Student Number

SCEGGS Darlinghurst

2012

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 1
Friday 16th March, 2012

Mathematics Extension 2

Outcomes Assessed: E2, E3, E4 and E9

General Instructions

- Working time – 80 minutes
- Write using black or blue pen; **diagrams in pencil and at least one-third of a page**
- Board-approved calculators, mathematical templates and geometrical instruments may be used
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work

Total marks – 43

- Attempt Questions 1-7
- Section A – 4 multiple choice questions to be done on answer sheet
- Section B – 3 questions to be answered in booklets. Start a new booklet for each question.

Question	Marks	Complex Numbers	Polynomials	Induction
M/C	/4	/3	/1	
5	12	/9	/3	
6	/14	/11	/3	
7	/13	/6	/3	/4
TOTAL	/43	/29	/10	/4

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\ x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

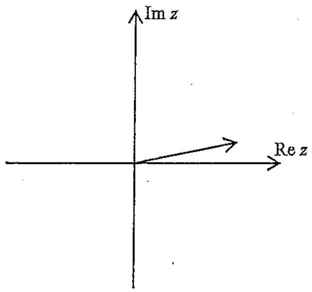
$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x>0$

Section A – Multiple Choice

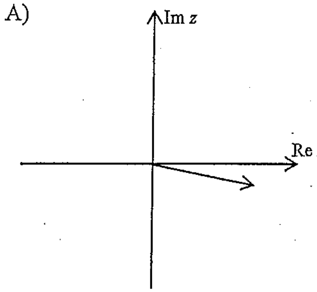
Question 1:

The complex number $z = 7 + i$ is represented on the Argand diagram below:

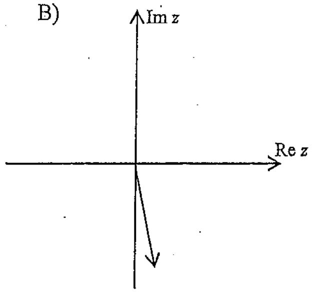


Which of the following Argand diagrams represents iz?

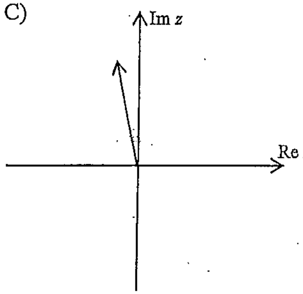
A)



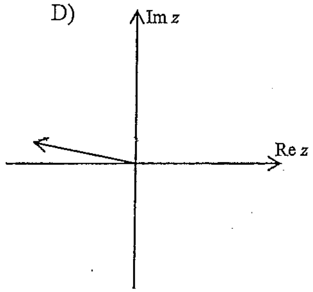
B)



C)



D)



Question 2:

If $z = 1 + 2i$ then $\mathrm{Re}(z^2)$ is

A) -3 B) 3

C) -4 D) 4

Question 3:

$1,\ \omega,\ \omega^2$ are the roots of the equation $z^3 - 1 = 0$, the value of:

$(1+\omega)(2+2\omega^2)$ is:

A) 2 B) -2

C) 2ω D) -2ω

Question 4:

The polynomial equation $4x^3 - 5x^2 + 2x - 1 = 0$ has roots α, β, γ.

Which of the following polynomial equations has roots $\frac{2}{\alpha}, \frac{2}{\beta}, \frac{2}{\gamma}$?

A) $x^3 - 2x^2 + 5x - 4 = 0$ B) $\frac{4}{x^3} - \frac{5}{x^2} + \frac{2}{x} - 1 = 0$

C) $x^3 - 4x^2 + 20x - 32 = 0$ D) $\frac{32}{x^3} - \frac{20}{x^2} + \frac{4}{x} - 1 = 0$

Section B

Marks

Question 5: (12 Marks)

a) Given $z = 1 + \sqrt{3}i$ find:

i) $|z|$ — 1

ii) $\arg(z)$ — 1

iii) z^3 — 1

b) The polynomial function $P(z) = z^4 - 4z^3 - 3z^2 + 50z - 52$ has $3 - 2i$ as a zero.

