Student Number: ___________

SCEGGS Darlinghurst

Term 2, 2008
Wednesday 18th June

EXTENSION 1 MATHEMATICS

Task Weighting : 30 %

Outcomes Assessed: HE2, HE3, HE4, HE6, HE7

General Instructions

- Time allowed - 75 minutes
- Write your student number at the top of each page
- **Start each question on a new page**
- Attempt **all** questions and show all necessary working.
- Marks may be deducted for careless or badly arranged work
- Mathematical templates, geometrical equipment and scientific calculators may be used.
- A table of Standard Integrals is attached at the back of this paper.

Question	Calculus	Communication	Reasoning	TOTAL
1	/7	/3		/13
2	/6		/5	/14
3	/8	/3	/4	/16
4	/6		/4	/14
TOTAL	**/27**	**/6**	**/13**	**/57**

Parent's Signature________________________________

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$

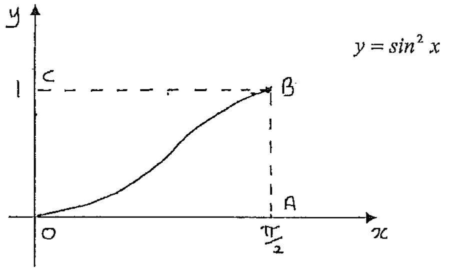
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 2 (14 Marks) START A NEW PAGE **Marks**

(a) Differentiate $x^2 \tan^{-1} 2x$ **2**

(b) The diagram shows the curve $y = \sin^2 x$ between $x = 0$ and $x = \frac{\pi}{2}$



Show that this curve divides the rectangle $OABC$ into two regions of equal area. **4**

(c) For the function $y = \ln(1 + e^x)$

(i) State the domain of $y = f(x)$ **1**

(ii) Find the inverse function $f^{-1}(x)$ **2**

(iii) State the domain of $f^{-1}(x)$ **1**

Question 2 continues on the next page

Question 2 (continued) **Marks**

(d) During recent heavy rainfall, Angus monitored the volume V, in kilolitres, of water in a dam on his rural property.

The rate of change of the volume of water in the dam after t hours is given by $\frac{dV}{dt} = k(V - 15000)$, where k is a constant.

(i) Show that $V = 15000 + Ae^{kt}$ satisfies the differential equation **1**

$$\frac{dV}{dt} = k(V - 15000).$$

(ii) Initially, the dam contained 150000 kilolitres and after 10 hours of pouring rain the volume increased to 375000 kilolitres.

Find the values of A and k. **2**

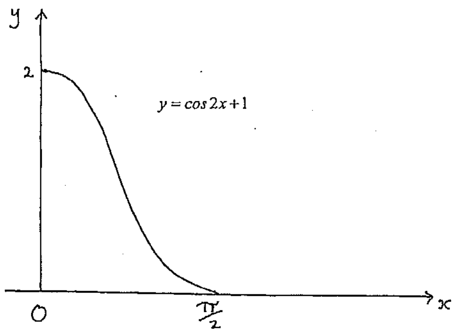
(iii) What volume of water will be in the dam after 2 days? (Answer to the nearest kilolitre.) **1**

Question 3 (16 marks) START A NEW PAGE **Marks**

(a) Use the substitution $u = \tan x$, or otherwise, to evaluate

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\, dx$$ 3

(b) The graph shows the curve $y = \cos 2x + 1$ for $0 \le x \le \frac{\pi}{2}$

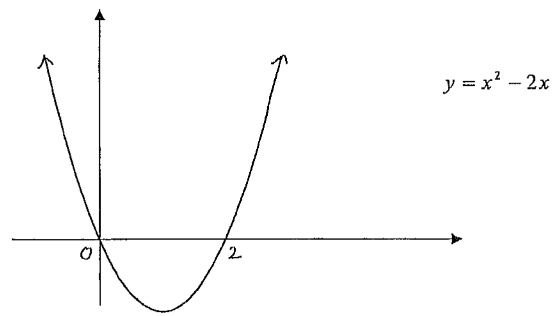


Find the exact volume of the solid of revolution formed when the region bounded by the curve $y = \cos 2x + 1$ and the x-axis between $x = 0$ to $x = \frac{\pi}{2}$ is rotated about the x-axis. 3

Question 3 continues on the next page.

Question 3 (continued) **Marks**

(c) The function $y = x^2 - 2x$ is shown in the diagram.



