Centre Number

Student Number

SCEGGS Darlinghurst

2012

HSC Assessment 2

Monday 4th June 2012

Mathematics Extension 1

General Instructions

- Time allowed – 75 minutes
- This paper is in **two** sections
- Write your Student Number where indicated
- Attempt **all** questions and show all necessary working
- One handwritten, A4 double-sided page of notes is allowed
- Marks may be deducted for careless or badly arranged work
- a table of standard integrals is provided at the back of this paper

Total marks – 49

Section I

5 marks

- Attempt Questions 1 – 5
- Allow about 15 minutes for this section
- Answer on the separate Multiple Choice Answer Sheet provided

Section II

44 marks

- Attempt Questions 6 – 9
- Allow about 1 hour for this section
- Answer in the booklets provided
- Start each question in a **new booklet**

Section	Question	Further Integration	Inverse Functions	Trigonometry General Solutions	Iterative Methods	Total
I	1 – 5					/5
II	6					/10
	7					/12
	8					/10
	9					/12
TOTAL						/49

Section I

5 marks
Attempt Questions 1 – 5
Allow about 15 minutes for this section

Use the multiple-choice answer sheet.

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ◯ B ⬤ C ◯ D ◯

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ⬤ B ⊗ C ◯ D ◯

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word ***correct*** and drawing an arrow as follows.

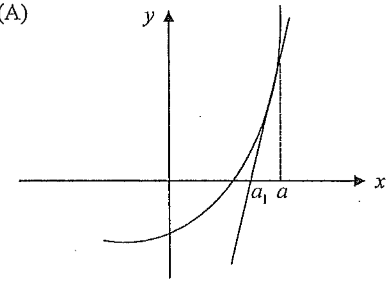
correct

A ⊗ B ⊗ C ◯ D ◯

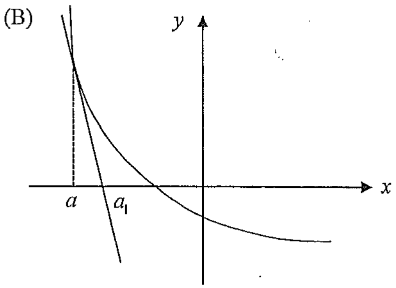
1 Newton's method relies on the x-intercept, a_1, of the tangent at $x = a$ being closer to the root than a.

Which of the diagrams below does not give a better approximation to the root at a_1 than a?

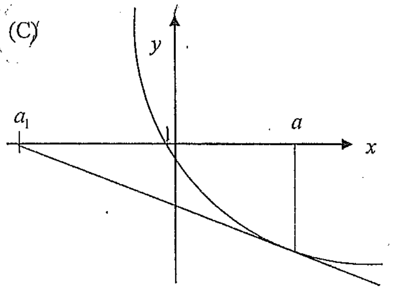
(A)



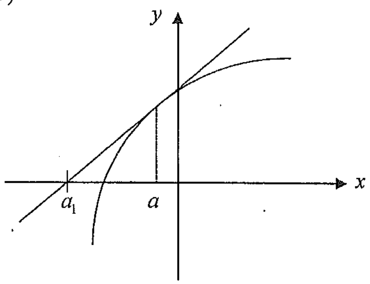
(B)



(C)



(D)



2 The general solution of the equation $\sin x = \frac{1}{\sqrt{2}}$ is

(A) $x = \pm\frac{\pi}{4} + 2n\pi$, where n is an integer.

(B) $x = \pm\frac{\pi}{6} + 2n\pi$, where n is an integer.

(C) $x = (-1)^n \frac{\pi}{4} + n\pi$, where n is an integer.

(D) $x = (-1)^n \frac{\pi}{6} + n\pi$, where n is an integer.

3 Which of the following is an expression for $\int \frac{e^x}{1+e^{2x}}\,dx$?

Use the substitution $u = e^x$.

