b) (i) $\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$

$= v \cdot \frac{dv}{dx}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2} \cdot 2v \cdot \frac{dv}{dx}$

$= v \cdot \frac{dv}{dx}$

$\therefore \frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

(ii) $\frac{dv}{dt} = \frac{-k}{x^2}$

sub $\frac{dv}{dt} = -g$, $x = R$,

$-g = \frac{-k}{R^2}$

$R^2 g = k$

(iii) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{R^2 g}{x^2}$

$\therefore \frac{1}{2}v^2 = \int -\frac{R^2 g}{x^2}\, dx$

$\frac{1}{2}v^2 = \frac{R^2 g}{x} + c$

when $x = R$, $v = u$

$\frac{1}{2}u^2 = \frac{R^2 g}{R} + c$

$\therefore c = \frac{1}{2}u^2 - Rg$

$\therefore \frac{1}{2}v^2 = \frac{R^2 g}{x} + \frac{1}{2}u^2 - Rg$

$\therefore v^2 = \frac{2R^2 g}{x} + u^2 - 2gR$

(iv) max distance, $v = 0$

$\frac{2R^2 g}{x} + u^2 - 2gR = 0$

$\frac{2R^2 g}{x} = 2gR - u^2$

$\therefore x = \frac{2R^2 g}{2gR - u^2}$

(v) $g = 9.8$, $R = 6400$

as $x \to \infty$, $u^2 = 2gR$

$u^2 = 2(9.8)(6400)$

$u = \pm 11200$

but $u > 0$ $\therefore u = 11200\,ms^{-1}$

THE SCOTS COLLEGE

2003
TRIAL HSC EXAMINATION

MATHEMATICS
EXTENSION 1

GENERAL INSTRUCTIONS

- Reading time - 5 minutes
- Working time - 2 hours
- Write using blue or black pen
- Board approved calculators may be used
- A table of integrals is provided
- All necessary working should be shown

- Start each question on a new booklet
- Attempt Questions 1 - 7
- All questions are of equal value

QUESTION 1

(a) Find the acute angle between the lines $2x-y=0$ and $x+3y=0$, giving the answer correct to the nearest minute. **[2]**

(b) Solve the inequality $\dfrac{x}{x-3}\le 3$ **[3]**

(c) If u, v and w are the roots of $x^3-4x+1=0$, find the value of $\dfrac{1}{u}+\dfrac{1}{v}+\dfrac{1}{w}$. **[3]**

(d) Solve the equation $\sin 2x=\tan x$ for $0\le x\le \pi$. **[4]**

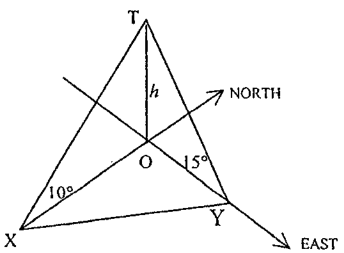
QUESTION 2 **[START A NEW BOOKLET]**

(a) A is the point $(-2,\ 1)$ and B is the point $(x,\ y)$. The point $P(13,\ -9)$ divides AB externally in the ratio 5 : 3. Find the values of x and y. **[3]**

(b) (i) Show that the equation of the normal to the parabola $x^2=4ay$ at the point $T(2at,\ at^2)$ is $x+ty=2at+at^3$. **[2]**

(ii) Hence show that there is only one normal to the parabola which passes through its focus. **[1]**

(c) A surveyor at X observes a tower due north. The angle of elevation to the top of the tower is 10°. He then walks 400m to a position Y which is due east of the tower. The angle of elevation from Y to the top of the tower is 15°.



(i) Write an expression for OY in terms of h. **[1]**

(ii) Calculate h to the nearest metre. **[4]**

(iii) Find the bearing of Y from X. **[1]**

$= \frac{1}{2}(3k^2+5k-2)$

$= \frac{1}{2}(3k+1)(k-2)$

$=$ RHS

$\therefore n=k+1$ is also true

Step 3. Since $n=1$, $n=k$ & $n=k+1$ are all true then $n=2, n=3, \ldots$ are true

$\therefore 1+4+7+\ldots+(3n-2)=\frac{n(3n-1)}{2}$.

