Student Number

SCEGGS Darlinghurst

2012
Higher School Certificate
Assessment Task 1
Friday 2nd March

Mathematics Extension 1

Task Weighting: 30%

General Instructions

- Time allowed – 70 minutes
- This paper has **four** multiple choice questions followed by **four** extended response questions.
- Attempt **all** questions.
- ~~Answer all questions on the pad paper provided~~
- Start each question **on a new page**
- Write your student number at the top of each page
- Mathematical templates, geometrical equipment and approved scientific calculators may be used

Question 1 - Multiple choice. (4 marks)

(Answer on the separate answer page provided.)

(a) A polynomial $P(x)$ has zeroes -2 and 5. Two factors of $P(x)$ are

A. $(x-2)$ and $(x+5)$
B. $(x-2)$ and $(x-5)$
C. $(x+2)$ and $(x-5)$
D. $(x+2)$ and $(x+5)$

(b) The equation of the tangent to the parabola $x^2 = 4ay$ at the point $(2at, at^2)$ is

A. $y = t^2x - at$
B. $y = tx - at^2$
C. $y = ax - at^2$
D. $y = 2tx - at^2$

(c) When $P(x) = x^3 + 2x^2 - 3x + k$ is divided by $(x-2)$, the remainder is -4.
The value of k is

A. -10
B. -14
C. 10
D. -28

(d) Given that $7x - x^2 \equiv (m-n)x^2 + (2m+n)x$, the values of m and n are

A. $m = 8$ *and* $n = -9$
B. $m = 8$ *and* $n = 9$
C. $m = 2$ *and* $n = -1$
D. $m = 2$ *and* $n = 3$

End of Multiple Choice

Question 2 (9 marks) START A NEW PAGE.

(a) Is $(x-4)$ a factor of $x^3 - 3x^2 + x - 12$?
Give a reason for your answer. 1

(b) The polynomial $x^3 - 2x + 5 = 0$ has roots α, β, γ.
Find

(i) $\alpha + \beta + \gamma$ 1

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$ 1

(iii) $\alpha^2 + \beta^2 + \gamma^2$ 1

(iv) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ 2

(c) (i) Sketch the polynomial $y = x^3 - x$, clearly showing any intercepts with the co-ordinate axes. 1

(ii) Hence, or otherwise, solve

$$x \leqslant \frac{1}{x}$$ 2

Question 3 (9 marks) START A NEW PAGE.

(a) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$.
$S(0, a)$ is the focus of the parabola.
The normal at P cuts the y-axis at N.
PF is the line drawn from P perpendicular to the directrix of the parabola.

(i) Show all this information on a diagram ($\frac{1}{3}$ of a page). 1

(ii) Show that the equation of the normal at P is $x + py = ap^3 + 2ap$. 2

(iii) State the coordinates of N and F. 1

(iv) Find the coordinates of the point M, the midpoint of NF. 1

(v) Show that the locus of M is the parabola $x^2 = 2a(y - \frac{1}{2}a)$ 1

(b) Prove by mathematical induction that for $n \geq 1$, 3

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

Question 4 (9 marks) START A NEW PAGE.

(a) When $P(x)$ is divided by $(x-2)(x+3)$, the quotient is $Q(x)$ and the remainder is $R(x)$.

$$\therefore P(x) = (x-2)(x+3)Q(x) + R(x)$$

(i) Explain why $R(x) = ax + b$ 1

(ii) When $P(x)$ is divided by $(x-2)$ the remainder is 8 and when $P(x)$ is divided by $(x+3)$ the remainder is -2.

What is the remainder when $P(x)$ is divided by $(x-2)(x+3)$? 3

(b) It is known that the equation of the normal to the parabola $x^2 = 4ay$ at $T(2at, at^2)$ is $x + ty = at^3 + 2at$. (Do NOT prove this.)

(i) Show that there is only one possible normal to the parabola that passes through the focus, S. 1

(ii) Give a geometrical description of the normal that passes through the focus. 1

(c) Prove by mathematical induction that $3^{3n} + 2^{n+2}$ is divisible by 5 for $n \geq 1$. 3

Question 5 (9 Marks) START A NEW PAGE.

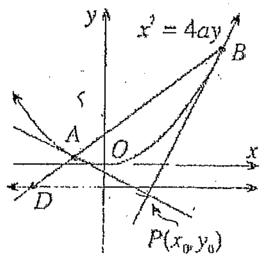
(a) The equation $x^3 + 2x^2 + kx - 6 = 0$ has roots $\alpha, \beta, \alpha + \beta$.

(i) Show that $\alpha + \beta = -1$. 1

(ii) Find the value of k. 3

(b) The chord of contact AB from the external point $P(x_0, y_0)$ to the parabola $x^2 = 4ay$ is given by $xx_0 = 2a(y + y_0)$

AB meets the directrix of the parabola at D.



