Name: ___ ___________

SCEGGS Darlinghurst

Term 1, 2004
Friday, 19th March

Extension 1 Mathematics

Task Weighting: 25%

General Instructions

- Time allowed - 60 minutes
- Write your name at the top of each page
- Start each question on a new page
- Attempt **all** questions
- Marks may be deducted for careless or badly arranged work
- Mathematical templates and geometrical equipment may be used
- Approved scientific calculators should be used
- A table of standard integrals is provided

	Com	Calc	Reas	
Question 1	/1	/6	/2	**/11**
Question 2	/1		/6	**/11**
Question 3	/5		/2	**/10**
Question 4		/4	/2	**/10**
TOTAL	/7	/10	/12	/42

Question 1 (11 marks) **Marks**

(a) (i) Factorise $P(x) = x^3 + 6x^2 + 11x + 6$ fully. **2**

(ii) Hence solve $P(x) \geq 0$. **1**

(b) Find:

(i) $\int x(x^2+4)^3\,dx$ using the substitution $u = x^2 + 4$. **3**

(ii) $\int_0^{\frac{\pi}{2}} \cos x\left(1-\sin^2 x\right)dx$ using the substitution $u = \sin x$. **3**

(c) Evaluate: **2**

$$\lim_{x \to 0} \frac{1-\cos 2x}{x^2}$$

Question 2 (11 marks) **Start a new page** **Marks**

(a) (i) Find the number of possible combinations if 8 people are chosen from a group of 10. **1**

(ii) *Explain* why your answer is also equal to ${}^{10}C_2$. **1**

(b) The remainder when $x^3 + ax + b$ is divided by $(x-2)(x+3)$ is $2x+1$. Find the values of a and b. **3**

(c) (i) Write $\sqrt{3}\sin\theta - \cos\theta$ in the form $R\sin(\theta - \alpha)$ where $R > 0$ and α is an acute angle measured in radians. **2**

(ii) Hence find the general solution of $\sqrt{3}\sin\theta - \cos\theta = \sqrt{3}$. **1**

(d) Show that : **3**

$$^{n+1}P_r = {}^nP_r + r \times {}^nP_{r-1}$$

Question 3 (10 marks) **Start a new page** **Marks**

(a) (i) Show that $f(x) = 2x^3 + 2x - 1$ has a root between $x = 0$ and $x = 1$. **1**

(ii) Using $x = 0$ as the first approximation, use TWO applications of Newton's Method to find a better approximation to the solution of $f(x) = 0$. **2**

(b) Prove by mathematical induction that: **4**

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n}{3}(2n+1)(2n-1)$$

for all positive integers, n.

(c) (i) How many different SIX letter words can be made using the letters of the word STRESS? **1**

(ii) How many different FIVE letter words can be made using the letters of the word STRESS? **2**

Question 4 (10 marks) **Start a new page** **Marks**

(a) The polynomial $P(x)=x^3-3x+2$ has three roots, α, β and γ.

Find the value of:

(i) $\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2$ **2**

(ii) $\alpha^2+\beta^2+\gamma^2$ **2**

(b) A committee of three women and seven men is seated randomly around a table. What is the probability that the three women are seated together? **2**

(c) Find the exact volume of the solid formed when the area bounded by the curve $y=\cos \pi x$, the x axis and the lines $x=0$ and $x=\frac{1}{2}$ is rotated about the x axis. **4**

End of Assessment

☺

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x>0$$

$$\int e^{ax} \, dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\frac{x}{a}, \quad a>0, \quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} \, dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} \, dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x>0$

YEAR 12 EXT 1 MATHS

ASSESSMENT TASK 1

March, 2004.

SOLUTIONS

QUESTION 1: (11 marks)

(a) (i) $P(x) = x^3 + 6x^2 + 11x + 6$

$P(-1) = -1 + 6 - 11 + 6$

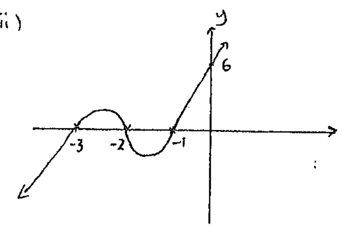
$= 0$

$\therefore (x+1)$ is a factor of $P(x)$. ✓

$\therefore P(x) = (x+1)(x^2+5x+6)$

$= (x+1)(x+2)(x+3)$ ✓

(ii)



