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Centre Number

Student Number

SCEGGS Darlinghurst

2011

HSC Assessment 2
17th June 2011

Extension 1 Mathematics

Assessment Outcomes:
Weighting: 30%

General Instructions

- Time allowed – 70 Minutes
- This paper has **four** questions
- Answer on the booklets provided
- Start each question on a **booklet**
- Write your Student Number on the front of each booklet
- Attempt **all** questions and show all necessary working
- One double-sided sheet of hand-written notes is allowed.
- Marks may be deducted for careless or badly arranged work
- A table of standard integrals is provided at the back of this paper

Total marks (53)

Question	Communication	Calculus	Reasoning	Marks
1	/3	/5	/2	/14
2	/5		/5	/13
3	/1	/3	/8	/14
4	/4	/3	/3	/12
TOTAL	/13	/11	/18	/53

HSC STANDARD INTEGRAL SHEET

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a \neq 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\ x > 0$

START A NEW BOOKLET

Marks

Question 1 (14 Marks)

a) A is the point $(3, -2)$, B is the point $(1, 4)$. Find the coordinates of the point P which divides AB externally in the ratio 5:2. **2**

b) Evaluate : **2**

$$\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$$

c) i) The line $y = mx$ makes an angle of 45^o with the line $y = 2x$. Show that $|m-2| = |1+2m|$ **2**

ii) Hence find the equations of the line $y = mx$ which make an angle of 45^o with the line $y = 2x$. **2**

d) i) Write an expression for $\tan(\alpha + \beta)$. **1**

ii) Hence evaluate exactly $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right)$ **2**

e) Using the substitution $u = 2 - x$ evaluate: **3**

$$\int_0^1 x\,(2-x)^3$$

START A NEW BOOKLET

Marks

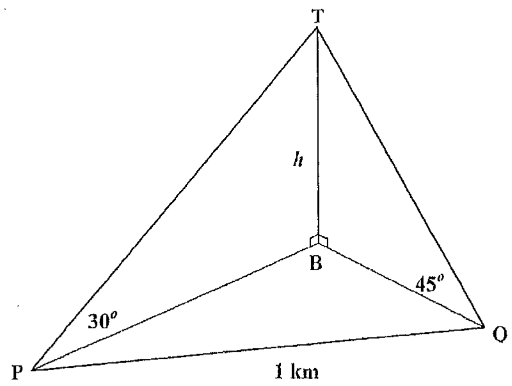
Question 2 (13 Marks)

a) i) Sketch $y = |x-2|$ and state the domain and range. **2**

ii) For the function $f(x) = |x-2|$, state the largest positive domain which will define an inverse function of $f(x)$. **1**

iii) Find $f^{-1}(x)$ in terms of x and state its domain. **2**

b)



The angle of elevation from a boat at P to a point T at the top of a vertical cliff is 30^o. The boat sails 1 km to a second point Q, from which the angle of elevation of T is 45^o. B is the point at the base of the cliff directly below T and h is the height of the cliff in metres. The bearings of B from P and Q are 050^oT and 290^oT respectively.

i) Show that $\angle PBQ = 120^o$ **1**

ii) Show that **3**

$$h = \frac{1000}{\sqrt{4+\sqrt{3}}} \text{ metres}$$

Marks

Question 2 (continued)

c) i) Express $2\cos x + 2\sqrt{3}\sin x$ in the form $A\cos(x - \alpha)$ where $A > 0$ *and* $0 < \alpha < \frac{\pi}{2}$ **2**

ii) Hence solve $2\cos x + 2\sqrt{3}\sin x = 1$ for $0 \le x \le \pi$ **2**
Round your answer to 2 decimal places.