Factorise $P(z)$ over the field of rationals. — 3

c) i) Find the five roots of the equation $z^5 + 1 = 0$ — 2

ii) Indicate the five roots on an Argand diagram. — 1

iii) Hence show that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$ — 3

Start a new booklet

Marks

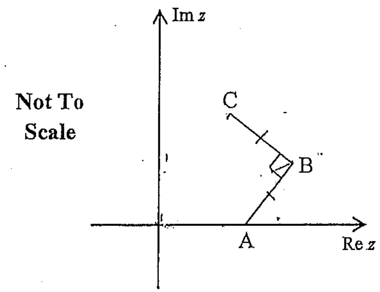
Question 6: (14 marks)

a) i) If $\frac{x+2}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$ find the values of A and B. — 2

ii) Hence, find: — 1

$$\int \frac{x+2}{(x+1)(x-2)}\,dx$$

b) The point A represents the complex number $z_1 = 2$ and the point B represents the complex number $z_2 = 3 + 2i$. The point C represents the complex number z_3 such that AB = BC, $\angle ABC = \frac{\pi}{2}$ and A, B, and C are in anti-clockwise order, as shown in the diagram.



i) Find the complex number z_3 represented by point C. — 2

ii) D is the point on the Argand diagram such that ABCD is a square. — 2
Find the complex number, z_4 represented by the point D

Question 6 continued on the next page.

Marks

Question 6 continued.

c) Consider the locus on an Argand diagram represented by **3**

$2|z| = z + \bar{z} + 4$, where $z = x + iy$.

Describe the locus geometrically

d) i) Sketch the region on an Argand diagram where the inequalities **2**

$|z| \leq 5$ and $\frac{-\pi}{4} \leq \arg(z - 5i) \leq 0$ hold simultaneously.

ii) Let α be the complex number of minimum modulus satisfying **2**

the inequalities given in part i).

Express α in the form $x + iy$.

Start a new booklet

Marks

Question 7: (13 Marks)

a) A sequence of numbers U_n is such that $U_1 = 3$, $U_2 = 21$ and **4**

$U_n = 7U_{n-1} - 10U_{n-2}$ for $n \geq 3$.

Use mathematical induction to show that $U_n = 5^n - 2^n$

b) Given that $2x^4 + 9x^3 + 6x^2 - 20x - 24 = 0$ has a triple root, **3**

find the triple root.

c) The n roots of unity $z_1, z_2, z_3 \dots\dots\dots z_n$ are represented by the points

$X_1, X_2, X_3 \dots\dots\dots X_n$ on an Argand diagram.

$X_1, X_2, X_3 \dots\dots\dots X_n$ are joined to form a polygon.

i) Show that the area of the polygon is given by $A_n = \frac{n}{2}\sin(\frac{2\pi}{n})$ **2**

ii) Show that the perimeter of the polygon is given by $P_n = 2n\sin(\frac{\pi}{n})$ **2**

iii) Show that $P_n \geq 2A_n$ for all positive integers n. **2**

End Of The Paper

Extension 2 Mathematics Assessment Task 1 – Solutions

Multiple Choice

Q1 $\times i$ rotates $90°$ anticlockwise

$\therefore C$

Q2 $(1+2i)^2 = 1+4i-4$

$= -3+4i$

$Re(z^2) = -3$

$\therefore A$

Q3 $(1+\omega)(2+2\omega^2)$

since $1+\omega+\omega^2=0$

$1+\omega = -\omega^2$

$1+\omega^2 = -\omega$

$\therefore (1+\omega)(2+2\omega^2) = -\omega^2 \times -2\omega$

$= 2\omega^3$

$= 2$

$\therefore A$

Q4 $y = \frac{2}{x}$

$\therefore x = \frac{2}{y}$

$4\left(\frac{2}{y}\right)^3 - 5\left(\frac{2}{y}\right)^2 + 2\left(\frac{2}{y}\right) - 1 = 0$