Com 1

$\therefore -3 \leq x \leq -2$ and $x \geq -1$. ✓

(b) (i) $\int x(x^2+4)^3\,dx$

$u = x^2 + 4$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$= \int u^3 \cdot \frac{1}{2}\,du$ ✓

$= \frac{1}{2}\int u^3\,du$

$= \frac{u^4}{8} + C$ ✓

$= \frac{(x^2+4)^4}{8} + C$ ✓

(ii) $\int_0^{\pi/2} \cos x(1-\sin^2 x)\,dx$

$u = \sin x$

$\frac{du}{dx} = \cos x$

when $x = 0$, $u = 0$ ✓

$x = \frac{\pi}{2}$, $u = 1$

$= \int_0^1 1 - u^2\,du$ ✓

$= \left[u - \frac{u^3}{3}\right]_0^1$

$= 1 - \frac{1}{3}$ ✓

$= \frac{2}{3}$

Calc 6

(c) $\lim_{x\to 0} \frac{1-\cos 2x}{x^2}$

$= \lim_{x\to 0} \frac{2\sin^2 x}{x^2}$ ✓

$= 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{x}\right)$

$= 2 \times 1 \times 1$

$= 2$ ✓

Reas 2

QUESTION 2: (11 marks)

(a) (i) $^{10}C_8 = 45$ ✓

(ii) Choosing 8 people to be in the group is the same as choosing 2 people who aren't.

OR $^{10}C_8 = \frac{10!}{8!\,2!}$

$^{10}C_2 = \frac{10!}{2!\,8!}$

$\therefore$ They are equal. ✓

Com 1

(b) When $x = 2$:

$8 + 2a + b = 2\times 2 + 1$

$2a + b = -3$ ① ✓

when $x = -3$:

$-27 - 3a + b = 2\times -3 + 1$

$-3a + b = 22$ ② ✓

$\therefore$ ① - ②:

$5a = -25$

$a = -5$

$\therefore b = 7$ ✓

(c) (i) $\sqrt{3}\sin\theta - \cos\theta$

$R\sin(\theta - \alpha) = R\sin\theta\cos\alpha - R\sin\alpha\cos\theta$

$\therefore R\cos\alpha = \sqrt{3}$

$R\sin\alpha = 1$

$\therefore R = \sqrt{3^2+1} = 2$ ✓

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore \alpha = \pi/6$ ✓

$\therefore \sqrt{3}\sin\theta - \cos\theta = 2\sin(\theta - \pi/6)$

(ii) $\therefore 2\sin(\theta - \pi/6) = \sqrt{3}$

$\sin(\theta - \pi/6) = \sqrt{3}/2$

$\theta - \pi/6 = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

$\theta = \frac{\pi}{2}$ or $\frac{5\pi}{6}$

So $\theta = \begin{cases} \frac{\pi}{2} \pm 2n\pi \\ \frac{5\pi}{6} \pm 2n\pi \end{cases}$ ✓

Reas 3

$= n\pi + (-1)^n\frac{\pi}{3} + \frac{\pi}{6}$

(d) $^nP_r = \frac{n!}{(n-r)!}$

$\therefore LHS = {}^{n+1}P_r$

$= \frac{(n+1)!}{(n+1-r)!}$

$RHS = \frac{n!}{(n-r)!} + r \times \frac{n!}{(n+1-r)!}$ ✓

$= \frac{n!(n+1-r)}{(n-r)!(n+1-r)} + \frac{r\times n!}{(n+1-r)!}$

$= \frac{n!n + n! - rn! + rn!}{(n+1-r)!}$ ✓

$= \frac{n!(n+1)}{(n+1-r)!}$

$= \frac{(n+1)!}{(n+1-r)!}$ ✓

Reas 3

QUESTION 3: (10 marks)

(a) (i) $f(0) = -1 < 0$

$f(1) = 3 > 0$

$\therefore$ The curve has a root between $x = 0$ and $x = 1$ because the sign of $f(x)$ changes and $f(x)$ is continuous. ✓ (Com 1)

* NB

(ii) $x = 0$, $f'(x) = 6x^2 + 2$

$$a_1 = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{-1}{2} = \frac{1}{2}$$ ✓

$$a_2 = \frac{1}{2} - \frac{f(\frac{1}{2})}{f'(\frac{1}{2})} = \frac{1}{2} - \frac{\frac{1}{4}}{3\frac{1}{2}} = \frac{3}{7}$$ ✓

(b) $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n}{3}(2n+1)(2n-1)$

Test $n=1$:

LHS $= 1^2 = 1$

RHS $= \frac{1}{3} \times 3 \times 1 = 1$ ✓

$\therefore$ True for $n = 1$;

Assume true for $n = k$:

ie $1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{k}{3}(2k+1)(2k-1)$

Aim to show $1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 + (2k+1)^2 = \frac{k+1}{3}(2k+3)(2k+1)$.