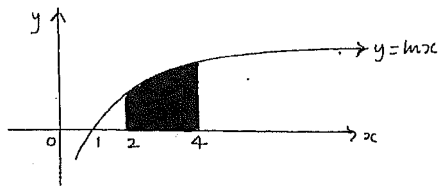
(i) State why the function $y = x^2 - 2x$ does not have an inverse function. 1

(ii) State the largest possible domain over which the function $y = x^2 - 2x$ is monotonic increasing. 1

(iii) Show that the inverse function $f^{-1}(x)$ over this restricted domain is given by $y = 1 + \sqrt{x+1}$ 2

(iv) On the same set of axes sketch $y = f(x)$ and $y = f^{-1}(x)$ showing all important features. 2

(b) The diagram shows the shaded area bounded by the curve $y = \ln x$, the x-axis and the lines $x = 2$ and $x = 4$.



4

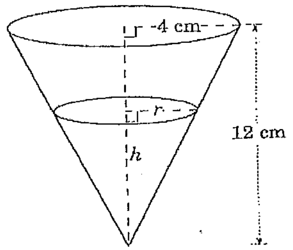
Show that the exact area of the shaded region is given by $(\ln 64 - 2)$ $units^2$

Question 4 (14 Marks) START A NEW PAGE **Marks**

(a) Use an appropriate double angle formula to find the exact value of 2

$$sin\left(2cos^{-1}\frac{6}{7}\right)$$

(b)



NOT TO SCALE

The diagram shows a conical drinking cup of height 12 cm and radius 4 cm. The cup is being filled with water at the rate of 3 cm^3 per second.
The height of the water at time t seconds is h cm and the radius of the water's surface is r cm.

(i) Show that $r = \frac{h}{3}$ 1

(ii) Show that the volume is given by $V = \frac{\pi}{27}h^3$ 1

(iii) Find the rate at which the height is increasing when the height of the water is 9 cm. 2

Question 4 (continued) **Marks**

(c) (i) Show that $\frac{d}{dx}\left(x\,sin^{-1}x+\sqrt{1-x^2}\right)=sin^{-1}x$ 2

(ii) Hence find the exact area bounded by the curve $y = sin^{-1}x$, the x-axis and the lines $x=0$ and $x=1$. 2

(d) (i) Show that $sin^{-1}x+cos^{-1}x=\frac{\pi}{2}$ 2

(ii) Given $sin^{-1}\left(-\frac{2}{3}\right)-cos^{-1}\left(-\frac{2}{3}\right)=k$ and by using the expressions

$sin^{-1}(-x)=-sin^{-1}(x)$ and $cos^{-1}(-x)=\pi-cos^{-1}(x)$

find an expression for $cos^{-1}\left(\frac{2}{3}\right)$ in terms of k. 2

End of paper.

12Ext① Task #2
18-06-08

16.06.2008

Question 1 /13 Calc /7 Comm /3

a) $\lim\limits_{x\to 0} \frac{\sin 7x}{5x}$

$= \frac{7}{5} \lim\limits_{x\to 0} \frac{\sin 7x}{7x}$

$= \frac{7}{5} \times 1$

$= \frac{7}{5}$

Comments.

Do you understand what you're doing? For very small angles $\sin x \doteqdot x \doteqdot \tan x$

✓ Well done but some confused working steps.

b) $u^2 = x+1$

$x = u^2 - 1$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

$\int x\sqrt{x+1}\,dx$

$= \int (u^2-1).\sqrt{u^2}\,.\,2u\,du$

$= \int (u^2-1).u.2u\,du$

$= \int 2u^2(u^2-1)\,du$

$= \int (2u^4 - 2u^2)\,du$ ✓

$= \frac{2u^5}{5} - \frac{2u^3}{3} + C$ ✓

$= \frac{2(\sqrt{x+1})^5}{5} - \frac{2(\sqrt{x+1})^3}{3} + C$ ✓

$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C$

Well done but some of you need to practise your index rules!

