! - 1$$

Question 4 begins on page 4 ...

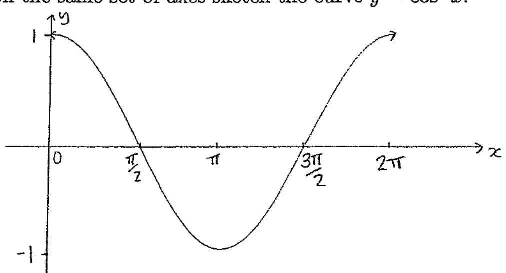
START A NEW PAGE

Question 4 (11 marks) **Marks**

(a) In the following questions you may use the identity

$$\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}.$$

i. Copy the sketch of $y = \cos x$ (below) into your writing booklet, and on the same set of axes sketch the curve $y = \cos^2 x$. **2**



ii. Show that $\cos^4 x = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$ **2**

iii. Hence, find the volume generated when the area bounded by the curves $y = \cos x$ and $y = \cos^2 x$, between $x = 0$ and $x = \frac{\pi}{2}$, is rotated about the x-axis. **3**

(b) Let each different arrangement of all the letters of PERMUTATION be called a word.

i. How many words are possible? **1**

ii. If a word is selected at random, what is the probability that all the vowels are together? **2**

iii. How many words are there in which the vowels occur in the order AEIOU from left to right, though not necessarily together? **1**

END OF ASSESSMENT

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 (10 marks)

(a) $\lim_{x \to 0} \frac{3x}{\sin 2x} = \lim_{x \to 0} \frac{3}{2} \cdot \frac{2x}{\sin 2x} = \frac{3}{2}$ ✓

Show working steps.

(b) $\angle ACB = 90°$ ($\angle$ in a semicircle $= 90°$) ✓

$\angle CAB = 25°$ ($\angle$ sum $\Delta ABC = 180°$) ✓

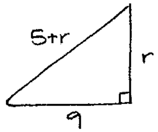
$\angle CKB = \angle CAB = 25°$ ($\angle$s in same seg. $=$) ✓

$\therefore k° = 25°$

Reas 3. Each reason required for the mark.

Learn the correct wording of the reasons.

(c) $\angle PTO = 90°$ (tangent $\perp$ radius at pt of tang) ✓



Pythagoras :

$(5+r)^2 = r^2 + 9^2$

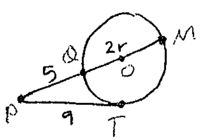
$25 + 10r + r^2 = r^2 + 81$

$10r = 56$

$r = 5.6$ cm ✓

This method is more successful except when algebraic errors occur in expanding $(5+r)^2$

OR



Product of the intercepts on the intersecting tangent & secant are equal. ✓

$5(5+2r) = 9^2$

$25 + 10r = 81$

$10r = 56$

$r = 5.6$ cm ✓

Note

point must be on circumference

$PT^2 = PQ \times PM$

not

$PT^2 = PQ \times QO$

or

$PT^2 = PQ \times PO$

(d) (i) $f(x) = 2\sin x - 10x + 5 = 0$

$f(0) = 2\sin 0 - 10 \times 0 + 5 = 5 > 0$

$f(1) = 2\sin 1 - 10 \times 1 + 5 = -3.3\ldots < 0$ ✓

& $f(x)$ is continuous. ✓

$\therefore 2\sin x - 10x + 5 = 0$ has a solution $0 < x < 1$

Calc. 4

must state curve is continuous. This word is very important.

(ii) $f'(x) = 2\cos x - 10$

$x_1 = 0.5$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ ✓

$= 0.5 - \frac{(2\sin 0.5 - 10 \times 0.5 + 5)}{(2\cos 0.5 - 10)}$

$= 0.616297\ldots$

$= 0.616$ to 3 sig. fig. ✓

Is your calculator in RADIAN mode?