(b) $x=3\sin 2t+4\cos 2t$

(i) $3\sin 2t+4\cos 2t$

$= R\sin(2t+\alpha)$

$= R\sin 2t\cos\alpha + R\cos 2t\sin\alpha$

$\therefore R\cos\alpha=3$ $\qquad R^2=3^2+4^2$

$R\sin\alpha=4$ $\qquad R=\sqrt{25}$

$\tan\alpha=\frac{4}{3}$ $\qquad R=5$

$\alpha=0{\cdot}93$

(ii) $x=5\sin(2t+0{\cdot}93)$

$\dot{x}=10\cos(2t+0{\cdot}93)$

$\ddot{x}=-20\sin(2t+0{\cdot}93)$

$=-4x$

$\therefore$ motion is S.H.

(iii) period $=\frac{2\pi}{2}$

$=\pi$

(iv) max disp when $\dot{x}=0$

$10\cos(2t+0{\cdot}93)=0$

$\cos(2t+0{\cdot}93)=0$

$2t+0{\cdot}93=\frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$2t=\pi-0{\cdot}93, \frac{3\pi}{2}-0{\cdot}93$

$t=0{\cdot}32, 1{\cdot}89\ldots$

At $t=0{\cdot}32$,

$x=5\sin(2(0{\cdot}32)+0{\cdot}93)$

$x=5$

Question 7

a) $\frac{d^2x}{dt^2}=0$ $\qquad$ Initially, $\dot{x}=V\cos\theta$, $\dot{y}=V\sin\theta$

$\frac{dx}{dt}=c_1$

$t=0,\ \frac{dx}{dt}=V\cos\theta,\quad \therefore \frac{dx}{dt}=V\cos\theta$

$x=\int V\cos\theta\,dt$

$x=Vt\cos\theta+c_2$

$t=0,\ x=0,\quad c_2=0.\ \therefore x=Vt\cos\theta$

$\frac{d^2y}{dt^2}=-10$

$\frac{dy}{dt}=\int -10\,dt$

$=-10t+c_3$

$t=0,\ \frac{dy}{dt}=V\sin\theta,\ \therefore c_3=V\sin\theta$

$\therefore \frac{dy}{dt}=-10t+V\sin\theta$

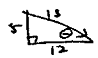
$y=\int -10t+V\sin\theta\,dt$

$y=-5t^2+Vt\sin\theta+c_4$

$t=0,\ y=10,\ \therefore c_4=10$

$\therefore y=-5t^2+Vt\sin\theta+10$

ii) $V=13$



sub $y=0$, $-5t^2+t\cdot 13\cdot\frac{5}{13}+10=0$

$-5t^2+5t+10=0$

$t^2-t-2=0$

$(t-2)(t+1)=0$

$\therefore t=2$ or $t=-1$ but $t\ge 0$

$\therefore t=2$

when $t=2$, $x=13\cdot 2\cdot\frac{12}{13}=24$m

Question 5

a) (i) $y = 2\tan^{-1}x$

$\frac{dy}{dx} = \frac{2}{1+x^2}$

greatest slope occurs at

$x = 0$, $\frac{dy}{dx} = 2$

(ii) $\frac{2}{1+x^2} = \frac{1}{3}$

$6 = x^2+1$

$x^2 = 5$

$x = \pm\sqrt{5}$

(iii) $A = \int_0^k f(x)\,dx$

(iv) As $k \to \infty$, ~~Area under~~ ~~Area~~ under curve

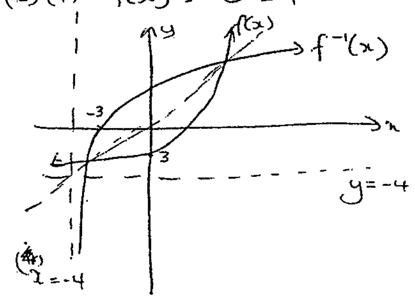
$A = 2\int_0^\infty f(x)\,dx$

$= 2.\left[2\tan^{-1}x\right]_0^\infty$

$= 2.\,2.\frac{\pi}{2}$

$= 2\pi$ sq units.