(i) Show that D has coordinates $\left(\frac{2a(y_0 - a)}{x_0}, -a\right)$. 2

(ii) Prove that PD subtends a right angle at the focus $S(0, a)$. 2

(iii) If the line $y = mx + b$ is parallel to the chord of contact, find an expression for the gradient m. 1

END OF PAPER

2012

Question 1 Multiple Choice

a) $P(-2)=0 \quad P(5)=0$

$\therefore (x+2)$ & $(x-5)$ are factors.

C

b) $y=\dfrac{x^2}{4a}$

$y'=\dfrac{2x}{4a}$

$=\dfrac{x}{2a}$

At $P(2at, at^2)$

$M_T=\dfrac{2at}{2a}$

$=t$

$y-y_1=m(x-x_1)$

$y-at^2=t(x-2at)$

$y-at^2=tx-2at^2$

$y=tx-at^2$

B

c) $P(2)=-4$

$2^3+2\times 2^2-3\times 2+k=-4$

$8+8-6+k=-4$

$10+k=-4$

$k=-14$

B

d) Match coefficients.

$m-n=-1 \quad ①$

$2m+n=7 \quad ②$

Add $\quad 3m=6$

$m=2$

Sub. in ①

$2-n=-1$

$-n=-3$

$n=3$

D

Question 2.

a) $P(x)=x^3-3x^2+x-12$

$P(4)=4^3-3\times 4^2+4-12$

$=4$

$P(4)\neq 0 \quad \therefore (x-4)$ is not a factor.

Done well by nearly all students. Some used long division which is not as efficient as using the factor theorem.

b) $x^3+0x^2-2x+5=0$

i) $\alpha+\beta+\gamma=-\dfrac{b}{a}$

$=0$

ii) $\alpha\beta+\alpha\gamma+\beta\gamma=\dfrac{c}{a}$

$=-2$

iii) $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$=0^2-2\times -2$

$=4$

(iv) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}=\dfrac{\alpha\gamma+\beta\gamma+\alpha\beta}{\alpha\beta\gamma}$

$\alpha\beta\gamma=-\dfrac{d}{a}$

$=-5$

$=\dfrac{-2}{-5}$

$=\dfrac{2}{5}$

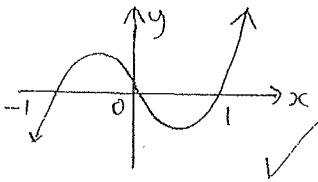
Most students knew the correct relationships but misread the equation. i.e. $-\dfrac{b}{a}=\dfrac{-(-2)}{1}$ instead of $-\dfrac{b}{a}=0$

This caused a compounding of errors

c) i) $y=x^3-x$

$=x(x^2-1)$

$=x(x-1)(x+1)$



Graph was done very well.

ii) $x\leq\dfrac{1}{x} \quad (x\neq 0)$

multiply both sides by x^2.

$x^2\times x\leq\dfrac{1}{x}\times x^2$

$x^3\leq x$

$x^3-x\leq 0$

from the graph

$x\leq -1 \quad , \quad 0<x\leq 1$

Note $x\neq 0$

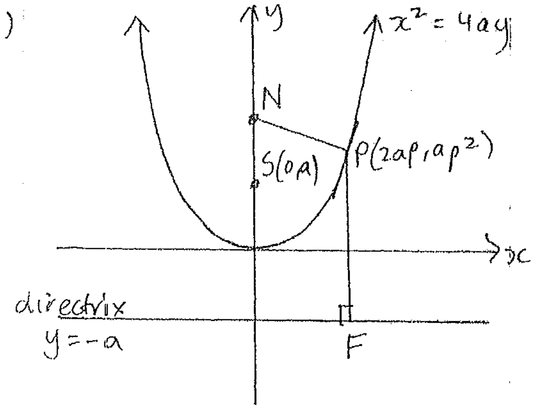
Some students missed the link between i) and ii).

The most common mistake was to multiply through by x to get

$x^2-1\leq 0$

Some students also missed that $x=0$ was not a solution.

a) i)



Please
- use a ruler for the axes & directrix
- Label x & y axes.

ii) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a}$

At $P(2ap, ap^2)$

$M_T = \frac{\cancel{4}ap}{4a}$

$= p$

$M_N = -\frac{1}{p}$

Equation of normal

$y - y_1 = m(x - x_1)$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = ap^3 + 2ap$

Very well done.

This is standard bookwork and if often just given in HSC exams recently.

iii) Normal cuts y axis at N.

$x = 0$

$py = ap^3 + 2ap$

$y = ap^2 + 2a$ $\qquad N(0, ap^2 + 2a)$

point F lies on directrix.