$\frac{32}{y^3} - \frac{20}{y^2} + \frac{4}{y} - 1 = 0$

$32 - 20y + 4y^2 - y^3 = 0$

$y^3 - 4y^2 + 20y - 32 = 0$

$\therefore x^3 - 4x^2 + 20x - 32 = 0$

$\therefore C$

Q5 a) $z = 1+\sqrt{3}i$

i) $|z| = \sqrt{1^2+(\sqrt{3})^2}$

$= 2$ ✓

ii) $\arg z = \tan^{-1}(\sqrt{3})$

$= \frac{\pi}{3}$ ✓

Done well by most students

iii) $z^3 = 2^3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3$

$= 8(\cos\pi + i\sin\pi)$

$= -8$ ✓ Comp /3

b) if $\alpha = 3-2i$ is a zero, because the coefficients are real $\bar{\alpha}$ is also a zero. ✓

$\therefore$ a factor of $P(z)$ is $(z-\alpha)(z-\bar{\alpha})$

now $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2Re(\alpha)z + |\alpha|^2$

$= z^2-6z+13$ ✓

Most students knew that $3+2i$ was also root but you should state why!

$$
\begin{array}{r}
z^2+2z-4 \\
z^2-6z+13 \overline{)\, z^4-4z^3-3z^2+50z-52} \\
\underline{z^4-6z^3+13z^2} \\
2z^3-16z^2+50z \\
\underline{2z^3-12z^2+26z} \\
-4z^2+24z-52 \\
\underline{-4z^2+24z-52} \\
0
\end{array}
$$
✓

Students must learn $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2Re(\alpha)z + |\alpha|^2$

$\therefore P(z) = (z^2-6z+13)(z^2+2z-4)$ Poly /3

c) $z^5+1=0$

$z^5=-1$

$z^5 = cis(\pi+2k\pi)$

$z = cis\left(\frac{\pi+2k\pi}{5}\right)$ ✓

Some students thought that if $z^5=-1$ then $z^5 = -1\,cis(2k\pi)$

$|z|$ is always positive

$k=0 \quad z_1 = cis\left(\frac{\pi}{5}\right)$

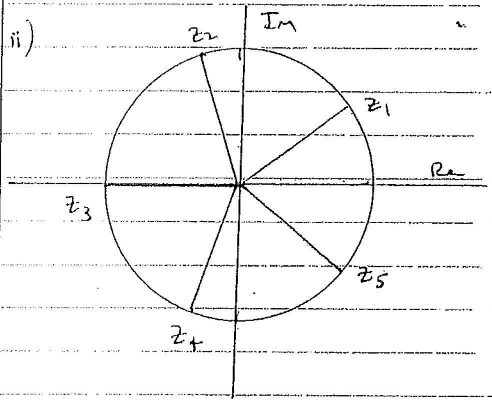
$k=1 \quad z_2 = cis\left(\frac{3\pi}{5}\right)$

$k=2 \quad z_3 = cis\left(\pi\right) = -1$ ✓

$k=3 \quad z_4 = cis\left(\frac{7\pi}{5}\right) = cis\left(-\frac{3\pi}{5}\right)$

$k=4 \quad z_5 = cis\left(\frac{9\pi}{5}\right) = cis\left(-\frac{\pi}{5}\right)$

ii)



✓

Students need to draw diagrams accurately. A math-o-mat would be beneficial. Since the roots are conjugate they must align.

iii) $z_1 + z_2 + z_3 + z_4 + z_5 = 0$ ✓

$\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right) + \left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right) + (-1)$
$+ \left(\cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}\right) + \left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right) = 0$

$2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} - 1 = 0$ ✓

$2\left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right) = 1$ ✓

$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

Comp /6

Students need to state why the sum equals zero. i.e. Σ roots $= -\frac{b}{a}$

You should also write $cis\frac{\pi}{5} = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$ etc. So that we can see the imaginary parts cancel.