must be in terms of x

Don't forget this last step. It's an easy mark. (Calc3)

c) i) $\frac{d}{dx}\left(e^{2x}(\cos x - 2\sin x)\right)$

$u = e^{2x}$ $\quad v = \cos x - 2\sin x$

$u' = 2e^{2x}$ $\quad v' = -\sin x - 2\cos x$

Product rule $\quad vu' + uv'$

$= 2e^{2x}(\cos x - 2\sin x) + e^{2x}(-\sin x - 2\cos x)$

$= \cancel{2e^{2x}\cos x} - 4e^{2x}\sin x$

$\quad - e^{2x}\sin x - \cancel{2e^{2x}\cos x}$

$= -5e^{2x}\sin x$

✓ Well done! Practise collecting like terms if you got this wrong. ✓ (Calc2)

ii) $\int_0^{\pi} e^{2x}\sin x\,dx$

$= -\frac{1}{5}\int_0^{\pi} -5e^{2x}\sin x\,dx$

$= -\frac{1}{5}\left[e^{2x}(\cos x - 2\sin x)\right]_0^{\pi}$ ✓

$= -\frac{1}{5}\left\{e^{2\pi}(\cos\pi - 2\sin\pi) - e^0(\cos 0 - 2\sin 0)\right\}$

$= -\frac{1}{5}\left\{e^{2\pi}(-1-0) - 1(1-0)\right\}$

$= -\frac{1}{5}(-e^{2\pi} - 1)$

$= \frac{1}{5}(e^{2\pi} + 1)$ ✓

well done. (Calc2)

d) $y = 3\cos^{-1} 2x$

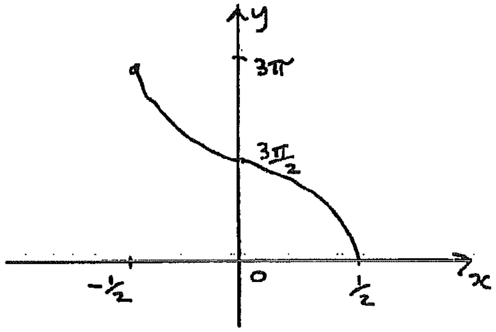
Domain $\quad -1 \le 2x \le 1$

$\quad -\frac{1}{2} \le x \le \frac{1}{2}$ ✓

Range $0 \le \cos^{-1}2x \le \pi$

$$0 \le 3\cos^{-1}2x \le 3\pi$$
$$0 \le y \le 3\pi$$

✓

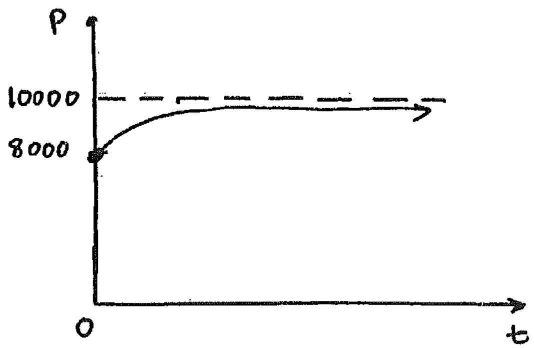


This part was overall very well done.

✓

(Comm1)

g) $P = 10000 - 2000e^{-0.03t}$



$t = 0$, $P = 10000 - 2000 = 8000$

$t \to \infty$, $e^{-0.03t} \to 0$ $\therefore$ $P \to 10000$

Not particularly well done. This is a 2U question! Think about how the graph evolves.

$e^{-0.03t} \to -e^{-0.03t}$

Then shift it up!

OR Substitute in a few values and find some points.

✓ Shape/start at 8000

✓ Horizontal Asymptote

(Comm2)

Question 2 Calc /6, Reas /5

a) $\frac{d}{dx}\left(x^2\tan^{-1}2x\right)$

$u = x^2$ $\quad v = \tan^{-1}2x$

$u' = 2x$ $\quad v' = \frac{1}{1+(2x)^2} \times 2$

$= \frac{2}{1+4x^2}$

using product rule $vu' + uv'$

$$= 2x\tan^{-1}2x + \frac{2x^2}{1+4x^2}$$

Some forgot to use the chain rule when differentiating $\tan^{-1}(2x)$

✓

✓ correct use of product rule (Calc2)

b) Area below curve

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 2x) \, dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{2}\left(\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right)$$
$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right)$$
$$= \frac{\pi}{4} \text{ units}^2$$

Generally well d

✓

✓

✓ (Calc3)

Area of rectangle OABC

$$= \frac{\pi}{2} \times 1$$
$$= \frac{\pi}{2} \text{ units}^2$$

$\therefore$ The curve divides the rectangle area in half.

✓ for calculating the area AND putting it all together. (Reas1)

c) i) $y = \ln(1+e^x)$

Domain $1+e^x > 0$

$e^x > -1$

This is true for all real x

$\therefore$ Domain is all real x

Parts (i) & (iii) were very poorly done!