Investigate $n = k+1$

$1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 + (2k+1)^2$

$= \frac{k}{3}(2k+1)(2k-1) + (2k+1)^2$ ✓ by assumption

$= (2k+1)\left(\frac{k}{3}(2k-1) + (2k+1)\right)$

$= \frac{1}{3}(2k+1)\left(k(2k-1) + 3(2k+1)\right)$ ✓

$= \frac{1}{3}(2k+1)(2k^2 - k + 6k + 3)$

$= \frac{1}{3}(2k+1)(2k^2 + 5k + 3)$

$= \frac{1}{3}(2k+1)(2k+3)(k+1)$ ✓

$= \frac{k+1}{3}(2k+3)(2k+1)$ as required.

If statement is true for $n = k$, it is also true for $n = k+1$.
Since it is true for $n = 1$, it is also true for $n = 2, 3, 4, \dots$ and hence for all positive integers by the principle of mathematical induction. (Com 4)

(c) (i) $\frac{6!}{3!}$ (since 3 e's). ✓

$(= 120)$.

(ii) 5 letter words total

= 5 letter words with 3 s's (and 2 of 3 other letters) + 5 letter words with 2 s's (and 3 other letters)

$= \left(\frac{5!}{3!} \times {}^3C_2\right) + \left(\frac{5!}{2!}\right)$ ✓ ✓ (Reas 2)

$= 60 + 60$

$= 120$

** NB This is not the same as $\frac{{}^6P_5}{3!}$ as not all 5 letter words contain 3 s's.

QUESTION 4: (10 marks)

(a) $P(x) = x^3 - 3x + 2$

$\alpha + \beta + \gamma = 0$ * NB no x^2 term !! $\therefore b = 0$.

$\alpha\beta + \beta\gamma + \alpha\gamma = -3$

$\alpha\beta\gamma = -2$

(i) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$

$= \alpha\beta\gamma(\alpha + \beta + \gamma)$ ✓

$= -2 \times 0$ ✓

$= 0$

(ii) $\alpha^2 + \beta^2 + \gamma^2$

$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$ ✓

$= 0^2 - 2 \times -3$

$= 6$ ✓

(b) Total number of arrangements $= 9!$

Total arrangements with 3 women together $= 3! \times 7!$

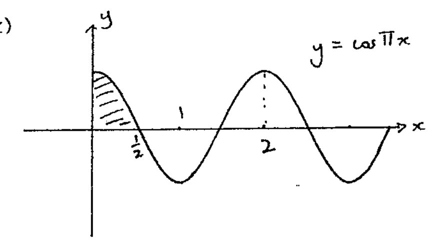
$= 30240$ ✓

$\therefore$ Probability $= \frac{30240}{362880}$ (Reas 2)

$= \frac{1}{12}$. ✓

* NB: Probability should always be < 1 !!

(c)



$$V = \pi \int_0^{\frac{1}{2}} (\cos \pi x)^2 \, dx$$ ✓

$$= \pi \int_0^{1/2} \cos^2 \pi x \, dx$$

but $\cos^2 A = \frac{1}{2}(\cos 2A + 1)$

$\therefore V = \pi \int_0^{\frac{1}{2}} \frac{1}{2}(\cos 2\pi x + 1)\, dx$ ✓ sub

$= \frac{\pi}{2}\left[\frac{1}{2\pi}\sin 2\pi x + x\right]_0^{\frac{1}{2}}$ ✓ int

$= \frac{\pi}{2}\left[\left(0 + \frac{1}{2}\right) - (0 + 0)\right]$

$= \frac{\pi}{4}$ units^3 ✓ eval.

(Calc 4)

* NB Quite a few people ended up with $\frac{\pi}{4}$ units^3 for the wrong reasons...
eg an incorrect substitution for $\cos^2 \pi x$, followed by an incorrect integration of this incorrect substitution.
This is not worth any marks!