[redacted]
Centre Number

Student Number

SCEGGS Darlinghurst

2007

HSC Assessment 3
29 May 2007

Mathematics

Assessment Outcomes: H2, H3, H4, H5, H6, H8 and H9

General Instructions
- Time allowed – 80 minutes
- This paper has **five** questions
- Write your Student Number at the top of each page
- Attempt **all** questions
- Marks may be deducted for careless or badly arranged work
- Mathematical templates, geometrical equipment and graphics calculators may be used
- A table of standard integrals is provided at the back of this paper

Weighting: 25%

Total marks (56)
- Attempt Questions 1 – 5
- START EACH QUESTION ON A NEW PAGE

Question	Communication	Calculus	Reasoning	Marks
1		/3		/11
2	/1		/5	/10
3	/4	/8		/12
4		/3	/5	/9
5	/6	/7		/13
TOTAL	/11	/21	/10	/55

Average: ______ **Standard Deviation:** ______ **Rank:** ______

Parents Signature: ....

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56 marks
Attempt Questions 1 – 5

Answer each question on the pad paper provided
Write your Student Number at the top of each page
Start each question on a new page

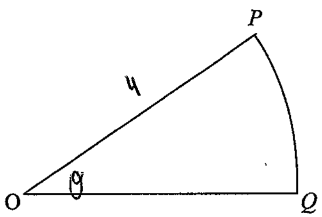
Marks

Question 1 (11 marks)

(a) Evaluate $\dfrac{\log e\, 5.2}{6.5}$ correct to two decimal places. **2**

(b) Solve $2^{x+1} = 6$ correct to three significant figures. **2**

(c) **2**



Not to scale

POQ is a sector of a circle, centre O and radius 4 cm. The length of the arc PQ is 6π cm.

Calculate the exact area of the sector POQ.

(d) Given that $\log_a 2 = 1.26$ and $\log_a 5 = 1.46$ find:

(i) $\log_a\left(\dfrac{16}{5}\right)$ **1**

(ii) $\log_a 5a$ **1**

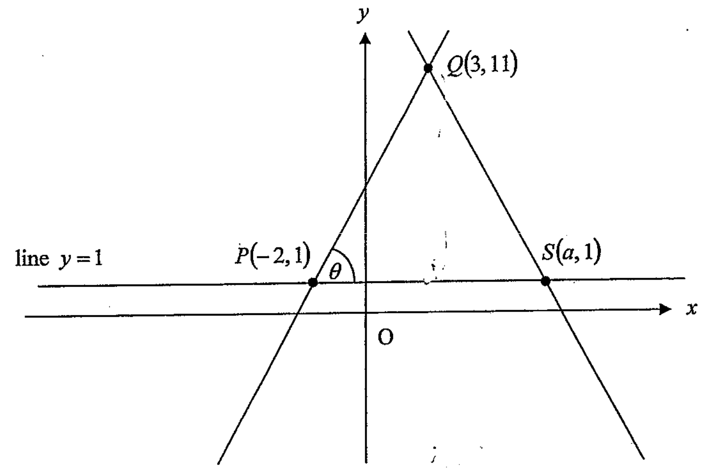
(e) Show that the equation of the tangent to the curve $y = e^{x^3}$ at the point whose x co-ordinate is 2 is given by: **3**

$$y = 12e^8 x - 23e^8$$

- **Start a new page**

Marks

Question 2 (10 marks)



In the diagram P is the point $(-2, 1)$ and Q is the point $(3, 11)$. Both points P and S lie on the line $y = 1$. Point S has co-ordinates $(a, 1)$.

(a) Find the equation of the line PQ. **2**

(b) The line PQ makes an angle of θ with the line $y = 1$. Show that $\theta = 63°$, correct to the nearest degree. **1**

(c) K is a point in the fourth quadrant such that $PQSK$ is a rhombus. Find the co-ordinates of the point K. **2**

(d) Explain why the x co-ordinate of point S is 8. **1**

(e) Find the perpendicular distance of S to the line PQ. **1**

(f) Find the exact length of PQ. **1**

(g) Hence or otherwise, determine the area of the rhombus $PQSK$. **2**

- **Start a new page**

Marks

Question 3 (12 marks)

(a) Differentiate with respect to x.