✓ (Reas 1)

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ii) $f^{-1}(x)$: $y = \log_e(1+e^x)$

Interchange x and y

$x = \log_e(1+e^y)$

make y the subject

use the definition $\begin{cases} x = \log_a N \\ a^x = N \end{cases}$

$e^x = 1+e^y$

$e^y = e^x - 1$ ✓

Take logs both sides or use the defn.

$\log_e e^y = \log_e(e^x-1)$

$y = \log_e(e^x-1)$ ✓

(Reas 2)

iii) **Domain of $f^{-1}(x)$**

$e^x - 1 > 0$

$e^x > 1$

Take logs both sides

$\log_e e^x > \log_e 1$

$x > 0$ ✓

(Reas 1)

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i) $V = 15000 + Ae^{kt}$

$\frac{dV}{dt} = k \cdot Ae^{kt}$

$= k(V - 15000)$

$\therefore V$ satisfies the equation.

Standard exponential growth question that was done well.

✓ (Calc)

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ii) When $t=0$ $V = 150000$

Find A

$150000 = 15000 + Ae^0$

$\therefore A = 135000$ ✓

When $t = 10$, $V = 375000$

$V = 15000 + Ae^{kt}$

$375000 = 15000 + 135000e^{10k}$

$360000 = 135000e^{10k}$

$e^{10k} = \frac{360000}{135000}$

$e^{10k} = \frac{8}{3}$

Take logs both sides.

$\ln e^{10k} = \ln(\frac{8}{3})$

$10k = \ln(\frac{8}{3})$

$k = \frac{1}{10}\ln(\frac{8}{3})$ ✓

$\doteq 0.09808\ldots$ (store in calculator)

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iii) After $t = 2$ days $= 48$ hours

$V = 15000 + 135000e^{k\times 48}$

$= 14\ 976\ 794$ kilolitres ✓

Question 3 16 Calc /8 Comm /3 Reas /4

a) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\,dx$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x\,dx$

Change limits

$x = \frac{\pi}{3}$ $u = \tan\frac{\pi}{3} = \sqrt{3}$

$x = \frac{\pi}{4}$ $u = \tan\frac{\pi}{4} = 1$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\,dx$

$= \int_1^{\sqrt{3}} \frac{du}{u}$

$= \Big[\ln u\Big]_1^{\sqrt{3}}$

$= \ln\sqrt{3} - \ln 1$

$= \ln\sqrt{3}$

First mark $\int \frac{1}{u}\,du$ was well done but after that...

Look at the standard integrals or your rule page! $\int \frac{1}{u}\,du = \ln u$

✓

No need to write $\ln(\tan x)$ because you changed the limits from x to u.

✓

✓

(Calc 3)

Volume about the x axis.

$V = \pi \int_a^b y^2\,dx$

$= \pi \int_0^{\frac{\pi}{2}} (\cos 2x + 1)^2\,dx$

$= \pi \int_0^{\frac{\pi}{2}} (\cos^2 2x + 2\cos 2x + 1)\,dx$

replace by

$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}(1+\cos 4x) + 2\cos 2x + 1\right)dx$

$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 4x + 2\cos 2x + 1\right)dx$

Note that the $\frac{1}{2}$ in this substitution only affects the $(1+\cos 4x)$ not the other terms in that integration.

Terrible! If you can't expand a quadratic you need to perfect this skill.

✓ $(a+b)^2 = a^2 + 2ab + b^2$.

Note $(\cos 2x)^2 \neq \cos^2 4x$

It is just $= \cos^2 2x$

$= \pi\left[\frac{1}{2}x + \frac{1}{2}\times\frac{1}{4}\sin 4x + 2\times\frac{1}{2}\sin 2x + x\right]_0^{\frac{\pi}{2}}$ ✓

$= \pi\left(\frac{\pi}{4} + \frac{1}{8}\sin 2\pi + \sin\pi + \frac{\pi}{2}\right) - 0$

$= \pi\left(\frac{\pi}{4} + 0 + 0 + \frac{\pi}{2}\right)$

$= \pi\left(\frac{3\pi}{4}\right)$

$= \frac{3\pi^2}{4}$ units^3

Well done if you got here.