(A) $\frac{-1}{2(1+e^x)^2} + c$

(B) $\frac{-e^x}{(1+e^x)^2} + c$

(C) $\tan^{-1} e^x + c$

(D) $\tan^{-1} e^{2x} + c$

4 Which of the following is not an odd function?

(A) $y = \sin^{-1} x$

(B) $y = \tan^{-1} x$

(C) $y = \cos^{-1} x$

(D) $y = 2\sin x$

5 Which of the following expressions is correct?

(A) $\tan^{-1} x = \cos^{-1} \dfrac{1}{\sqrt{1-x^2}}$

(B) $\tan^{-1} x = \cos^{-1} \dfrac{1}{\sqrt{1+x^2}}$

(C) $\tan^{-1} x = \cos^{-1} \dfrac{x}{\sqrt{1-x^2}}$

(D) $\tan^{-1} x = \cos^{-1} \dfrac{x}{\sqrt{1+x^2}}$

End of Section I

BLANK PAGE

Section II

44 marks
Attempt Questions 6 – 9
Allow about 60 minutes for this section

Answer each question in a NEW BOOKLET.
Write your STUDENT NUMBER at the top of each page.

Marks

Question 6 (10 marks)

(a) Find $\int \frac{-1}{\sqrt{3-4x^2}}\, dx$ **2**

(b) Let $f(x) = 2\cos^{-1}(x-1)$

(i) State the domain and range of $f(x)$. **2**

(ii) Sketch the graph of $y = f(x)$, indicating clearly the endpoints of the graph. **2**

(c) (i) Differentiate $\tan^{-1}(x^3)$. **2**

(ii) Hence find $\int_{-1}^{1} \frac{x^2}{1+x^6}\, dx$. **2**

Leave your answer in exact form.

- Start a new booklet

Marks

Question 7 (12 marks)

(a) Evaluate $\int_0^{\frac{\pi}{4}} \cos x \sin^3 x\ dx$ **2**

(b) If $\alpha = \sin^{-1}\frac{1}{\sqrt{5}}$ and $\beta = \sin^{-1}\frac{1}{\sqrt{10}}$

(i) show that $\sin(\alpha+\beta) = \frac{1}{\sqrt{2}}$ **2**

(ii) Hence, find the exact value of $\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}}$ **1**

(c) (i) Show that $\frac{x+2}{x+1} = 1 + \frac{1}{x+1}$ **1**

(ii) Sketch the curve $f(x) = \frac{x+2}{x+1}$ for $x > -1$. **2**

Use at least $\frac{1}{3}$ of a page and show any asymptotes and intercepts clearly.

(iii) Find $f^{-1}(x)$ in terms of x. **2**

(iv) Sketch $y = f^{-1}(x)$ on the same diagram as drawn in part (ii). **2**

■ Start a new booklet

Question 8 (10 marks) **Marks**

(a) Use the substitution $u = \sqrt{x}$ to find **3**

$$\int \frac{1}{\sqrt{x}\,\sqrt{1-x}}\,dx$$

(b) (i) Express $\cos x - \sin x$ in the form $R\cos(x+\alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

(ii) Hence, solve $\cos x - \sin x = 1$ for $0 \le x \le 2\pi$. **2**

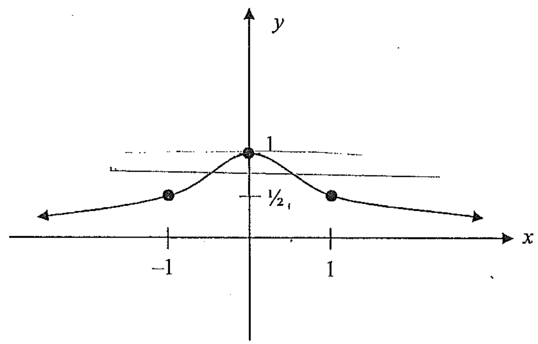
(iii) Find all possible solutions for $\cos x - \sin x = 1$. **1**

(c) Show that $\log_e x = \sin x$ has a root between $x = 2$ and $x = 3$. **2**

■ Start a new booklet

Question 9 (12 marks) **Marks**

The diagram shows the function $f(x) = \dfrac{1}{1+x^2}$.