(i) $\cos\frac{x}{3}$ **2**

(ii) $x\tan x$ **2**

(iii) $\dfrac{e^x-1}{e^x+1}$ (Give your answer in simplest form.) **2**

(b) Jessica was asked to evaluate $\int_0^{\frac{\pi}{4}} \cos 5x\, dx$. As part of her working, she incorrectly wrote that $\sin\frac{5\pi}{4} = \frac{1}{\sqrt{2}}$.

(i) Explain why Jessica is incorrect. **1**

(ii) Find the exact value of $\int_0^{\frac{\pi}{4}} \cos 5x\, dx$ **2**

(c) (i) Without using calculus, sketch the curve $y = e^x + 1$ showing all important features. **1**

(ii) On the same set of axes, draw the line $y = x$. Use your graph to determine the number of solutions to the equation $e^x + 1 - x = 0$. **2**

- **Start a new page**

Marks

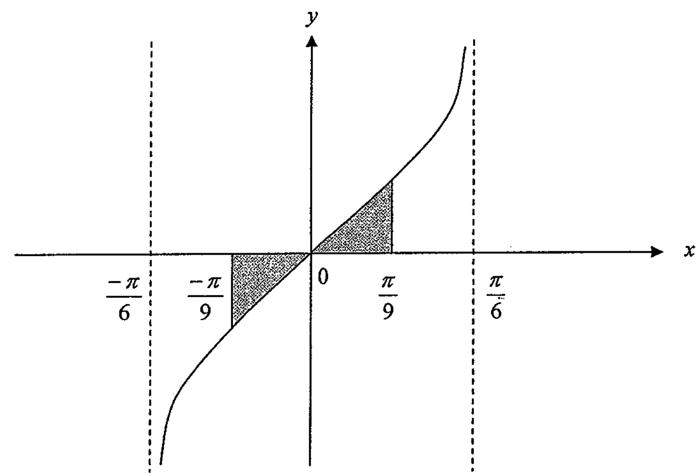
Question 4 (9 marks)

(a) (i) State the domain of $y = \log_a x$. **1**

(ii) Solve for x. **3**

$$\log_2 x + \log_2(x-1) = 1$$

(b) The diagram shows part of the graph of the function $y = \tan 3x$.



The shaded region is bounded by the curve, the x axis and the lines $x = \frac{-\pi}{9}$ and $x = \frac{\pi}{9}$. The region is rotated about the x axis to form a solid of revolution.

(i) By using the identity $1 + \tan^2 x = \sec^2 x$ show that the volume of the solid is given by **2**

$$V = 2\pi \int_0^{\frac{\pi}{9}} \left(\sec^2 3x - 1\right) dx$$

(ii) Find the exact value of this volume. **3**

- **Start a new page**

Marks

Question 5 (13 marks)

(a) Evaluate $\int_{-1}^{3} \frac{x}{x^2+3}\,dx$. **3**

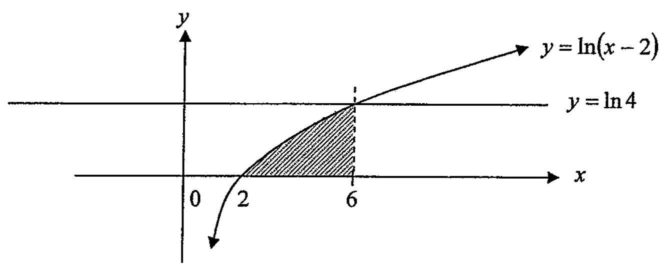
Give your answer in exact form.

(b) (i) Using about one-third of a page, sketch the curve $y = 4\sin 2x$ for $-\pi \le x \le \pi$. **2**

(ii) On the same diagram, sketch $y = \frac{x}{2}$ and shade the region represented by $\int_{0}^{\frac{\pi}{4}} \left(4\sin 2x - \frac{x}{2}\right) dx$. **2**

(iii) Find the exact value of the integral in part (ii). **2**

(c) The diagram shows the curve $y = \ln(x-2)$ and the line $y = \ln 4$.



The shaded area bounded by the curve $y = \ln(x-2)$, the line $x = 6$ and the x axis is shown.