(b)(i) $f(x) = e^x - 4$



(ii) $f(x)$ & $f^{-1}(x)$ are reflections along the line $y = x$.

Points of intersection are $y = e^x - 4$ & $y = x$ hold true

ie $e^x - 4 = x$

$e^x - x - 4 = 0$.

(iv) let $f(x) = e^x - x - 4$

$f(1) = e - 1 - 4 < 0$

$f(2) = e^2 - 2 - 4 > 0$

$\therefore$ root lies between $x = 1$ & $x = 2$

$f'(x) = e^x - 1$ $\quad x_1 = 1.5$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1.5 - \frac{f(1.5)}{f'(1.5)}$

$= 1.79$ (2 dp).

Question 6

a) Step 1: Need to prove $n=1$ is true

LHS $= 1$, RHS $= \frac{1(3(1)-1)}{2}$

$= 1$

$=$ LHS

$\therefore n=1$ is true

Step 2: Assume that $n=k$ is true

ie $1+4+7+\dots+(3k-2) = \frac{k(3k-1)}{2}$

Need to prove that $n=k+1$ is true

ie $1+4+7+\dots+(3k-2)+(3k+1)$

$= \frac{(k+1)(3k+2)}{2}$

LHS $= 1+4+7+\dots+(3k-2)+(3k+1)$

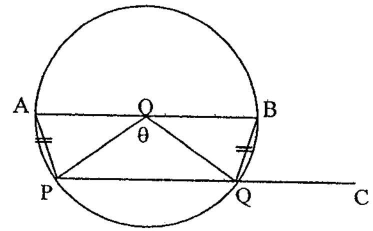
$= \frac{k(3k-1)}{2} + (3k-1)$

$= \frac{1}{2}(3k^2 - k + 6k - 2)$

QUESTION 3 **[START A NEW BOOKLET]**

(a) Evaluate $\int_0^{2\pi} \cos^2 2x\,dx$. **[3]**

(b) The points A, B, P and Q lie on the circle with centre at O.



AB is a diameter and PC passes through Q.

AP is equal to BQ and $\angle POQ = \theta$

(i) Express $\angle AOP$ in terms of θ. **[1]**

(ii) Prove that AB is parallel to PC. **[2]**

(c) By graphing or some other justification, simplify **[3]**

(i) $\sin^{-1}x + \sin^{-1}(-x)$

(ii) $\tan^{-1}x + \tan^{-1}(-x)$

(iii) $\sin^{-1}x - \cos^{-1}(-x)$

(d) Find $\int_0^2 2x\sqrt{1-\frac{x}{2}}\,dx$ using the substitution $u = 1 - \frac{x}{2}$ **[3]**

QUESTION 4 **[START A NEW BOOKLET]**

(a) The surface area of a cube is increasing at a rate of 10cm^2 per second. Find the rate of increase of the volume of the cube when the edge of the cube has length 12cm. **[4]**

(b) N is the number of animals in a certain population at time t years. The population size N satisfies the equation $\frac{dN}{dt} = -k(N-1000)$ for some constant k.