$F(2ap, -a)$

(must have both points)

Note that point F lies vertically below P so it has the same x-coordinate as P. (You should know that!)

iv) $N(0, ap^2 + 2a)$ $\quad F(2ap, -a)$

$M = \left(\frac{0 + 2ap}{2}, \frac{ap^2 + 2a - a}{2}\right)$

$= \left(ap, \frac{ap^2 + a}{2}\right)$

v) $x = ap$ $\quad (1)$

$y = \frac{ap^2 + a}{2}$ $\quad (2)$

Eliminate p

rearrange (1)

$p = \frac{x}{a}$

$y = \frac{a}{2}(p^2 + 1)$

$y = \frac{a}{2}\left(\left(\frac{x}{a}\right)^2 + 1\right)$

$y = \frac{a}{2} \times \frac{x^2}{a^2} + \frac{a}{2}$

$y = \frac{x^2}{2a} + \frac{a}{2}$

$(\times 2a)$ $\quad 2ay = x^2 + a^2$

$x^2 = 2ay - a^2$

factorise

$x^2 = 2a\left(y - \frac{1}{2}a\right)$

Your working must be accurate to get this mark. You have to show the required result exactly.

) Prove that for $n \geq 1$

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{3^3} + \dots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

This part was well done for first 2 marks.

Show true for $n=1$

$LHS = \frac{1}{2}$

$RHS = 2 - \frac{3}{2}$

$= \frac{1}{2}$

$LHS = RHS$ $\therefore$ True for $n=1$ ✓

Assume true for $n=k$

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{3^3} + \dots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$$

Prove true for $n=k+1$

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{3^3} + \dots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} = 2 - \frac{k+1+2}{2^{k+1}} = 2 - \frac{k+3}{2^{k+1}}$$

$$LHS = \underbrace{\frac{1}{2} + \frac{2}{2^2} + \frac{3}{3^3} + \dots + \frac{k}{2^k}}_{\text{using the assumption}} + \frac{k+1}{2^{k+1}}$$

$= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}$ ✓

$= 2 - \frac{2(k+2)}{2 \times 2^k} + \frac{k+1}{2^{k+1}}$

$= 2 \quad \frac{-2k-4+k+1}{2^{k+1}}$

$= 2 \quad \frac{-k-3}{2^{k+1}}$

$= 2 - \frac{k+3}{2^{k+1}}$ ✓

$= RHS$

Many errors here with negative signs. Your working must be perfect to get this mark.

$\therefore$ True for $n=k+1$ if true for $n=k$.

$\therefore$ The statement is true for $n=1, n=2, \dots$

By the principle of mathematical induction it is true for all $n \geq 1$.

Question 4

a) $P(x) = (x-2)(x+3)Q(x) + R(x)$

i) When the polynomial is divided by a quadratic $(x-2)(x+3) = (x^2+x-6)$, the degree of the remainder must be one or zero. The remainder must be linear form. ✓

For those students who answered this question your explanation was sound. For those who didn't, this is a standard concept that must be revised.

ii) $P(2) = 8$, $P(-3) = -2$

$P(x) = (x-2)(x+3)Q(x) + ax+b$

$P(2) = 0 + 2a+b = 8$ (1)

$P(-3) = 0 \quad -3a+b = -2$ (2)

Solve simultaneously

(1)−(2) $5a = 10$

$a = 2$ ✓

sub. in (1) $4+b = 8$

$b = 4$ ✓

$\therefore$ Remainder is $(ax+b) = 2x+4$ ✓

Done well by students who attempted it.

b) i) Normal $x+ty = at^3+2at$ passes through $S(0,a)$

$0 + at = at^3 + 2at$

$at^3 + at = 0$

$at(t^2+1) = 0$

$at = 0 \qquad t^2+1=0$

$t=0 \qquad$ No real solution. ✓

$\therefore$ There is only one normal through $S(0,a)$

ii) $t=0$ gives point $T(2at, at^2) = T(0,0)$

$\therefore$ The normal is from the origin.

It is the y axis ✓

Many silly errors in this question which prevented students from arriving at the correct factorisation. Most common was $a(t^2+1)=0$

It was also surprising to see Ext 1 students arrive at a solution to $t^2+1=0$.

Think about your answers!!

Question 4 (continued)

) Prove that $3^{3n}+2^{n+2}$ is divisible by 5 for $n\geq 1$

Show true for $n=1$

$$3^3+2^3=27+8$$
$$=35$$
$$=5\times 7$$

$\therefore$ True for $n=1$

Assume true for $n=k$

$3^{3k}+2^{k+2}=5P$ where P is an integer

Prove true for $n=k+1$

$3^{3(k+1)}+2^{k+1+2}=5Q$ where Q is an integer.

$$\text{LHS}=3^{3k+3}+2^{k+3}$$
$$=3^3\times 3^{3k}+2^{k+3}$$
using the assumption
$$=27\left(5P-2^{k+2}\right)+2^{k+3}$$
$$=27\times 5P-27\times 2^{k+2}+2\times 2^{k+2}$$
$$=5\times 27P-25\times 2^{k+2}$$
$$=5\left(27P-5\times 2^{k+2}\right)$$
$$=5Q$$

(where Q is an integer since P is an integer)

If the statement is true for $n=k$, then it is true for $n=k+1$.