(i) Explain why the shaded area is given by **2**

$$A = 6\ln 4 - \int_{0}^{\ln 4} \left(e^{y} + 2\right) dy$$

(ii) Find the exact value of this area. **2**

End of paper

Year 12 Assessment task #3 2007

Question 1

a) 0.25 (2 d.p.) ✓✓

This part was very well done! ☺

b) $2^{x+1} = 6$

Take logs both sides

$\log_e 2^{x+1} = \log_e 6$

$(x+1).\log_e 2 = \log_e 6$ ✓

$x+1 = \dfrac{\log_e 6}{\log_e 2}$

$x = \dfrac{\log_e 6}{\log_e 2} - 1$

$\doteqdot 1.58$ (3 s.f.) ✓

This is a standard question. Also well done!

Another method is to use the definition of a logarithm and then the change of base rule.

$\log_2 6 = x+1$

$x = \log_2 6 - 1$

$x = \dfrac{\log_e 6}{\log_e 2} - 1$

c) Find θ

$l = r\theta$

$6\pi = 4.\theta$

$\theta = \dfrac{6\pi}{4}$

$\theta = \dfrac{3\pi}{2}$ ✓

Find Area

$A = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 4^2 \times \frac{3\pi}{2}$

$= 12\pi \quad cm^2$ ✓

- Note the exact area is required.
- Some silly mistakes in this calculation. Be careful of cancelling fractions. Maybe use your calculator.

d) i) $\log_a\left(\frac{16}{5}\right) = \log_a 16 - \log_a 5$

$= \log_a 2^4 - \log_a 5$

$= 4\log_a 2 - \log_a 5$

$= 4 \times 1.26 - 1.46$

$= 3.58$ ✓

ii) $\log_a 5a = \log_a 5 + \log_a a$

$= 1.46 + 1$

$= 2.46$ ✓

Know your log. laws.

$\log_a \frac{m}{n} = \log_a m - \log_a n$

$\log_a m^p = p\log_a m$

$\log_a mn = \log_a m + \log_a n$

$\log_a a = 1$

Some crazy working was shown in part d). Some more practice might be a good idea.

e) $y = e^{x^3}$

$y' = e^{x^3} \times 3x^2$

$= 3x^2 e^{x^3}$

when $x = 2$, gradient tangent ✓

$m_1 = 3 \times 2^2 \times e^{2^3}$

$= 12e^8$

At $x = 2$, $y = e^{2^3} = e^8$ ✓

point $(2, e^8)$

Equation of tangent

$y - y_1 = m(x - x_1)$

$y - e^8 = 12e^8(x-2)$

$y - e^8 = 12e^8 x - 24e^8$ ✓

$y = 12e^8 x - 23e^8$

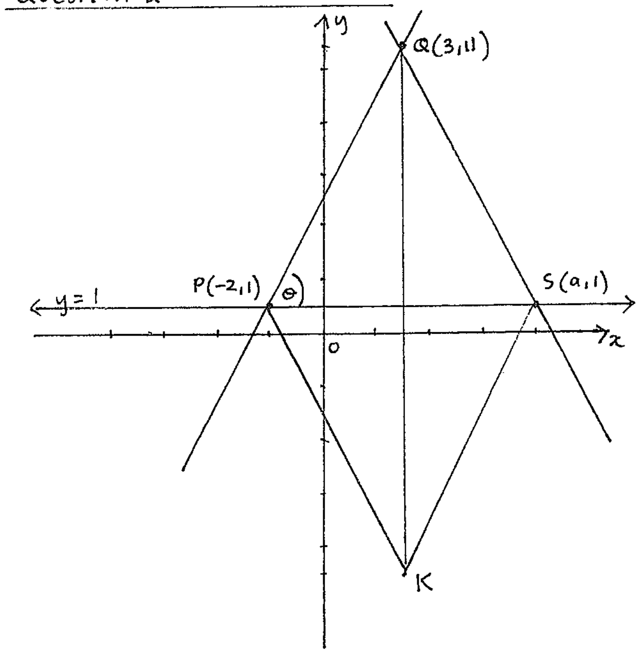
Calc 3

Some mistakes with finding y' and then substituting the $x = 2$

Note that $e^{2^3} = e^8$ not e^6.

Some creative fudging here to try to get the answer. It will not fool the HSC markers!