START A NEW BOOKLET

Marks

Question 3 (14 Marks)

a) Find the term independent of x in the binomial expansion of: **3**

$$\left(x - \frac{3}{x^2}\right)^{12}$$

b) Find the general solution for $2\sin x + 1 = 0$ **2**

c) Using the substitution $x = u^2 - 2$ find: **3**

$$\int \frac{x\,dx}{\sqrt{x+2}}$$

d) Consider the geometric series:

$$S = 1 + (1+x) + (1+x)^2 \ldots\ldots\ldots + (1+x)^n$$

i) Write down the expansion of $(1+x)^{n+1}$ **1**

ii) Show that: **2**

$$S = \frac{(1+x)^{n+1} - 1}{x}$$

iii) Hence show that: **2**

$$S = \binom{n+1}{1} + \binom{n+1}{2}x + \ldots\ldots \binom{n+1}{r+1}x^r \ldots\ldots \binom{n+1}{n+1}x^n$$

iv) Explain why: **1**

$$\binom{1}{1} + \binom{2}{1} + \ldots\ldots \binom{n}{1} = \binom{n+1}{2}$$

START A NEW BOOKLET

Marks

Question 4 (12 Marks)

a) Consider the function $f(x) = \cos^{-1}(x-1)$

i) Find the domain of the function. **1**

ii) Sketch the graph of $y = f(x)$ showing clearly the co-ordinates of the endpoints. **2**

iii) The region of the first quadrant bounded by the curve $y = f(x)$ and the co-ordinate axes is rotated through one complete revolution about the y – axis. Find the exact value of the volume of the solid of revolution. **3**

b) Let $f(x) = \tan^{-1}\left(\frac{2}{x}\right) - \tan^{-1}(-2x)$ where $x > 0$

i) Show that: **2**

$$f'(x) = \frac{-6x^2+6}{(x^2+4)(1+4x^2)}$$

ii) By solving $f'(x) = 0$ show that the inverse function does not exist. **3**

ii) Find the largest possible domain for $f(x)$ so that its inverse function $f^{-1}(x)$ exists. **1**

END OF THE PAPER

Extension 1 HSC Assessment Task 2 2011

Q1 a) $(3,-2)\quad(1,4)$

$5:-2$

$x=\frac{5\times1+-2\times3}{5+-2}$ $\quad y=\frac{5\times4+-2\times-2}{5+-2}$

$=-\frac{1}{3}$ ✓ $\quad =\frac{24}{3}$

$=8$ ✓

$\therefore P\left(-\frac{1}{3},8\right)$

Careless substitution into the formula cost some students marks.

b) $\int_0^{\sqrt3}\frac{dx}{\sqrt{4-x^2}}=\sin^{-1}\left(\frac{x}{2}\right)\Big]_0^{\sqrt3}$ ✓

$=\sin^{-1}\left(\frac{\sqrt3}{2}\right)-\sin^{-1}0$

$=\frac{\pi}{3}-0$

$=\frac{\pi}{3}$ ✓ Calc - 2

Generally well done but some students were unable to recall the answer even with the aid of their notes.

c) i) $y=mx \qquad y=2x$

gradient $=m \qquad$ gradient $=2$

$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$\tan45^\circ=\frac{|m-2|}{|1+2m|}$ ✓

$1=\frac{|m-2|}{|1+2m|}$ ✓ Comm - 2

$|1+2m|=|m-2|$

Well done

ii) $|1+2m|=|m-2|$

$1+2m=m-2$ or $1+2m=-m+2$

$m=-3$ or $3m=1$

$m=\frac{1}{3}$

$y=-3x$ ✓ or $y=\frac{1}{3}x$ ✓

This question proved surprisingly difficult for students with many only giving one equation

d) i) $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ ✓ Comm - 1

No excuse!

ii) let $\alpha=\tan^{-1}\left(\frac{1}{4}\right)$ $\quad$ let $\beta=\tan^{-1}\left(\frac{3}{5}\right)$

$\therefore\tan\alpha=\frac{1}{4}$ $\quad$ $\tan\beta=\frac{3}{5}$

$\tan(\alpha+\beta)=\frac{\frac{1}{4}+\frac{3}{5}}{1-\frac{1}{4}\times\frac{3}{5}}$ ✓

