Name: ....

Teacher: ....

SCEGGS Darlinghurst

2005
Higher School Certificate
Assessment Task 2
Friday 10th June

Mathematics Extension 1

Task Weighting: 35%

General Instructions

- Time allowed – 75 minutes
- This paper has **four** questions
- Attempt **all** questions
- Write using blue or black pen
- Answer all questions on the pad paper provided
- Begin each question **on a new page**
- Write your name at the top of each page
- Mathematical templates, geometrical equipment and approved scientific calculators may be used
- A table of standard integrals is provided

Question	Comm	Reason	Calculus	Marks
1	/4		/3	/15
2		/7	/7	/15
3		/3	/3	/16
4	/3	/5	/4	/14
TOTAL	/7	/15	/17	/60

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 (15 marks) **Marks**

(a) The interval AB lies between $A(-1,4)$ and $B(5,-3)$. Find the co-ordinates of the point P which divides the interval AB externally in the ratio $1:3$. **2**

(b) Find $\int \frac{x}{\left(x^2+2\right)^2}\,dx$ using the substitution $u = x^2 + 2$. **3**

(c) (i) State the domain and range of $y = 3\sin^{-1}2x$. **2**

(ii) Sketch $y = 3\sin^{-1}2x$, showing all important features. **1**

(d) Solve the inequality $\frac{2x}{x-2} \le 3$. **3**

(e) (i) Sketch $f(x) = \sqrt{x-2}$. **1**

(ii) Explain why the function $f(x) = \sqrt{x-2}$ has an inverse function $f^{-1}(x)$. **1**

(iii) Write down the equation of the inverse function $f^{-1}(x)$. **1**

(iv) On the same set of axes as your graph for (i), sketch $y = f^{-1}(x)$. **1**

START A NEW PAGE

Question 2 (15 marks) **Marks**

(a) Find the exact value of $\tan\left(\sin^{-1}\left(-\frac{1}{3}\right)\right)$. **2**

(b) (i) Sketch the curve $y = 1 + \cos x$ for $0 \le x \le 2\pi$. **1**

(ii) Calculate the exact volume generated when the arc of $y = 1 + \cos x$ between $x = 0$ and $x = \frac{3\pi}{2}$ is rotated about the x-axis. **3**

(c) Use the substitution $u = 1 - x$ to find $\int_0^1 \frac{x}{\sqrt{1-x}}\,dx$. **4**

(d) (i) Find $\frac{d}{dx}\left[\sqrt{1-x^2} + x\sin^{-1}x\right]$. **3**

(ii) Hence, evaluate $\int_0^{1/2} \sin^{-1}x\,dx$.
Leave your answer in exact form. **2**

START A NEW PAGE

Question 3 (16 marks) **Marks**

(a) (i) Find the exact gradients of the tangents to the curve $y = \sin^{-1}\frac{x}{2}$ at $x = 0$ and $x = 1$. **3**

(ii) Find the acute angle between these two tangents correct to the nearest minute. **2**

(b) (i) Show that $3\sin\theta + \cos\theta = 2$ can be written as $3t^2 - 6t + 1 = 0$ if $t = \tan\frac{\theta}{2}$. **2**

(ii) Hence, or otherwise, solve $3\sin\theta + \cos\theta = 2$ for $0° \le \theta \le 360°$. Leave your answers correct to the nearest minute. **3**

(c) Show that $\cos^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{-3}{4}\right) = \frac{\pi}{2}$. **3**

(d) Evaluate $\displaystyle\int_0^{5/2} \frac{dx}{\sqrt{25 - 2x^2}}$. **3**

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Question 4 (14 marks) **Marks**

(a) Find the general solution/s in radians for which $3\cos 2x = 2 + \sin x$. **3**

(b) Use the substitution $u = \sin x$ to show $\displaystyle\int_0^{\pi/6} \frac{\cos x}{4\sin^2 x + 1}\,dx = \frac{\pi}{8}$. **4**