Question 2



a) $m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{11-1}{3+2}$

$= \frac{10}{5}$

$= 2$ ✓

Equation PQ

$y - y_1 = m(x - x_1)$

$y - 1 = 2(x+2)$

$y - 1 = 2x + 4$

$y = 2x + 5$ ✓

Done well by nearly all students

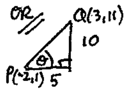
b) $m_{PQ} = 2$

using $m = \tan\theta$

$\tan\theta = 2$

$\therefore \theta = 63°$ (to nearest degree)

OR



$\tan\theta = \frac{10}{5}$

$\tan\theta = 2$ ✓

Reas 1

Nearly all students knew the relationship between the angle and the gradient.

Just be careful writing $\theta = \tan^{-1} 2$ this was left out quite a few times.

c)

K has coordinates $(3, -9)$ ✓✓

Reas 2

Most students realised they needed to subtract 10 but wrote $(3, -10)$

d) Since PQSK is a rhombus, ΔPQS is an isosceles triangle. The diagonals of a rhombus bisect at right angles. The diagonals bisect at the point $(3,1)$.

$\therefore$ The point S is $(8,1)$ as shown clearly on the diagram on previous page. ✓

Comm 1

This was done poorly and reflects the need to work on questions that require explanations.

Students tried to describe how they calculated 'a' without discuss the geometrical reasons why it works.

e) PQ: $2x - y + 5 = 0$ $S(8,1)$

$pd = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}} \right|$

$= \left| \frac{2\times8 - 1 + 5}{\sqrt{4+1}} \right|$

$= \frac{20}{\sqrt{5}}$ ✓

$= \frac{20}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= 4\sqrt{5}$ units

Done well

f) $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(3+2)^2 + (11-1)^2}$

$= \sqrt{5^2 + 10^2}$

$= \sqrt{125}$ ✓

$= 5\sqrt{5}$ units

Done well

g) Area rhombus PQSK

Here are three methods you could choose.

$A = \frac{1}{2}$ product of diagonals

$= \frac{1}{2} \times QK \times PS$

$= \frac{1}{2} \times 10 \times 20$

$= 100\ u^2$ ✓✓

$A = \text{area } \Delta PQK \times 2$

$A = \frac{1}{2} ab \sin C \times 2$

$= \frac{1}{2} \times \sqrt{125} \times \sqrt{125} \times \sin 63° \times 2$

$\doteq 100\ u^2$

$A = \text{base} \times \text{perp. height}$

$= \sqrt{125} \times 4\sqrt{5}$

$= 100\ u^2$

Reas 2

Done reasonably well although quite a few students quoted $A = \frac{1}{2}xy$ as the area but didn't know what x and y stood for.

I saw $A = \frac{1}{2} \times 4\sqrt{5} \times 5\sqrt{5}$ quite often

Question 3

a) i) $y = \cos\frac{x}{3}$

$= \cos\frac{x}{3}$

$y' = -\frac{1}{3}\sin\frac{x}{3}$

✓ ✓

Calc 2

This question was very well done, although some integrated instead of differentiated.

ii) $y = x\tan x$

$u = x$ $\quad v = \tan x$

$u' = 1$ $\quad v' = \sec^2 x$

using product rule

$y' = vu' + uv'$

$= \tan x + x.\sec^2 x$ ✓ ✓

Calc 2

Again, well done.

iii) $y = \dfrac{e^x - 1}{e^x + 1}$

$u = e^x - 1$ $\quad v = e^x + 1$

$u' = e^x$ $\quad v' = e^x$

using quotient rule

$$y' = \frac{vu' - uv'}{v^2}$$

$$= \frac{e^x(e^x+1) - e^x(e^x-1)}{(e^x+1)^2}$$ ✓

$$= \frac{(e^x)^2 + e^x - (e^x)^2 + e^x}{(e^x+1)^2}$$

$$= \frac{2e^x}{(e^x+1)^2}$$ ✓

Calc 2

- Most got the first mark for correct substitution into the quotient rule.
- Very poor simplifying to get a second mark.
- Note that $e^{x^2} = e^{(x^2)}$, if you want to square e^x, you must write $(e^x)^2 = e^{2x}$.
- Many did not include brackets in their first line & this caused them grief.

b) i) Jessica is incorrect because $\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$

$\frac{5\pi}{4} = 225°$, This angle is in quadrant 3.