$\therefore$ True for $n=1$, $n=2$, and so on.

$\therefore$ The statement is true for all $n\geq 1$ by the principle of mathematical induction.

This question was done well by most candidates.

Again silly errors were costly for some. They usually occurred in the manipulation of indices

i.e. $2^{k+3}=4\times 2^k$

etc

Question 5

a) $x^3+2x^2+kx-6=0$

i) Sum one at a time

$$\alpha+\beta+\gamma=-\frac{b}{a}$$
$$\alpha+\beta+\alpha+\beta=-\frac{2}{1}$$
$$2\alpha+2\beta=-2$$
$$\boxed{\alpha+\beta=-1} \quad (1)$$

The quickest way

Since $\alpha+\beta=-1$, it is a root.

$\therefore P(-1)=0$

$$-1+2-k-6=0$$
$$-k-5=0$$
$$k=-5$$

Well done if you did it!

ii) Sum Two at a time

$$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$$
$$\boxed{\alpha\beta+\alpha(\alpha+\beta)+\beta(\alpha+\beta)=\frac{k}{1}} \quad (2)$$

Product

$$\alpha\beta\gamma=-\frac{d}{a}$$
$$\alpha\beta(\alpha+\beta)=--\frac{6}{1}$$
$$\boxed{\alpha\beta(\alpha+\beta)=6} \quad (3)$$

Here's another way.

You could factorise this to make it easier.

$$\alpha\beta+(\alpha+\beta)^2=k$$
sub. in $\alpha+\beta=-1$
$$\alpha\beta+(-1)^2=k$$
$$\alpha\beta+1=k.$$

sub. in $\alpha+\beta=-1$ to make it easier

$$-\alpha\beta=6$$
$$\therefore \alpha\beta=-6.$$

$$\therefore k=\alpha\beta+1$$
$$=-6+1$$
$$=-5$$

Solve simultaneously (1) and (3)

$$\alpha+\beta=-1$$
$$\beta=-1-\alpha$$

Sub in (3)

$$\alpha(-1-\alpha)\times -1=6$$
$$-\alpha(-1-\alpha)=6$$
$$\alpha+\alpha^2=6$$
$$\alpha^2+\alpha-6=0$$
$$(\alpha+3)(\alpha-2)=0$$

$\alpha=-3$ $\quad$ $\alpha=2$

$\beta=-1+3=2$ $\quad$ $\beta=-1-2=-3$.

$\therefore$ The roots are $\overset{\alpha}{2}, \overset{\beta}{-3}, \overset{\alpha+\beta}{-1}$

$$\therefore k=\alpha\beta+\alpha(\alpha+\beta)+\beta(\alpha+\beta)$$
$$=2\times -3+2\times -1-3\times -1$$
$$=-6-2+3$$
$$=-5$$

Question 5 (continued)

b) i) Chord of contact $xx_0 = 2a(y+y_0)$

directrix $y = -a$

You must show by substitution and rearranging to get this mark.

Find x

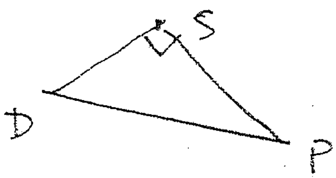
$$xx_0 = 2a(-a+y_0)$$

$$x = \frac{2a}{x_0}(y_0 - a)$$

$$D\left(\frac{2a(y_0-a)}{x_0}, -a\right)$$

ii) $P(x_0, y_0)$ $D\left(\frac{2a(y_0-a)}{x_0}, -a\right)$ $S(0, a)$

← Write out the three points clearly before you start especially if you make mistakes with signs.



PD makes a right angle at S

$m_{SD} \times m_{SP}$

$$= \frac{-a-a}{\frac{2a(y_0-a)}{x_0} - 0} \times \frac{y_0 - a}{x_0 - 0}$$

$$= \frac{-2ax_0}{2a(y_0-a)} \times \frac{y_0-a}{x_0}$$

$$= -1$$

$\therefore$ $DS \perp SP$

$\therefore$ $\angle DSP = 90^\circ$

iii) $y = mx + b$ matches the gradient of chord of contact.

An easy question if you got there.

$$xx_0 = 2a(y+y_0)$$

$$xx_0 = 2ay + 2ay_0$$

$$2ay = xx_0 - 2ay_0$$

$$y = \frac{x_0}{2a}x - y_0$$

Since parallel, $m = \frac{x_0}{2a}$