Question 2 (12 marks)

(a) $\cot\theta + \tan\frac{\theta}{2}$

$= \frac{1}{\tan\theta} + \tan\frac{\theta}{2}$

$= \frac{1-t^2}{2t} + t$ ✓ [since $\tan\theta = \frac{2t}{1-t^2}$, $t = \tan\frac{\theta}{2}$]

$= \frac{1-t^2+2t^2}{2t}$

$= \frac{1+t^2}{2t}$

$= \frac{1}{\sin\theta}$ ✓ [since $\sin\theta = \frac{2t}{1+t^2}$]

$= \operatorname{cosec}\theta$

This question was very well done.

(b) (i) $\sqrt{3}\sin t + \cos t$ — Reas. 4

$R\sin t\cos\alpha + R\cos t\sin\alpha = R\sin(t+\alpha)$

$R\cos\alpha = \sqrt{3}$ ①

$R\sin\alpha = 1$ ②

$\frac{②}{①}$: $\tan\alpha = \frac{1}{\sqrt{3}}$, $\alpha = \frac{\pi}{6}$ ✓

$①^2 + ②^2$: $R^2 = \sqrt{3}^2 + 1^2$, $R = 2$ ✓

$\therefore \sqrt{3}\sin t + \cos t = 2\sin(t + \frac{\pi}{6})$

(ii) $\sqrt{3}\sin t + \cos t = 1$

$2\sin(t+\frac{\pi}{6}) = 1$

$\sin(t+\frac{\pi}{6}) = \frac{1}{2}$

$(t+\frac{\pi}{6}) = \pi n + (-1)^n(\sin^{-1}\frac{1}{2})$

$t + \frac{\pi}{6} = \pi n + (-1)^n\frac{\pi}{6}$

$t = \pi n + (-1)^n\frac{\pi}{6} - \frac{\pi}{6}$ ✓✓

[✓✗ for solutions between 0 & 2π]

Those that used the formula did much better - although you have to be careful to use the formula correctly at the right spot.

(c) Prove $5^n + 2(11)^n$ is a multiple of 3. — Comm. 3

* Prove true for $n=1$ ✓

$5^1 + 2(11)^1 = 27 = 3\times 9$ $\therefore$ true for $n=1$

* Assume true for $n=k$

ie. assume $5^k + 2(11)^k = 3M$ [M integer]

$5^k = 3M - 2(11)^k$

* Prove true for $n=k+1$

ie. prove $5^{k+1} + 2(11)^{k+1} = 3 \times \text{stuff}$

$5^{k+1} + 2(11)^{k+1}$

$= 5\times 5^k + 2\times 11\times 11^k$

$= 5(3M - 2\times 11^k) + 22\times 11^k$ ✓

$= 15M - 10\times 11^k + 22\times 11^k$

$= 15M + 12\times 11^k$

$= 3(5M + 4\times 11^k)$ which is a multiple of 3 ✓

* Therefore, if it's true for $n=k$, it's true for $n=k+1$. Since it's true for $n=1$, it's true for $n=2,3,\dots$

$\therefore$ by PMI it's true for all integers $n>0$.

* Please learn your index rules — never ever multiply bases!

* And don't substitute twice ($5^k = 3M - 2(11)^k$ & $2(11)^k = 3M - 5^k$)

It just won't fall out if you do.

(d) $W_1W_2W_3\ M_1M_2M_3M_4M_5$ — Reas. 3

(i) $3! \times 5!$ ways to carry out interviews ✓

($3!$: arrange women 1st; $5!$: arrange men next.)

* AND ⇒ MULTIPLY

* OR ⇒ ADD

(ii) # ways to choose 2, at least 1 W = total # ways choose 2 − # ways to choose so no women ✓

$= \binom{8}{2} - \binom{5}{2}$ ✓

$= 18$

OR # ways ≥1 W = # ways W,M + # ways WW ✓

$= \binom{3}{1}\binom{5}{1} + \binom{3}{2} = 18$ ✓

In general, this question was done well.