(a) Explain why the inverse of $f(x)$ is not a function. **1**

(b) What is the largest domain containing $x = 1$ for which $f(x)$ has an inverse function? **1**

(c) On the same set of axes, sketch $y = f(x)$ and $y = f^{-1}(x)$ for this domain. (Use at least $\frac{1}{3}$ of a page for your diagram.) **2**

(d) Find an expression for the inverse function $y = f^{-1}(x)$ in terms of x. **2**

(e) The graphs of $y = f(x)$ and $y = f^{-1}(x)$ meet at exactly one point A. Mark point A on your diagram and explain why the x-co-ordinate of A satisfies the equation $x^3 + x - 1 = 0$. **1**

Question 9 continues on the next page

Question 9 (continued) **Marks**

(f) Taking $x = 0.5$ as the first approximation, use one application of Newton's method to obtain another approximation to the solution of $x^3 + x - 1 = 0$. (Answer correct to 3 decimal places.) **2**

(g) Use Simpson's rule with five function values to approximate $\int_0^1 \frac{1}{1+x^2}\,dx$. **2**

Express your answer in simplest fraction form.

(h) Find the exact value of $\int_0^1 \frac{1}{1+x^2}\,dx$ and using your answer from part (g), find an approximation for π in simplest fraction form. **1**

End of paper

Year12 Ext1 HSC Task #2 2012.

① the root is further away in this case. (C)

II

② $\sin x = \frac{1}{\sqrt{2}}$

$x = (-1)^n \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + n\pi$ where $n \in \mathbb{Z}$

$x = (-1)^n \left(\frac{\pi}{4}\right) + n\pi$ (C)

T

③ $\int \frac{e^x}{1+e^{2x}}\,dx$

$u = e^x$

$\frac{du}{dx} = e^x$

$du = e^x dx$

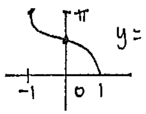
$= \int \frac{e^x\,dx}{1+(e^x)^2}$

$= \int \frac{du}{1+u^2}$

$= \tan^{-1} u + C$

$= \tan^{-1} e^x + C$ (C)

Int

④ 

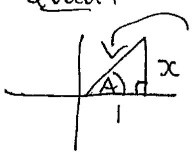
$y = \cos^{-1} x$ is not odd (C)

Inv

⑤ Let $A = \tan^{-1} x$

$\tan A = x$ for $-\frac{\pi}{2} < A < \frac{\pi}{2}$

∴ Quad I



By pythagoras.

$= \sqrt{1+x^2}$

∴ $\cos A = \frac{1}{\sqrt{1+x^2}}$

$A = \cos^{-1} \frac{1}{\sqrt{1+x^2}}$ (B)

Inv

Section II Question 6.

a)

Method A

$\int \frac{-1}{\sqrt{3-4x^2}}\,dx$

$= \frac{1}{2}\int \frac{-2}{\sqrt{\sqrt{3}^2-(2x)^2}}\,dx$

$= \frac{1}{2}\cos^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$

or

$= -\frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$

Method B

$\int \frac{-1}{\sqrt{3-4x^2}}\,dx$

$= \int \frac{-1}{\sqrt{4\left(\frac{3}{4}-x^2\right)}}\,dx$

$= \int \frac{-1}{2\sqrt{\frac{3}{4}-x^2}}\,dx$

$= -\frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2-x^2}}\,dx$

$= -\frac{1}{2}\sin^{-1}\left(\frac{x}{\frac{\sqrt{3}}{2}}\right) + C$

$= -\frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$

Use standard integrals.

$\int \frac{f'(x)}{\sqrt{a^2 - f(x)^2}}\,dx$

$= \sin^{-1}\frac{f(x)}{a} + C$

or $= -\cos^{-1}\frac{f(x)}{a} + C$

This is a standard question, you must be able to do/expect in Trial or HSC.