(i) Verify where A is constant, that $N = 1000 + Ae^{-kt}$ is a solution of the equation. **[2]**

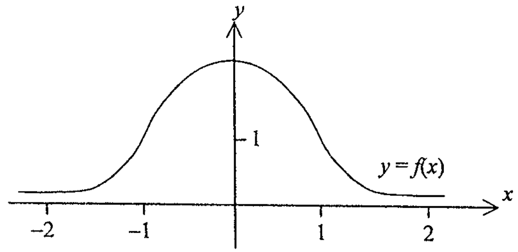
(ii) Initially there are 2500 animals but after 2 years there are only 2200 left. Find the values of A and k, to 2 decimal places. **[2]**

(iii) Find when the number of animals has fallen to 1300. **[2]**

(iv) Sketch the graph of the population size against time. **[2]**

QUESTION 5 **[START A NEW BOOKLET]**

(a) The graph below shows the derivative of $y = 2\tan^{-1} x$.



(i) Where does $y = 2\tan^{-1} x$ have its greatest slope and what is this slope? [2]

(ii) Calculate the x values correspond with $\frac{dy}{dx} = \frac{1}{3}$? [1]

(iii) Write an integral that represents the area in the first quadrant bounded by this curve, the x axis and $x = k$, where $k > 0$. [1]

(iv) By considering the limit as $k \to \infty$ determine the total area bounded by this curve and the x axis. [1]

(b) (i) Sketch the graph of function $f(x) = e^x - 4$. [1]

(ii) On the same diagram sketch the graph of the inverse function f^{-1}. [2]

(iii) Explain why the x coordinate of any point of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ satisfies the equation $e^x - x - 4 = 0$. [1]

(iv) Show that the equation $e^x - x - 4 = 0$ has a root between $x = 1$ and $x = 2$. Use one application of Newton's method to approximate the root, to 2 decimal places. [3]

QUESTION 6 **[START A NEW BOOKLET]**

(a) Prove by Mathematical Induction that $1 + 4 + 7 + \ldots + (3n-2) = \frac{n(3n-1)}{2}$ for all positive integers n. [5]

(b) A particle moves in a straight line so that its displacement x from a fixed point 0 at time t is given by $x = 3\sin 2t + 4\cos 2t$.

(i) If the motion is expressed in the form of $x = R\sin(2t + \alpha)$ where α is in radians, evaluate the constants R and α, to 2 decimal places. [3]

(ii) Show that the motion is Simple Harmonic. [1]

(iii) What is the period of oscillation? [1]

(iv) Determine the maximum displacement from the centre of motion. [2]

d) $u = 1 - \frac{x}{2}$ $\quad x = 0,\ u = 1$

$x = 2,\ u = 0$

$\frac{du}{dx} = -\frac{1}{2}$ $\quad x = 2(1-u)$

$\int_0^2 2x\sqrt{1 - \frac{x}{2}}\,dx$

$= 2 \cdot -2\int_1^0 2(1-u)\sqrt{u}\,du$

$= -8\int_0^1 u^{\frac{1}{2}}(1-u)\,du$

$= 8\int_1^0 u^{\frac{1}{2}} - u^{3/2}\,du$

$= 8\left[\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right]_0^1$

$= 8\left(\frac{2}{3} - \frac{2}{5} - 0\right)$

$= \frac{32}{15}$

Question 4

a) $\frac{dA}{dt} = 10$ $\quad x = 12$

$\frac{dV}{dt} = ?$

$A = 6x^2$ $\quad V = x^3$

$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$

$10 = 12x \cdot \frac{dx}{dt}$

$x = 12,\ 10 = 144 \cdot \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{10}{144}$

$\frac{dx}{dt} = \frac{5}{72}$

$\therefore \frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$

$= 3x^2 \cdot \frac{dx}{dt}$

$= 3(12^2)\ \frac{5}{72}$

$= 30\ cm^3/s$

(b)

(i) $N = 1000 + Ae^{-kt}$

$\frac{dN}{dt} = Ae^{-kt} \cdot -k$

$= -k(N - 1000)$

(ii) $t = 0,\ N = 2500$

$t = 2,\ N = 2200$

$2500 = 1000 + A$

$A = 1500$

$2200 = 1000 + 1500e^{-2k}$

$1500e^{-2k} = 1100$

$e^{-2k} = \frac{11}{15}$

$-2k = \ln\left(\frac{11}{15}\right)$

$k = \frac{\ln\left(\frac{11}{15}\right)}{-2}$

$k = 0.16$ (2dp)