$\sin\theta$ is negative in quadrant 3. ✓

Comm 1

Poorly done! Explain, means you cannot simply state the correct answer.

ii) $\displaystyle\int_0^{\pi/4} \cos 5x \, dx$

$= \left[\frac{1}{5}\sin 5x\right]_0^{\pi/4}$ ✓

$= \frac{1}{5}\sin\frac{5\pi}{4} - \frac{1}{5}\sin 0$

$= \frac{1}{5} \times -\frac{1}{\sqrt{2}} - \frac{1}{5} \times 0$

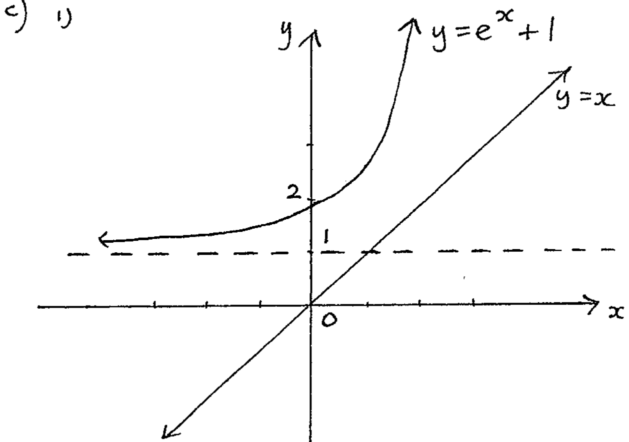
$= -\frac{1}{5\sqrt{2}}$ ✓

Calc 2

Integration step was well done.

Many did not know how to evaluate $\sin\left(\frac{5\pi}{4}\right)$. Note that $\sin\left(\frac{5\pi}{4}\right) \neq 5\sin\left(\frac{\pi}{4}\right)$

c) i)



Comm 1

✓ must show asymptote at $y=1$ and intercept at $(0,2)$

- Many forgot the asymptote at $y = 1$

ii) Sketch line $y = x$

$e^x - x + 1 = 0$

$e^x + 1 = x$ ✓

By drawing the two graphs $y = e^x + 1$ and $y = x$, the solutions are given by the points of intersection.

Since the two graphs will never intersect, there are no possible solutions. ✓

- Many did not get this last mark because their sketch of $y = e^x + 1$ was too "flat"

Comm 2

Question 4

a) i) $y = \log_a x$

domain: $x > 0$ ✓

ii) $\log_2 x + \log_2(x-1) = 1$

$\log_2 x(x-1) = 1$

by definition

$2^1 = x(x-1)$ ✓

$2 = x^2 - x$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$ ✓

$x = 2 \qquad x = -1$

But $x = -1$ is not a solution because $\log_2(-1)$ is undefined

$\therefore x = 2$ is the only solution ✓

Reas 3

Some people got the answer $x = 2$ but it was usually from incorrect methods. Sorry, no marks in that case.

(1) Correct quadratic equation

(1) Two solutions found from a quadratic equation

(1) Explanation about why $x \neq -1$ using domain of logarithm.

b) i) Volume about the x axis

$V = \pi \int_a^b y^2 \, dx$

$= \pi \int_{-\pi/9}^{\pi/9} \tan^2 3x \, dx$ ✓

Since $y = \tan 3x$ is an odd function

$= 2\pi \int_0^{\pi/9} \tan^2 3x \, dx$ ✓

$= 2\pi \int_0^{\pi/9} (\sec^2 3x - 1) \, dx$

Reas 2

must be stated that $y = \tan 3x$ is odd. That is why the area below the x axis is equal to the area above the x axis.

ii) $V = 2\pi \left[\frac{1}{3}\tan 3x - x \right]_0^{\pi/9}$ ✓

$= 2\pi \left\{ \frac{1}{3}\tan\left(3 \times \frac{\pi}{9}\right) - \frac{\pi}{9} - 0 \right\}$

$= 2\pi \left\{ \frac{1}{3}\tan\frac{\pi}{3} - \frac{\pi}{9} \right\}$ ✓

$= 2\pi \left\{ \frac{1}{3} \times \sqrt{3} - \frac{\pi}{9} \right\}$

$= \frac{2\sqrt{3}\pi}{3} - \frac{2\pi^2}{9} \quad u^3$ ✓

Calc 3

You have to realise that $(\sec^2 3x - 1)$ are separate functions $\neq \sec^2(3x-1)$