✓ (Calc 3)

c) i) The horizontal line test fails. There is not one-to-one correspondence

✓ Very well done. (Comm 1)

ii) $x > 1$

✓ ($x \geq 1$ is OK)

iii) $y = x^2 - 2x$

Interchange x and y

$x = y^2 - 2y$

Complete the square

$x + 1 = y^2 - 2y + 1$

$x + 1 = (y-1)^2$

$y - 1 = \pm\sqrt{x+1}$

$y = 1 \pm \sqrt{x+1}$

This question comes straight from your textbook examples! Completing the square method is used in questions like this.

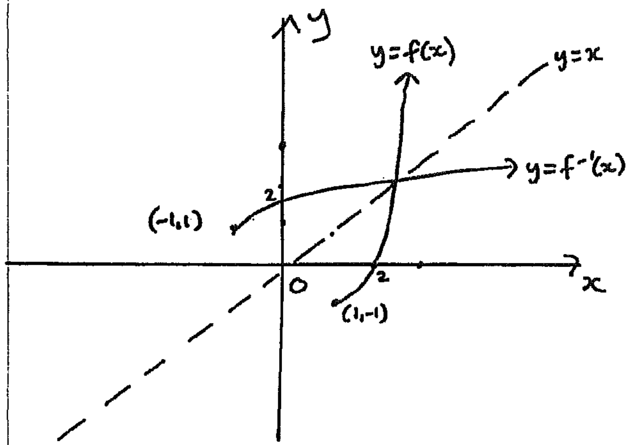
✓

Since the range of the inverse function is $y > 1$

$\therefore y = 1 + \sqrt{x+1}$

← This mark was harder to get. Do you understand the logic.

✓ (Reas 2)

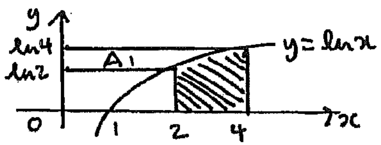


Although not all Originals and Inverse Functions intersect on the line $y=x$, this one does. ✓

✓ intersect on $y=x$

Well done.

(Comm 2)



Find area to the y axis. A_1

$y=\log_e x$

$x=e^y$

$$A_1=\int_{\ln 2}^{\ln 4} e^y\,dy$$

$$=\left[e^y\right]_{\ln 2}^{\ln 4}$$

$$=e^{\ln 4}-e^{\ln 2}$$

$$=4-2$$

$$=2 \quad u^2$$

Shaded area = Large Rectangle $- A_1 -$ Small rectangle

$$=4\ln 4-2-2\ln 2$$

$$=\ln\frac{4^4}{2^2}-2 \quad =\ln 64-2$$

This is definitely a harder 2U question.

Everyone should be able to find area to the y axis.

Note $\int \ln x\,dx \neq \frac{1}{x}$ ✗

✓ but $\int \frac{1}{x}\,dx=\ln x$. ✓

Clear up your rules please!

✓

(Calc 2)

✓ Draw a diagram.

✓ Use some labelled (coloured) sections.

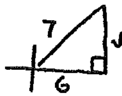
(Reas 2)

Question 4 Calc /6, Reas /4

a) $\sin\left(2\cos^{-1}\frac{6}{7}\right)$

Let $A=\cos^{-1}\frac{6}{7}$

$\cos A=\frac{6}{7}$



← By Pythagoras $\sqrt{7^2-6^2}=\sqrt{13}$

$$\sin\left(2\cos^{-1}\frac{6}{7}\right)=\sin 2A$$

$$=2\sin A\cos A$$

$$=2\times\frac{\sqrt{13}}{7}\times\frac{6}{7}$$

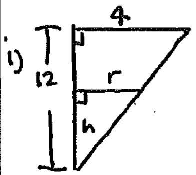
$$=\frac{12\sqrt{13}}{49}$$

It was surprising how many students could not correctly applying Pythagoras or calculate 7x7

✓ Overall, however this was done well

✓

(b) i)



Using similar Δs

$$\frac{12}{4}=\frac{h}{r}$$

$$\therefore\ 3r=h$$

$$r=\frac{h}{3}$$

It is not good enou[gh] to simply state $h=3\times r$ because $12=3\times 4$.