(c) (i) State the domain and range of $y = 2\cos^{-1}(1 - x)$. **2**

(ii) Sketch the curve $y = 2\cos^{-1}(1 - x)$. **1**

(iii) On the same set of axes, sketch $y = -\pi x + 2\pi$. **1**

(iv) Explain why: **1**

$$\int_0^2 2\cos^{-1}(1 - x)\,dx = \int_0^2 (-\pi x + 2\pi)\,dx$$

(v) Show that $\pi = \frac{1}{2}\int_0^2 2\cos^{-1}(1 - x)\,dx$ **2**

End of paper

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1a) $A=(-1,4)$ $B(5,-3)$

external ratio $-1:3$

$P=\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$

$=\left(\frac{-1\times5+3\times-1}{-1+3}, \frac{-1\times-3+3\times4}{-1+3}\right)$

$=\left(-4, 7\tfrac{1}{2}\right)$ ✓✓

b) $\int \frac{x}{(x^2+2)^2}\,dx$ $u=x^2+2$

$\frac{du}{dx}=2x$

$dx=\frac{du}{2x}$

$=\int \frac{x}{u^2}\cdot\frac{du}{2x}$ ✓ Calc

$=\frac{1}{2}\int u^{-2}\,du$

$=\frac{1}{2}\left[-u^{-1}\right]+c$ ✓ Calc

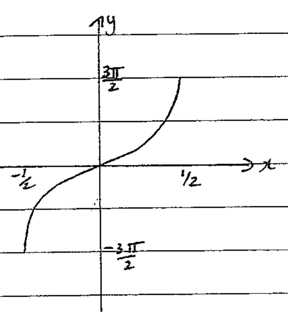
$=\frac{-1}{2(x^2+2)}+c$ ✓ Calc

c) i) $y=3\sin^{-1}2x$

$D: -\tfrac{1}{2}\le x\le \tfrac{1}{2}$ ✓

$R: -\frac{3\pi}{2}\le y\le\frac{3\pi}{2}$ ✓

ii)



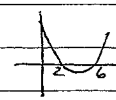
✓ com

d) $\frac{2x}{x-2}\le 3$

$2x(x-2)\le 3(x-2)^2$

$2x^2-4x\le 3x^2-12x+12$

$x^2-8x+12\ge 0$ ✓

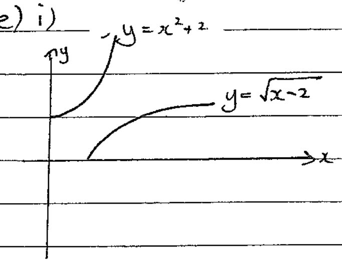


$(x-6)(x-2)\ge 0$

$x<2,\ x\ge 6$

✓ ✓

e) i)



✓ com

ii) It passes the horizontal line test. ✓ com

iii) f^{-1}: $x=\sqrt{y-2}$

$x^2=y-2$

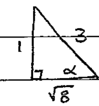
$y=x^2+2$ ✓

iv) see graph in (i) ✓ com

2a) $\tan\left(\sin^{-1}\left(-\frac{1}{3}\right)\right)$

Let $\sin^{-1}\left(-\frac{1}{3}\right)=\alpha$

$\therefore \sin\alpha=\frac{-1}{3}$



$\sqrt{8}=2\sqrt{2}$

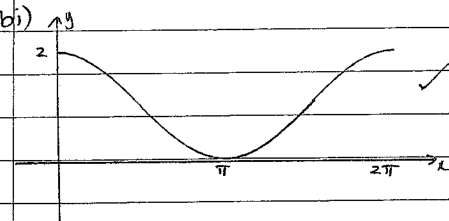
4th quad.

$\therefore \tan\left(\sin^{-1}\left(-\frac{1}{3}\right)\right)$

$=\tan\alpha$

$=\frac{-1}{2\sqrt{2}}$ ✓✓ Reas

b) i)



✓

ii) $V=\pi\int_0^{3\pi/2}(1+\cos x)^2\,dx$

$=\pi\int_0^{3\pi/2}(1+2\cos x+\cos^2 x)\,dx$ ✓ Calc

$=\pi\int_0^{3\pi/2}\left(1+2\cos x+\frac{1}{2}(\cos 2x+1)\right)dx$

$=\pi\left[x+2\sin x+\frac{1}{2}\left(\frac{1}{2}\sin 2x+x\right)\right]_0^{3\pi/2}$ ✓ Calc