$=\frac{17/20}{1-3/20}$

$=1$

$\therefore\alpha+\beta=\tan^{-1}1$ Reas - 2

$=\frac{\pi}{4}$

$\therefore\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)=\frac{\pi}{4}$ ✓

Many students failed to realise that α,β are angles. Students conclude their working at $\tan(\alpha+\beta)$, thereby not answering the question.

e) $\int_0^1 x(2-x)^3\,dx$ $\quad u=2-x \quad x=0\ u=2$

$du=-dx \quad x=1\ u=1$

$dx=-du$

$=\int_2^1(2-u)\times u^3\times-du$ ✓

$=\int_2^1 u^4-2u^3\,du$

$=\left[\frac{u^5}{5}-\frac{u^4}{2}\right]_2^1$ ✓

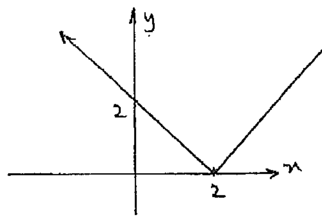
$=\left(\frac{1}{5}-\frac{1}{2}\right)-\left(\frac{32}{5}-\frac{16}{2}\right)$ Calc - 3

$=\frac{13}{10}$ ✓

The more successful candidates worked with $\frac{du}{dx}$ (rather than $\frac{dx}{du}$)

Most errors were algebraic rather than conceptual and often involved handling the negative.

Q2 i)



D: all real x Comm - 2

R: $y \geq 0$ ✓

ii) $x \geq 2$ ✓

iii) $f^{-1}(x)$: $y = x - 2$

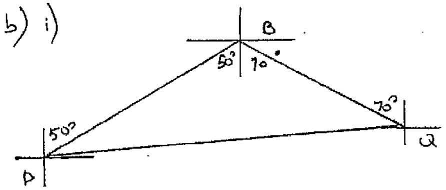
interchange x and y

$x = y - 2$ ✓

$y = x + 2$

$\therefore f^{-1}(x) = x + 2 \quad x \geq 0$ ✓ Reas - 2

b) i)



$\angle PBQ = 50° + (360° - 290°)$ ✓ Comm-1

$= 120°$

ii) $\tan 45° = \frac{BQ}{h}$ $\tan 60° = \frac{BP}{h}$

$BQ = h \tan 45°$ $BP = h \tan 60°$

$= h$ ✓ $= \sqrt{3}h$

$PQ^2 = PB^2 + BQ^2$

$1000^2 = (\sqrt{3}h)^2 + h^2 - 2 \times \sqrt{3}h \times h \cos 120°$ ✓

$= 3h^2 + h^2 - 2\sqrt{3}h^2 \times -\frac{1}{2}$

$= 4h^2 + \sqrt{3}h^2$ Reas - 3

$1000^2 = h^2(4 + \sqrt{3})$ ✓

$h^2 = \frac{1000^2}{4+\sqrt{3}} \quad \therefore h = \frac{1000}{\sqrt{4+\sqrt{3}}}$ as $h > 0$

Done very well by nearly all students

Done poorly as many students interchange x and y inside the absolute values $x = |y - 2|$

Done well although some students thought that P was east of Q and used angle sum of a triangle.

Candidates who had difficulty with this question obtained an incorrect expression for BP usually $\frac{h}{\sqrt{3}}$.

c) i) $2\cos x + 2\sqrt{3}\sin x \equiv A\cos(x - \alpha)$

$\equiv A\cos x\cos\alpha + A\sin x\sin\alpha$

$\therefore A\cos\alpha = 2 \ldots ①$ $A\sin\alpha = 2\sqrt{3} \ldots ②$

$② \div ①$ $\frac{A\sin\alpha}{A\cos\alpha} = \frac{2\sqrt{3}}{2}$

$= \sqrt{3}$

$\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$ ✓

$①^2 + ②^2 = A^2\cos^2\alpha + A^2\sin^2\alpha = 2^2 + (2\sqrt{3})^2$

$A^2(\cos^2\alpha + \sin^2\alpha) = 16$

$A^2 = 16$ Comm-2

$A = 4$ ✓ $A > 0$

$\therefore 2\cos x + 2\sqrt{3}\sin x = 4\cos(x - \frac{\pi}{3})$

Done very well.