You must state y[ou] ✓ are using simil[ar] Δs!

ii) Volume of a cone

$$V=\frac{1}{3}\pi r^2 h$$

$$=\frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$$

$$=\frac{1}{3}\pi\times\frac{h^2}{9}\times h$$

$$=\frac{\pi h^3}{27}$$

✓

iii) $\frac{dV}{dt} = 3 \text{ cm}^3/\text{sec}$

$\frac{dV}{dh} = \frac{3\pi h^2}{27} = \frac{\pi h^2}{9}$

Find $\frac{dh}{dt}$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$3 = \frac{\pi h^2}{9} \times \frac{dh}{dt}$

when $h = 9$

$\frac{dh}{dt} = \frac{3}{\frac{\pi \times 9^2}{9}}$

$= \frac{3}{9\pi}$

$= \frac{1}{3\pi}$

$\doteqdot 0.106$ cm/sec

✓ ✓

Don't forget the units!

(Calc 2)

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c) $\frac{d}{dx}\left(x\sin^{-1}x + \sqrt{1-x^2}\right)$

$\frac{d}{dx}\left((x\sin^{-1}x + (1-x^2)^{1/2}\right)$

product rule

$u = x \quad v = \sin^{-1}x$

$u' = 1 \quad v' = \frac{1}{\sqrt{1-x^2}}$

$vu' + uv'$

$= \sin^{-1}x \cdot 1 + x \cdot \frac{1}{\sqrt{1-x^2}} + \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x$

$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$

$= \sin^{-1}x$

Please be sure to set out your work clearly. You don't get marks for fudging & if your work is unclear it will look like you're fudging it.

✓ correct use of product rule

✓

(Calc 2)

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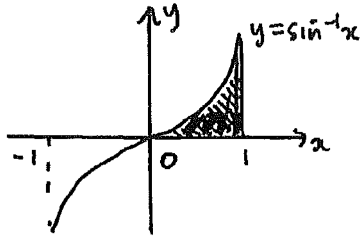
c) ii)



$\text{Area} = \int_0^1 \sin^{-1}x \, dx$

$= \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^1$

$= \left(1\sin^{-1}1 + \sqrt{1-1}\right) - \left(0 + \sqrt{1}\right)$

$= \left(\frac{\pi}{2} - 1\right)$ units^2

This was an easy two marks for most.

(Calc 2)

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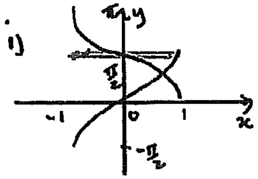
d) i)



It can be shown graphically by adding the ordinates that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

OR show $\frac{d}{dx}(\sin^{-1}x + \cos^{-1}x) = 0$

$\therefore \sin^{-1}x + \cos^{-1}x = \text{constant}$

then evaluate that constant by substituting eg $x = 0$

While this is a nice way to convince yourself $\sin^{-1}x + \cos^{-1}x = \pi/2$, it does not constitute a proof. You really need to show that the function you get by adding the ordinates is a constant. ✓✓ (Reas 2)

Note: there are lots of ways to prove (i)

ii) $\sin^{-1}\left(-\frac{2}{3}\right) - \cos^{-1}\left(-\frac{2}{3}\right) = k$

Replace

$\sin^{-1}\left(-\frac{2}{3}\right) = -\sin^{-1}\left(\frac{2}{3}\right)$

$\cos^{-1}\left(-\frac{2}{3}\right) = \pi - \cos^{-1}\left(\frac{2}{3}\right)$

from (i) $\sin^{-1}\left(\frac{2}{3}\right) + \cos^{-1}\left(\frac{2}{3}\right) = \frac{\pi}{2}$

$\therefore \sin^{-1}\left(-\frac{2}{3}\right) - \cos^{-1}\left(-\frac{2}{3}\right) = k$

$-\sin^{-1}\left(\frac{2}{3}\right) - \left(\pi - \cos^{-1}\left(\frac{2}{3}\right)\right) = k$

$-\left(\frac{\pi}{2} - \cos^{-1}\left(\frac{2}{3}\right)\right) - \pi + \cos^{-1}\left(\frac{2}{3}\right) = k$

$2\cos^{-1}\left(\frac{2}{3}\right) - \frac{\pi}{2} - \pi = k$

Those who ended up with the result $\cos^{-1}(2/3) = k + \sin^{-1}\left(\frac{2}{3}\right) + \pi$ received 1/2.

To get the extra mark you needed to link this part with part (i). (Reas 2) ✓ ✓

$2\cos^{-1}\left(\frac{2}{3}\right) = k + \frac{3\pi}{2}$

$\therefore \cos^{-1}\left(\frac{2}{3}\right) = \frac{k}{2} + \frac{3\pi}{4}$