✓ ✓

Be careful with negative signs.

b) $f(x) = 2\cos^{-1}(x-1)$

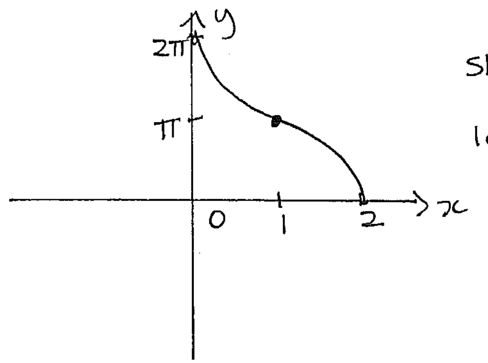
i) domain

$-1 \le x-1 \le 1$

$0 \le x \le 2$ ✓

range $0 \le \frac{f(x)}{2} \le \pi$

$\therefore 0 \le f(x) \le 2\pi$ ✓

ii) 

Shape ✓

location ✓

You must have the correct shape to get this mark.

The point $(1, \pi)$ is where the curve changes concavity.

Question 6 (continued)

c) i) $\frac{d}{dx}\tan^{-1}(x^3)$

$$= \frac{1}{1+(x^3)^2} \times 3x^2$$

$$= \frac{3x^2}{1+x^6}$$

$\boxed{\frac{d}{dx}\left(\tan^{-1} f(x)\right) = \frac{1}{1+f(x)^2} \times f'(x)}$

chain rule

ii) $\therefore \int_{-1}^{1} \frac{x^2}{1+x^6}\,dx$

$$= \frac{1}{3}\int_{-1}^{1} \frac{3x^2}{1+x^6}\,dx$$

$$= \frac{1}{3}\left[\tan^{-1}(x^3)\right]_{-1}^{1}$$

$$= \frac{1}{3}\left[\tan^{-1}1 - \tan^{-1}(-1)\right]$$

$$= \frac{1}{3}\left[\tan^{-1}1 + \tan^{-1}1\right]$$

$$= \frac{2}{3}\tan^{-1}1$$

$$= \frac{2}{3} \times \frac{\pi}{4}$$

$$= \frac{\pi}{6}$$

There were some issues with the placement of the 3 and $\frac{1}{3}$. Other than that, this question was well done by most students.

$\boxed{\text{Since odd function } \tan^{-1}(-x) = -\tan^{-1}x}$

Question 7.

a) $\int_0^{\frac{\pi}{4}} \cos x \sin^3 x\,dx$

$$= \left[\frac{\sin^4 x}{4}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left\{\left(\sin\frac{\pi}{4}\right)^4 - (\sin 0)^4\right\}$$

$$= \frac{1}{4} \times \left(\frac{1}{\sqrt{2}}\right)^4 - 0$$

$$= \frac{1}{4} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{16}$$

Use reverse chain rule

$\boxed{\int f'(x) \times f(x)^n\,dx = \frac{f(x)^{n+1}}{n+1} + C}$

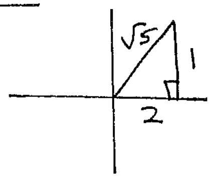
Many students did not realise you could reverse the chain rule here.

The most common mistake was to take a sin out and convert $\sin^2 x$ using double angle result. Possible but very difficult

b) i) $\alpha = \sin^{-1}\frac{1}{\sqrt{5}}$

$\sin\alpha = \frac{1}{\sqrt{5}}$ for $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$

$\therefore$ quad 1

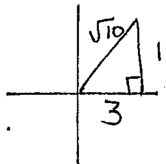


By Pythagoras.

$\therefore \cos\alpha = \frac{2}{\sqrt{5}}$

$\beta = \sin^{-1}\frac{1}{\sqrt{10}}$

$\sin\beta = \frac{1}{\sqrt{10}}$



$\therefore \cos\beta = \frac{3}{\sqrt{10}}$

Most students realised that they needed triangles to find $\cos\alpha$ and $\cos\beta$. Some silly mistakes here.