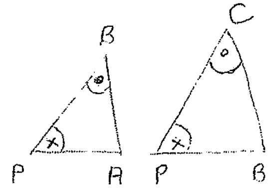
Question 3 (10 marks)

(a)(i) $\angle PBA = \angle TPA$ ($\angle$ bet. tang. & chord = $\angle$ in alt. seg.) ✓

$\angle TPA = \angle PCB$ (alt. $\angle$s $//$ =) ✓

$\therefore \angle PBA = \angle PCB$

Reas. 4



(ii) In ΔPBA & ΔPCB

$\angle PBA = \angle PCB$ (part i)

$\angle BPA = \angle CPB$ (common) ✓

$\therefore \Delta PBA \parallel\!| \Delta PCB$ (equiangular)

Note the order of the letters on the similar Δs put them/draw them in the same orientation

$\therefore \dfrac{PA}{PB} = \dfrac{PB}{PC}$ (corr sides in similar Δs in same ratio) ✓

$PB^2 = PA \times PC$

(b) (i) # unrestricted seating arrangements $= \underset{\text{way to seat 1st person}}{1} \times \underset{\text{ways to arrange rest}}{9!}$

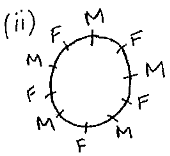
$= 9!$ ✓

$= 362\,880$

Reas 3

This question was very well done.

(ii)

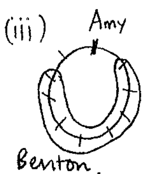


ways $= \underset{\text{way to seat 1st (say, M)}}{1} \times \underset{\text{ways to seat males}}{4!} \times \underset{\text{ways to seat females}}{5!}$

$= 4!\,5!$ ✓

$= 2880$

(iii)



ways $= \underset{\text{ways to seat 1st (Amy) say}}{1} \times \underset{\text{ways to seat Benton}}{7} \times \underset{\text{ways to seat rest}}{8!}$

$= 7 \times 8!$ ✓

OR, # ways = total # ways − # ways A, B together

$= 9! - 1 \times 8! \times 2!$ ✓

$= 282\,240$

(c) Prove $1\times1! + 2\times2! + \dots + n\times n! = (n+1)! - 1$

Comm 3

* Prove true for $n=1$

LHS $= 1\times1! = 1$ ✓

RHS $= (1+1)! - 1 = 1$ $\therefore$ true for $n=1$

Please set working out as two separate parts

LHS =

RHS =

$\therefore$ LHS = RHS

* Assume true for $n=k$

ie. assume $1\times1! + 2\times2! + \dots + k\times k! = (k+1)! - 1$

* Prove true for $n=k+1$

ie. prove $\underbrace{1\times1! + \dots + k\times k!} + (k+1)(k+1)! = (k+2)! - 1$

LHS $= (k+1)! - 1 + (k+1)(k+1)!$ ✓ using assumption

$= (k+1)! + (k+1)(k+1)! - 1$

Rearrange if necessary and factorize out $(k+1)!$.

$= (k+1)!\,(1 + k + 1) - 1$

$= (k+1)!\,(k+2) - 1$

$= (k+2)! - 1$

$=$ RHS ✓

No fudging please. Yes! It's obvious if you do!

$\therefore$ If true for $n=k$, it's true for $n=k+1$,

since true for $n=1$, it's true for $n=2,3,\dots$

$\therefore$ By PMI, it's true for all integers $n>0$.