Q6 a) i) $\frac{x+2}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$

$x+2 = A(x-2) + B(x+1)$

sub $x = -1$

$1 = -3A$

$A = -\frac{1}{3}$ ✓

sub $x = 2$

$4 = 3B$

$B = \frac{4}{3}$ ✓

Generally done well just a few silly errors

ii) $\int \frac{x+2}{(x+1)(x-2)}\,dx = -\frac{1}{3}\int\frac{dx}{x+1} + \frac{4}{3}\int\frac{dx}{x-2}$

$= -\frac{1}{3}\ln|x+1| + \frac{4}{3}\ln|x-2| + c$ ✓

$= \ln\left(\sqrt[3]{\left|\frac{(x-2)^4}{x+1}\right|}\right) + c$

Some students need to revise 2unit integration!

b) i) $\vec{AB} = \vec{AO} + \vec{OB}$

$= -2 + 3 + 2i$

$= 1 + 2i$ ✓

$\vec{BC} = i(1+2i)$

$= -2 + i$

$\vec{OC} = \vec{OB} + \vec{BC}$

$= 3 + 2i + -2 + i$

$= 1 + 3i$

$\therefore z_3 = 1 + 3i$ ✓

Most students had some idea on how to do this question. There were issues with direction of vectors and multiplying by i.

ii) $\vec{AD} = \vec{BC}$

$\vec{OD} = \vec{OA} + \vec{AD}$

$= 2 + \vec{BC}$ ✓

$= 2 + -2 + i$

$= i$

$z_4 = i$ ✓

c) $z = x + iy$

$|z| = \sqrt{x^2+y^2}$ $\quad \bar{z} = x - iy$

$\therefore 2|z| = z + \bar{z} + 4$

$2\sqrt{x^2+y^2} = x + iy + x - iy + 4$ ✓

$2\sqrt{x^2+y^2} = 2x + 4$

$4x^2 + 4y^2 = 4x^2 + 16x + 16$

$4y^2 = 16x + 16$

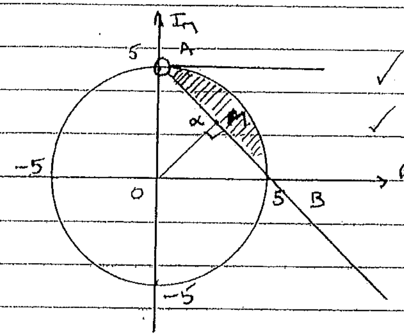
$y^2 = 4x + 4$ ✓

$y^2 = 4(x+1)$ ✓

half a sideways parabola vertex $(-1, 0)$ focus $(0, 0)$

Done well by some but I saw many silly errors which is frustrating. The most common $(2x+4)^2 = 4x^2 + 16$ Stay calm when doing questions!

d) i)



✓ graphs

✓ region.

Some students were sloppy. The arg locus not passing thru' $(5, 0)$ Not putting an open circle at $(0, 5)$ etc

AB has equation $y = -x + 5$

OM ⊥ AB passing thru' O

$\therefore$ OM has equation $y = x$

$\therefore$ solve simultaneously

$y = x$

$y = -x + 5$

$2x = 5$ ✓

$x = 5/2$

$y = 5/2$

$\therefore M\left(\frac{5}{2}, \frac{5}{2}\right)$

$\therefore \alpha = \frac{5}{2} + \frac{5}{2}i$ ✓

A few people knew how to start it but again there were silly errors.