$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{1}{\sqrt{5}} \times \frac{3}{\sqrt{10}} + \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{10}}$$

$$= \frac{3}{\sqrt{50}} + \frac{2}{\sqrt{50}}$$

$$= \frac{5}{\sqrt{50}}$$

$$= \frac{5}{5\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

The most frustrating issue here was that I saw $\sqrt{5} \times \sqrt{10} = \sqrt{15}$ so many times!!

b) ii) Since $\sin(\alpha+\beta) = \frac{1}{\sqrt{2}}$

$\therefore \alpha+\beta = \sin^{-1}\frac{1}{\sqrt{2}}$

$= \frac{\pi}{4}$

Done well if students made it this far.

c) i)

Method A

$LHS = \frac{x+2}{x+1}$ split up

$= \frac{x+1+1}{x+1}$

$= \frac{x+1}{x+1} + \frac{1}{x+1}$

$= 1 + \frac{1}{x+1}$

$= RHS$

Method B

$RHS = 1 + \frac{1}{x+1}$

common denom

$= \frac{x+1+1}{x+1}$

$= \frac{x+2}{x+1}$

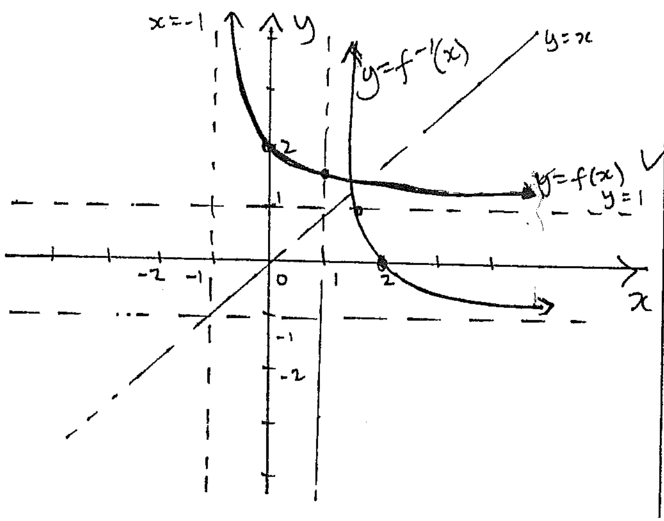
$= LHS.$

Done very well!

ii) $f(x) = \frac{x+2}{x+1} = 1 + \frac{1}{x+1}$ hyperbola

Shift up one unit.

vertical asymptote $x=-1$



$y=f(x)$.

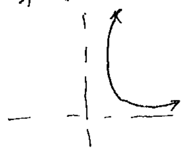
domain $x>-1$

range $y>1$

Done reasonably well although many students didn't read the question and graphed over the total domain!

must show asymptotes clearly

Some of you need to buy a ruler and be careful drawing curves that are asymptotic



is not acceptable

c) iii) $f(x) = \frac{x+2}{x+1}$

$y = \frac{x+2}{x+1}$

Swap x and y

$x = \frac{y+2}{y+1}$

Make y the subject

$x(y+1) = y+2$

$xy + x = y+2$

$xy - y = 2-x$

$y(x-1) = 2-x$

$y = \frac{2-x}{x-1}$

or

$y - xy = x-2$

$y(1-x) = x-2$

$y = \frac{x-2}{1-x}.$

Either of these.
(Signs are opposite)

Done well by most students.

It was pleasing to see students complete this even though they may have had trouble with the sketch!

iv) Sketch $y=f^{-1}(x)$ on same diagram.

domain $x>1$

range $y>-1$

must show asymptotes clearly.

Done well

Question 8.

a) $u = \sqrt{x}$ $\quad \therefore x = u^2$

$\therefore u = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$du = \frac{1}{2\sqrt{x}}\,dx$

This part was not very well done.

Once you have found the substitution parts, have a careful look at each part of the original and rearrange it & set it up ready for the substitution.

$\int \frac{1}{\sqrt{x}\,\sqrt{1-x}}\,dx$

set up.

$= 2\int \frac{1}{\sqrt{1-x}}\,\frac{dx}{2\sqrt{x}}$

You never have to put in variables only numbers.

$= 2\int \frac{1}{\sqrt{1-u^2}} \cdot du$

You have to recognise that this will integrate to an inverse sine function.

$= 2\sin^{-1} u + C$

$= 2\sin^{-1}\sqrt{x} + C$

must be written in terms of x for final answer.

b) i) $\cos x - \sin x$

$= R\cos(x+\alpha)$

$= R\cos x\cos\alpha - R\sin x\sin\alpha$

Equate parts

$R\cos\alpha = 1$ ①

$R\sin\alpha = 1$ ②

Find R	Find α
$R = \sqrt{1^2+1^2}$	①/②
$= \sqrt{2}$	$\tan\alpha = 1$
	$\alpha = \frac{\pi}{4}$

$\therefore \cos x - \sin x = \sqrt{2}\cos(x+\frac{\pi}{4})$

In Extension 1, This is a basic standard technique. You must be able to do auxiliary angle method. in Trial / HSC.