ii) $2\cos x + 2\sqrt{3}\sin x = 1$

becomes $4\cos(x - \frac{\pi}{3}) = 1$

$\cos(x - \frac{\pi}{3}) = \frac{1}{4}$ ✓

$x - \frac{\pi}{3} = \cos^{-1}(\frac{1}{4})$

$x = \cos^{-1}(\frac{1}{4}) + \frac{\pi}{3}$

$= 2.02^c$ (to 2 dp) ✓

Done well, although many students gave their answer in degrees. Some students did not read the question carefully giving two answers for x when $0 \leq x \leq \pi$.

Q3 a) $\left(x - \frac{3}{x^2}\right)^{12} = \sum_{r=0}^{12} \binom{12}{r} x^{12-r} \left(\frac{-3}{x^2}\right)^r$

$\therefore x^{12-r} \times (x^{-2})^r = x^0$ ✓

$x^{12-r-2r} = x^0$

$12 - 3r = 0$

$3r = 12$

$r = 4$ ✓

$\therefore$ term is $\binom{12}{4} \cdot (-3)^4$

$= 40095$ ✓

Reas - 3

Some careless errors with the negative power but generally well done.

The term must be evaluated.

b) $2\sin x + 1 = 0$

$2\sin x = -1$

$\sin x = -\frac{1}{2}$

$x = \pi n + (-1)^n \sin^{-1}\left(-\frac{1}{2}\right)$ ✓

$= \pi n + (-1)^n\left(-\frac{\pi}{6}\right)$ ✓

It was insufficient to merely copy a formula from your notes. You had to know how to use it.

c) $\displaystyle\int \frac{x\,dx}{\sqrt{x+2}}$ $\qquad x = u^2 - 2 \to u = \sqrt{x+2}$, $\quad dx = 2u\,du$

$= \displaystyle\int \frac{(u^2-2)\times 2\cancel{u}\,du}{\cancel{\sqrt{u^2}}}$ ✓

$= \displaystyle\int 2u^2 - 4\,du$

$= \frac{2u^3}{3} - 4u + c$ ✓

Calc - 3

$= \frac{2}{3}\sqrt{(x+2)^3} - 4\sqrt{x+2} + c$ ✓

The algebraic precision required for this substitution eluded many students.

d) i) $(1+x)^{n+1} = \binom{n+1}{0} + \binom{n+1}{1}x + \binom{n+1}{2}x^2 + \dots + \binom{n+1}{n+1}x^{n+1}$ ✓

Comm-1

Well done

ii) $S = \underbrace{1 + (1+x) + (1+x)^2 + \dots + (1+x)^n}$ ✓

G.P. with $a = 1$ $r = (1+x)$ $(n+1)$ terms

$S = \frac{1\left[(1+x)^{n+1} - 1\right]}{(1+x)-1}$ ✓

$= \frac{(1+x)^{n+1} - 1}{x}$

Only a few students thought to sum the GP

iii) $S = \frac{(1+x)^{n+1} - 1}{x}$

$= \frac{\cancel{\binom{n+1}{0}} + \binom{n+1}{1}x + \binom{n+2}{2}x^2 + \dots \binom{n+1}{n+1}x^{n+1} - \cancel{1}}{x}$ ✓

$= \frac{\binom{n+1}{1}x + \binom{n+2}{2}x^2 + \dots + \binom{n+1}{n+1}x^{n+1}}{x}$ ✓

$= \binom{n+1}{1} + \binom{n+2}{2}x + \dots + \binom{n+1}{n+1}x^n$

Not a well done question.

iv) As $1 + (1+x) + (1+x)^2 + \dots + (1+x)^n$

$= \binom{n+1}{1} + \binom{n+1}{2}x + \binom{n+1}{3}x^2 + \dots + \binom{n+1}{n+1}x^n$

equating coefficients of x then ✓

$\binom{n+1}{2} = \binom{1}{1} + \binom{2}{1} + \dots + \binom{n}{1}$

Reas - 5

Very few students made this connection.