$=\pi\left[\left(\frac{3\pi}{2}+2\sin\frac{3\pi}{2}+\frac{1}{4}\sin 3\pi+\frac{3\pi}{4}\right)-0\right]$

$=\pi\left(\frac{9\pi}{4}-2\right)$ ✓ Calc

c) $\int_0^1 \frac{x}{\sqrt{1-x}}\,dx$ $u=1-x$

$\frac{du}{dx}=-1$

$dx=-du$

when $x=1$ $u=0$

$x=0$ $u=1$

$u=1-x \Rightarrow x=1-u$

$\int_1^0 \frac{1-u}{\sqrt{u}}\cdot -du$ ✓✓ Calc

$\int_0^1\left(u^{-1/2}-u^{1/2}\right)du$ ✓ Calc

$=\left[2u^{1/2}-\frac{2}{3}u^{3/2}\right]_0^1$

$=\left(2(1)-\frac{2}{3}(1)\right)-0$

$=1\tfrac{1}{3}$ ✓ Calc

d) i) $\frac{d}{dx}\left((1-x^2)^{1/2}+x\sin^{-1}x\right)$ Reas ✓✓

$=\frac{1}{2}(1-x^2)^{-1/2}\times-2x+x\times\frac{1}{\sqrt{1-x^2}}+\sin^{-1}x$

$=\frac{-x}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}+\sin^{-1}x$

$=\sin^{-1}x$ ✓ Reas

ii) $\int_0^{1/2}\sin^{-1}x\,dx$

$=\left[\sqrt{1-x^2}+x\sin^{-1}x\right]_0^{1/2}$ ✓ Reas

$=\left(\sqrt{1-\frac{1}{4}}+\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)\right)-\left(\sqrt{1-0}+0\right)$

$=\frac{\sqrt{3}}{2}+\frac{\pi}{12}-1$ ✓ Reas

3a) i) $y = \sin^{-1}\frac{x}{2}$

$y' = \frac{1}{\sqrt{4-x^2}}$ ✓

at $x=0$ $y' = \frac{1}{2}$ ✓

$x=1$ $y' = \frac{1}{\sqrt{3}}$ ✓

ii) $\tan\theta = \left|\frac{\frac{1}{2} - \frac{1}{\sqrt{3}}}{1 + \frac{1}{2}\times\frac{1}{\sqrt{3}}}\right|$ ✓

$= 0.060023\ldots$

$\theta = 3^\circ 26'$ ✓

b i) $3\sin\theta + \cos\theta = 2$

$3\left(\frac{2t}{1+t^2}\right) + \left(\frac{1-t^2}{1+t^2}\right) = 2$ ✓

$6t + 1 - t^2 = 2 + 2t^2$ ✓

$3t^2 - 6t + 1 = 0$

b ii) $3t^2 - 6t + 1 = 0$

$t = \frac{6 \pm \sqrt{36 - 4\times3\times1}}{6}$

$= \frac{6 \pm \sqrt{24}}{6}$

$= 1.8164\ldots,\ 0.1835\ldots$

$\therefore \tan\frac{\theta}{2} = 1.8164\ldots$ $\quad \tan\frac{\theta}{2} = 0.1835\ldots$

$\frac{\theta}{2} = 61^\circ 10'$ $\quad \frac{\theta}{2} = 10^\circ 24'$

$\theta = 122^\circ 20'$ $\quad \theta = 20^\circ 48'$

Test $\theta = 180^\circ$

$3\sin 180^\circ + \cos 180^\circ$

$= -1$

$\therefore$ $\theta = 180^\circ$ is not a solution

$\therefore$ $\theta = 20^\circ 48',\ 122^\circ 20'$

c) $\cos^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{-3}{4}\right) = \frac{\pi}{2}$

Let $\cos^{-1}\left(\frac{3}{5}\right) = \alpha$

then $\cos\alpha = \frac{3}{5}$ $\quad$ 4, 5, 3, α

1st quad ✓ Reas

Let $\tan^{-1}\left(\frac{-3}{4}\right) = \beta$

$\tan\beta = \frac{3}{4}$ $\quad$ 3, 5, 4, β

4th quad

$\sin(\alpha - \beta)$

$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \frac{4}{5}\times\frac{4}{5} - \frac{3}{5}\times\frac{-3}{5}$