(1) Correct integration

(1) Correct substitution AND simplification

(1) Correct evaluation of $\tan\pi/3 = \sqrt{3}$

Question 5

a) $\int_{-1}^{3} \frac{x}{x^2+3} \, dx$

$= \frac{1}{2} \int_{-1}^{3} \frac{2x}{x^2+3} \, dx$

$= \frac{1}{2} \left[\ln(x^2+3) \right]_{-1}^{3}$ ✓✓

$= \frac{1}{2} \left\{ \ln(9+3) - \ln((-1)^2+3) \right\}$

$= \frac{1}{2} \left\{ \ln 12 - \ln 4 \right\}$

$= \frac{1}{2} \ln \frac{12}{4}$ ✓

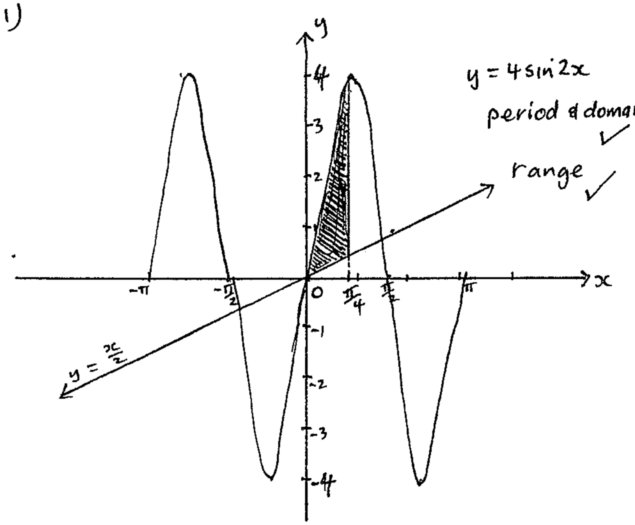
$= \frac{1}{2} \ln 3$

Calc 3

The integration was well done but after substitution many were confused as to how to simplify using log laws

$\left(\frac{12}{4} = 3 \right)$!

b) i)



period & domain ✓

range ✓

Comm 2

- When sketching both $y = 4\sin 2x$ & $y = \frac{x}{2}$ on the same number plane please use the same scale.
- Note that the region required was from $x = 0$ to $x = \frac{\pi}{4}$

ii) $y = \frac{x}{2}$

x	0	$\frac{\pi}{2}$	π
y	0	0.8	1.6

✓ line

✓ region shaded.

Comm 2

iii) $\int_0^{\pi/4} \left(4\sin 2x - \frac{x}{2}\right) dx$ $= \left[4 \times -\frac{1}{2}\cos 2x - \frac{1}{2} \times \frac{x^2}{2} \right]_0^{\pi/4}$ ✓ $= \left[-2\cos 2x - \frac{1}{4}x^2 \right]_0^{\pi/4}$ $= \left\{ -2\cos\left(2\times\frac{\pi}{4}\right) - \frac{1}{4}\times\left(\frac{\pi}{4}\right)^2 \right\} - \left\{ -2\cos 0 - 0 \right\}$ $= -2\cos\frac{\pi}{2} - \frac{1}{4}\times\frac{\pi^2}{16} - -2\times 1$ $= 0 - \frac{\pi^2}{64} + 2$ $= 2 - \frac{\pi^2}{64} \quad u^2$ ✓	Calc 2 • Most were ok on the integration although remember to divide by 2 not × by 2 when integrating $\sin 2x$ • After substitution there were many errors in the simplification process careful with - brackets - +/-
c) i) Area $= \int_2^6 \ln(x-2)\, dx$ this cannot be found easily so the area must be calculated to the y axis. <u>Make x the subject</u> $y = \ln(x-2)$ $y = \log_e(x-2)$ By definition $e^y = x-2$ $x = e^y + 2$ ✓ <u>Shaded Area</u> = Rectangle − Area to y axis ✓ $= 6\times\ln 4 - \int_0^{\ln 4} (e^y + 2)\, dy$	Comm 2 * This <u>must</u> be clearly shown.
ii) $= 6\ln 4 - \left[e^y + 2y \right]_0^{\ln 4}$ ✓ $= 6\ln 4 - \left\{ e^{\ln 4} + 2\ln 4 - (e^0 + 0) \right\}$ $= 6\ln 4 - e^{\ln 4} - 2\ln 4 + 1$ $= 4\ln 4 - 4 + 1$ $= 4\ln 4 - 3 \quad u^2$. ✓	Calc 2 • Again the integration was fine but lots of errors after substitution, especially when expanding brackets.