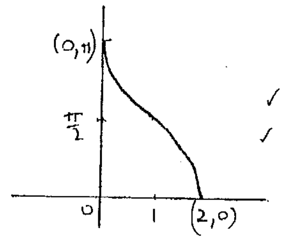
Some students successfully used the sum of an AP!

Q4 a) i) D: $-1 \le x - 1 \le 1$

$0 \le x \le 2$ ✓

(i) (ii) Well done.

ii)



✓ graph

✓ endpoints

Comm - 2

iii) $y = \cos^{-1}(x-1)$

$\cos y = x - 1$

$x = \cos y + 1$

$V = \pi \displaystyle\int_0^{\pi} (\cos y + 1)^2\,dy$

$= \pi \displaystyle\int_0^{\pi} \cos^2 y + 2\cos y + 1\,dy$

$= \pi \displaystyle\int_0^{\pi} \frac{1}{2}(\cos 2y + 1) + 2\cos y + 1\,dy$ ✓

$= \pi \displaystyle\int_0^{\pi} \frac{1}{2}\cos 2y + 2\cos y + \frac{3}{2}\,dy$

$= \pi \left[\frac{1}{4}\sin 2y + 2\sin y + \frac{3y}{2}\right]_0^{\pi}$ ✓

(iii) Find x carefully.

$(\cos y + 1)^2$ is NOT $\cos^2 y + 1$

Maintain limits on the y axis.

$$= \pi\left[\left(\frac{1}{4}\sin 2\pi + 2\sin\pi + \frac{3\pi}{2}\right) - \left(\frac{1}{4}\sin 0 + 2\sin 0 + \frac{3 \cdot 0}{2}\right)\right]$$

$$= \pi\left[0 + 0 + \frac{3\pi}{2} - 0 - 0 - 0\right]$$

$$= \frac{3\pi^2}{2} \text{ units}^3$$ ✓ Calc - 3

b) i) $f(x) = \tan^{-1}\left(\frac{2}{x}\right) - \tan^{-1}(-2x)$

$$f'(x) = \frac{-2x^{-2}}{1 + \left(\frac{2}{x}\right)^2} - \frac{-2}{1 + (-2x)^2}$$ ✓

$$= \frac{-2}{x^2\left(1 + \frac{4}{x^2}\right)} + \frac{2}{1 + 4x^2}$$

$$= \frac{-2}{x^2 + 4} + \frac{2}{1 + 4x^2}$$

$$= \frac{-2(1 + 4x^2) + 2(x^2 + 4)}{(x^2+4)(1+4x^2)}$$ ✓

$$= \frac{-2 - 8x^2 + 2x^2 + 8}{(x^2+4)(1+4x^2)}$$

$$= \frac{-6x^2 + 6}{(x^2+4)(1+4x^2)}$$ Com - 2

ii) $f'(x) = 0$ $\quad \frac{-6x^2+6}{(x^2+4)(1+4x^2)} = 0$

$\therefore -6x^2 + 6 = 0$

$x^2 = 1$

$x = \pm 1$

$x = 1 \quad x > 0$ ✓

$\therefore$ stationary point at $x = 1$ Reas - 3

test $x = 1$

x	$\frac{1}{2}$	1	2
$f'(x)$	+	0	−

✓

$\therefore$ at $x = 1$ there is a max t.p.

$\therefore$ $f(x)$ is not monotonically increasing ✓

$\therefore$ an inverse does not exist

d) $x \geq 1$ ✓

(ii) Remember $x > 0$.

Test to prove that there is a maximum stationary point

The stationary points are not sufficient as they could be horizontal inflexions