$= 1$ ✓Reas

$\sin(\alpha-\beta) = 1$

$\alpha - \beta = \sin^{-1}(1)$

$= \frac{\pi}{2}$ ✓Reas

$\therefore$

$\cos^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{-3}{4}\right) = \pi/2$

d) $\int_0^{5/2} \frac{dx}{\sqrt{25-2x^2}}$

$= \int_0^{5/2} \frac{dx}{\sqrt{2\left(\frac{25}{2} - x^2\right)}}$

$= \frac{1}{\sqrt{2}}\int_0^{5/2} \frac{dx}{\sqrt{\frac{25}{2} - x^2}}$ ✓Calc

$= \left[\frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}x}{5}\right]_0^{5/2}$ ✓Calc

$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}}{5}\times\frac{5}{2} - \frac{1}{\sqrt{2}}\sin^{-1}0$

$= \frac{\pi}{4\sqrt{2}}$ ✓ Calc

4a) $3\cos 2x = 2 + \sin x$

$3(1 - 2\sin^2 x) = 2 + \sin x$

$3 - 6\sin^2 x = 2 + \sin x$

$6\sin^2 x + \sin x - 1 = 0$

$(3\sin x - 1)(2\sin x + 1) = 0$ ✓Reas

$\sin x = \frac{1}{3}$ or $\sin x = -\frac{1}{2}$

$x = \sin^{-1}\frac{1}{3}$ $\quad x = \frac{-\pi}{6}, \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

if n is an integer

$x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{3}\right)$, ✓Reas

$= n\pi + (-1)^n\left(\frac{-\pi}{6}\right)$ ✓Reas

b) $\int_0^{\pi/6} \frac{\cos x}{4\sin^2 x + 1}\,dx$ $\quad u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

when $x = \pi/6$ $\quad u = \frac{1}{2}$

$x = 0$ $\quad u = 0$

$\int_0^{1/2} \frac{\cos x}{4u^2+1}\cdot\frac{du}{\cos x}$ ✓Calc

$\int_0^{1/2} \frac{1}{4u^2+1}\,du$

$\frac{1}{4}\int_0^{1/2} \frac{1}{u^2 + 1/4}\,du$ ✓Calc

$\left[\frac{1}{4}\times 2\tan^{-1} 2u\right]_0^{1/2}$ ✓Calc

$= \frac{1}{2}\tan^{-1}(1) - \frac{1}{2}\tan^{-1}(0)$

✓ Calc

$= \frac{\pi}{8}$

c i) $y = 2\cos^{-1}(1-x)$

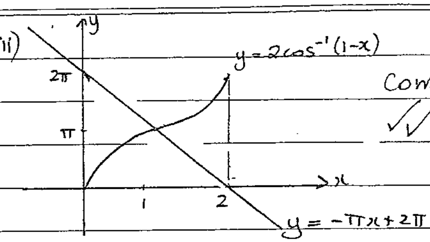
D: $-1 \le 1-x \le 1$

$-2 \le -x \le 0$

$0 \le x \le 2$ ✓

R: $0 \le y \le 2\pi$ ✓

c ii)



Com ✓✓

iii) See sketch.

iv) $\int_0^2 2\cos^{-1}(1-x)\,dx$ is the area under the curve $y = 2\cos^{-1}(1-x)$ between $x=0$ and $x=2$.

$\int_0^2 -\pi x + 2\pi\,dx$ is the area under the line $y = -\pi x + 2\pi$ between $x=0$ and $x=2$.

As the curve has rotational symmetry about the point $(1, \pi)$ these areas are equal. ✓Com

v) $\frac{1}{2}\int_0^2 2\cos^{-1}(1-x)\,dx$

$= \frac{1}{2}\int_0^2 (-\pi x + 2\pi)\,dx$ ✓Reas

$= \frac{1}{2}\times$ Area of Δ

$= \frac{1}{2}\times\frac{1}{2}\times2\times2\pi$ ✓ Reas

$= \pi$