Question 4 (11 marks)

(a)(i) $y = \cos^2 x$

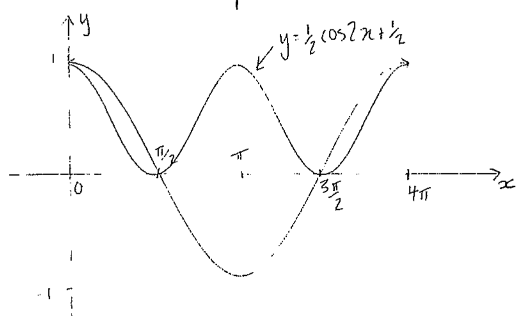
$= \frac{1}{2}\cos 2x + \frac{1}{2}$

stretch, squash, shift up.

✓ 2 correct transformations

✓ all correct.

Comm 2.



(ii) LHS $= \cos^4 x$

$= (\cos^2 x)^2$

$= \left(\frac{1}{2}\cos 2x + \frac{1}{2}\right)^2$

$= \frac{1}{4}\cos^2 2x + \frac{1}{2}\cos 2x + \frac{1}{4}$ ✓

$= \frac{1}{4}\left(\frac{1}{2}\cos 4x + \frac{1}{2}\right) + \frac{1}{2}\cos 2x + \frac{1}{4}$

$= \frac{1}{8}\cos 4x + \frac{1}{8} + \frac{1}{2}\cos 2x + \frac{1}{4}$

$= \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$ ✓

Reas 2.

* $(A+B)^2 = A^2 + \underline{\underline{2AB}} + B^2$

* If they give you some information then USE IT!

(iii) $V = \pi \int_0^{\pi/2} y_{top}^2 - y_{bottom}^2 \, dx$

$= \pi \int_0^{\pi/2} \cos^2 x - \cos^4 x \, dx$

$= \pi \int_0^{\pi/2} \frac{1}{2}\cos 2x + \frac{1}{2} - \left(\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx$

$= \pi \int_0^{\pi/2} \frac{1}{8} - \frac{1}{8}\cos 4x \, dx$ ✓

$= \frac{\pi}{8}\int_0^{\pi/2} 1 - \cos 4x \, dx$

* $V = \pi\int top^2 - bottom^2$

<u>NOT</u> $\pi \int (top - bottom)^2$

(this got you into a mess)

* use the previous parts of the question.

$V = \frac{\pi}{8}\int_0^{\pi/2} 1 - \cos 4x \, dx$

$= \frac{\pi}{8}\left[x - \frac{\sin 4x}{4}\right]_0^{\pi/2}$ ✓

$= \frac{\pi}{8}\left[\left(\frac{\pi}{2} - \frac{\sin 2\pi}{4}\right) - \left(0 - \frac{\sin 0}{4}\right)\right]$

$= \frac{\pi^2}{16}$ cube units ✓

Calc. 3

(b) (i) P E R M U T A T I O N

words $= \frac{11!}{2!}$ ✓

Reas. 4

(ii) Prob(all vowels together)

$= \frac{\text{\# ways vowels all together}}{\text{total \# ways}}$ ✓

$= \dfrac{\frac{7!}{2!} \times 5!}{\frac{11!}{2!}}$ ✓

$= \frac{7! \times 5!}{11!}$

All vowels together:

E, U, A, I, O

P R M T T N

$\frac{7!}{2!}$ ways to arrange group (2 Ts)

×5! ways to arrange vowels.

* always be on the lookout for repeated letters!
* vowels can be rearranged (they are not simply written on a card 'AEIOU')

* Remember to read the question carefully to see if they want a probability or not.

(iii) _ _ _ _ _ _ _ _ _ _ _

11 positions

→ $\binom{11}{5}$ ways to choose 5 positions for the vowels A, E, I, O, U, & only 1 way to arrange those vowels.

→ $\frac{6!}{2!}$ ways to then arrange consonants.

∴ # words $= \binom{11}{5} \times \frac{6!}{2!}$ ✓

* Hard question! Well done to those who solved it.

*Alternative solution: $\frac{1}{5!}$ of the total arrangements will have the vowels in order

∴ # words $= \frac{1}{5!} \times \frac{11!}{2!}$