Question 8 (continued)

b) ii) $\cos x - \sin x = 1$

$\sqrt{2}\cos(x+\frac{\pi}{4}) = 1$

$\cos(x+\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

This is now a 2u problem

domain

$0 \le x \le 2\pi$

$\frac{\pi}{4} \le x+\frac{\pi}{4} \le 2\pi+\frac{\pi}{4}$

Note the domain when $\frac{\pi}{4}$ is added.

quad 1 & 4

$x+\frac{\pi}{4} = \frac{\pi}{4},\ 2\pi-\frac{\pi}{4},\ 2\pi+\frac{\pi}{4}$

You can do another revolution so you get 3 answers.

$\therefore x = 0,\ \frac{7\pi}{4}-\frac{\pi}{4},\ 2\pi+\frac{\pi}{4}-\frac{\pi}{4}$

$x = 0,\ \frac{3\pi}{2},\ 2\pi$

Well done to those who got this correct with all 3 answers.

iii) $\cos x - \sin x = 1$

$\cos(x+\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

Find general solution.

You had a rule page!

$x+\frac{\pi}{4} = \pm\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) + 2n\pi$

$x = \pm\frac{\pi}{4} + 2n\pi - \frac{\pi}{4}$

Everyone should have been able to do this using the rule.

$\therefore$

$x = \frac{\pi}{4}-\frac{\pi}{4}+2n\pi$ or $x = -\frac{\pi}{4}-\frac{\pi}{4}+2n\pi$

$x = 0+2n\pi$ $\qquad x = -\frac{\pi}{2}+2n\pi$

$= 2n\pi$

where $n \in \mathbb{Z}$ $\qquad$ where $n \in \mathbb{Z}$.

Note

If it had been worth 2 marks, the simplification steps would have been necessary.

No simplification needed this time but you should still do the basics. ie. $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

Question 8 continued.

c) $\log_e x = \sin x$.

Let $f(x) = \log_e x - \sin x$.

$f(2) = \log_e 2 - \sin 2$
$\doteqdot -0.21615\ldots$

$f(3) = \log_e 3 - \sin 3$
$\doteqdot 0.95749\ldots$

Since $f(2) < 0$ and $f(3) > 0$ and the curve is continuous there is a root between $x = 2$ and $x = 3$.

Calculator must be in radians for this mark.

This is an easy question and it is the prelude to finding the root by halving the interval or Newton's method.

You must include this in your conclusion. Make sure you do in the Trial / HSC.

Question 9.

a) The inverse is not a function because the horizontal line test cuts $y=f(x)$ more than once.

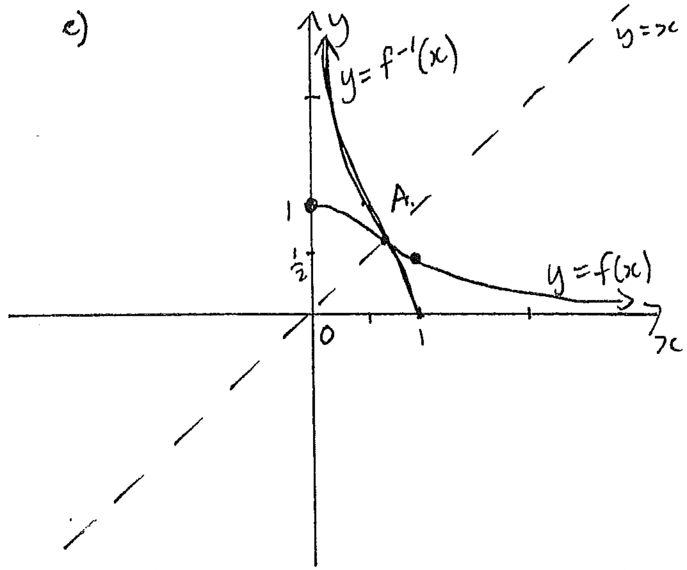
Most students had some idea of how to answer this question but wrote poor reasons! i.e. $f^{-1}(x)$ does not fail the horizontal line test!!

b) $x > 0$ is largest domain.
or $x \geq 0$

Done well.