Q7 a) for $n = 1$ $\quad u_1 = 3$

using the result: $u_1 = 5^1 - 2^1$

$= 3$

for $n = 2$ $\quad u_2 = 21$

using the result: $u_2 = 5^2 - 2^2$

$= 21$

$\therefore$ the result is true for $n = 1$ and $n = 2$

for $n = 3$ $\quad u_3 = 7u_2 - 10u_1$

$= 7 \times 21 - 10 \times 3$

$= 117$

the result is $u_3 = 5^3 - 2^3$

$= 125 - 8$

$= 117$ ✓

$\therefore$ result is true for $n = 3$

Done well by nearly all candidates

Assume the result is true for $n = k$

i.e. $u_k = 5^k - 2^k$

Show the result is true for $n = k+1$

i.e. $u_{k+1} = 5^{k+1} - 2^{k+1}$

$u_{k+1} = 7u_k - 10u_{k-1}$

$= 7(5^k - 2^k) - 10(5^{k-1} - 2^{k-1})$ ✓ using the assumption

$= 7 \times 5^k - 7 \times 2^k - \frac{10 \times 5^k}{5} + \frac{10 \times 2^k}{2}$ ✓

$= 5 \times 5^k - 2 \times 2^k$ ✓

$= 5^{k+1} - 2^{k+1}$

$\therefore$ the result is true for all integers n by the principle of mathematical induction

b) $P(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24$

$P'(x) = 8x^3 + 27x^2 + 12x - 20$

$P''(x) = 24x^2 + 54x + 12$

$\therefore$ to find triple root let $P''(x) = 0$ ✓

$24x^2 + 54x + 12 = 0$

$4x^2 + 9x + 2 = 0$

$(4x+1)(x+2) = 0$

$x = -\frac{1}{4} \quad x = -2$

$P(-2) = 2\times(-2)^4 + 9(-2)^3 + 6(-2)^2 - 20(-2) - 24$

$= 32 - 72 + 24 + 40 - 24$ ✓

$= 0$

since $P(-2) = P''(-2) = 0$

the triple root is $x = -2$. ✓

Done well. I would like to have seen the solution to $P''(x)=0$ n more solutions. Also I would like to see substitution not just $P(-2)=0$

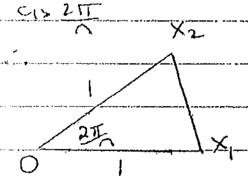
c) i) $z^n = 1$

$z^n = cis(2k\pi)$

$z = cis\left(\frac{2k\pi}{n}\right)$

$z_1 = cis\,0$

$z_2 = cis\,\frac{2\pi}{n}$



Area of $\Delta X_1OX_2 = \frac{1}{2}\times 1\times 1\times \sin\frac{2\pi}{n}$ ✓

$= \frac{1}{2}\sin\frac{2\pi}{n}$

In an n sided polygon there are n triangles

$\therefore A_n = n\times\frac{1}{2}\sin\frac{2\pi}{n}$ ✓

$= \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)$

Should have seen why the angle is $\frac{2\pi}{n}$. Since it was given to you there needs to be more explanation.

Otherwise the completion of the area and cosine rule was sound

ii) let l = length of one side in the polygon

using the cosine rule

$l^2 = 1^2 + 1^2 - 2\times1\times1\times\cos\frac{2\pi}{n}$ ✓

$= 2 - 2\cos\frac{2\pi}{n}$

$= 2 - 2\left(1 - 2\sin^2\frac{\pi}{n}\right)$

$= 4\sin^2\frac{\pi}{n}$

$l = 2\sin\frac{\pi}{n}$ ✓

$\therefore$ there are n sides

$\therefore P_n = 2n\sin\frac{\pi}{n}$

iii) $P_n - 2A_n$

$= 2n\sin\frac{\pi}{n} - n\sin\left(\frac{2\pi}{n}\right)$

$= 2n\sin\frac{\pi}{n} - n\times 2\times\sin\frac{\pi}{n}\cos\frac{\pi}{n}$ ✓

$= 2n\sin\frac{\pi}{n}\left(1 - \cos\frac{\pi}{n}\right)$

now $\sin\frac{\pi}{n} \geq 0$ and $\left(1-\cos\frac{\pi}{n}\right) \geq 0$ for all n

$\therefore P_n - 2A_n \geq 0$ ✓

$\therefore P_n \geq 2A_n$

It was pleasing to see many of you attempt this question.