Question 9 (continued)

c)



$y=f(x)$
domain $x \geq 0$
range $y > 0$

$y=f^{-1}(x)$
domain $x > 0$
range $y \geq 0$

Be careful as many students left $y=f(x)$ for the whole domain.

d) $f(x) = \dfrac{1}{1+x^2}$

$y = \dfrac{1}{1+x^2}$

swap x and y.

$x = \dfrac{1}{1+y^2}$

make y the subject.

$x(1+y^2) = 1$

$x + xy^2 = 1$

$xy^2 = 1 - x$

$y^2 = \dfrac{1-x}{x}$

$y = \pm\sqrt{\dfrac{1-x}{x}}$

$\therefore y = \sqrt{\dfrac{1-x}{x}}$ for $x \geq 0, y \geq 0$.

Done well by nearly all students but most did not explain why this is the case!

Take positive case only.

Question 9 (continued)

e) Mark point A on diagram.

$y=f(x)$ and $y=f^{-1}(x)$ intersect on the line $y=x$.

$\therefore$ Solve simultaneously

$y=f(x)=\frac{1}{1+x^2}$

and $y=x$.

$$\therefore \frac{1}{1+x^2}=x$$
$$1=x+x^3$$
$$\therefore x^3+x-1=0$$

Most students tried to solve $\frac{1}{1+x^2}=\sqrt{\frac{1-x}{x}}$ and got in to difficulties. Poorly done!

f) $f(x)=x^3+x-1$

$f'(x)=3x^2+1$

$$f(0.5)=(0.5)^3+0.5-1$$
$$=-0.375$$

$$f'(0.5)=3\times0.5^2+1$$
$$=1.75$$

<u>Using Newton's method</u>

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$
$$=0.5-\frac{-0.375}{1.75}$$
$$=-0.714 \quad (3\text{d.p})$$

Done very well by nearly all students. Just a few careless errors.

Question 9 continued.

g) $\int_0^1 \frac{1}{1+x^2}\,dx$ $\qquad h=\frac{1-0}{4}=\frac{1}{4}$

$y=\frac{1}{1+x^2}$

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y	1	$\frac{1}{1+(\frac{1}{4})^2}$	$\frac{1}{1+(\frac{1}{2})^2}$	$\frac{1}{1+(\frac{3}{4})^2}$	$\frac{1}{2}$
Simplified	1	$\frac{16}{17}$	$\frac{4}{5}$	$\frac{16}{25}$	$\frac{1}{2}$
weightings	1	4	2	4	1

Using Simpson's rule

$$\int_0^1 \frac{1}{1+x^2}\,dx$$
$$\doteqdot \frac{\frac{1}{4}}{3}\left\{1+\frac{1}{2}+4\left(\frac{16}{17}+\frac{16}{25}\right)+2\times\frac{4}{5}\right\}$$
$$=\frac{8011}{10200}$$

(Answer as a fraction)

Using Simpson's Rule $\int \doteqdot \frac{b-a}{6}\left[f(a)+4\times f\left(\frac{a+b}{2}\right)+f(b)\right]$ was more successful than using the algorithm $\frac{h}{3}\left[f(\text{first})+f(\text{last}) \; 2f(\text{evens})+4(\text{odds})\right]$ If you use this algorithm learn how to do it properly!!!

h) $\int_0^1 \frac{1}{1+x^2}\,dx$

$$=\left[\tan^{-1}x\right]_0^1$$
$$=\tan^{-1}1-\tan^{-1}0$$
$$=\frac{\pi}{4}$$

Equating parts h) and g)

$$\therefore \frac{\pi}{4}=\frac{8011}{10200}$$
$$\pi=\frac{8011}{2550}$$
$$=3\frac{361}{2550}$$

Done well for those who attempted it!