(iii) $1300 = 1000 + 1500e^{-0.16t}$

$1500e^{-0.16t} = 300$

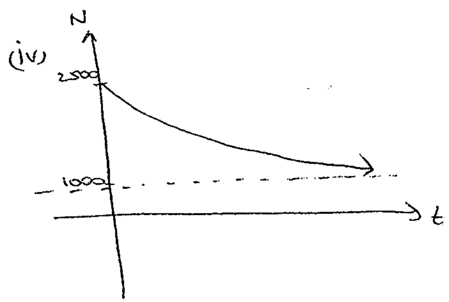
$e^{-0.16t} = \frac{1}{5}$

$-0.16t = \ln\left(\frac{1}{5}\right)$

$t = \frac{\ln\left(\frac{1}{5}\right)}{-0.16}$

$t = 10.06$ years

(iv)



$$\frac{h^2}{\tan^2 10^\circ} + \frac{h^2}{\tan^2 15^\circ} = 160000$$

$$h^2\left(\frac{1}{\tan^2 10^\circ} + \frac{1}{\tan^2 15^\circ}\right) = 160000$$

$$h^2 = 3471.345..$$

$$h = 58.918...$$

$$h = 59\text{m}$$

(iii) $\cancel{\tan}\theta = \frac{04}{400}$

$$\sin\theta = \frac{59}{400\tan 15^\circ}$$

$$\theta \doteq 033^\circ T.$$

Question 3.

a) $\cos^2 x = \frac{1}{2}(\cancel{2}\cos 2x - 1)$

$\cos^2 2x = \frac{1}{2}(\cancel{2}\cos 4x + 1)$

$\therefore \int_0^{2\pi} \cos^2 2x \, dx$

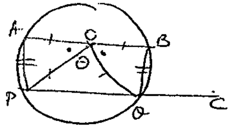
$= \frac{1}{2}\int_0^{2\pi} \cancel{2}\cos 4x + 1 \; dx$

$= \frac{1}{2}\left[\frac{\sin 4x}{\cancel{2}4} + x\right]_0^{2\pi}$

$= \frac{1}{2}\left(\frac{\sin 8\pi}{4} + 2\pi - \left(\frac{\sin 0}{2} - 0\right)\right)$

$= \pi.$

(b) (i)



$\Delta AOP \equiv \Delta BOQ$ (SSS)

$\therefore \angle AOP = \angle BOQ$

$\therefore \angle AOP = \frac{180 - \theta}{2}$

$= 90^\circ - \frac{\theta}{2}$

(ii) $\angle OPQ = \frac{180-\theta}{2}$ (base $\angle$s of isosceles Δ)

$= 90 - \frac{\theta}{2}$

$\therefore \angle AOP = \angle OPQ$ & they are alternate.

$\therefore AB \parallel PQ.$

c) (i) $\sin^{-1}(x) + \sin^{-1}(-x)$

$= \sin^{-1}(x) - \sin^{-1}(x)$

$= 0$

(ii) $\tan^{-1}(x) + \tan^{-1}(-x)$

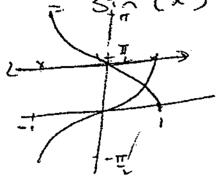
$= \tan^{-1}(x) - \tan^{-1}(x)$

$= 0$

(iii) $\sin^{-1}(x) - \cos^{-1}(-x)$

$= \sin^{-1}(x) - \cos^{-1}(x)$

$= -\frac{\pi}{2}.$



QUESTION 7 [START A NEW BOOKLET]

(a) A projectile has an initial velocity V and an angle of projection θ.

(i) Assuming $\frac{d^2y}{dt^2} = -10$, $\frac{d^2x}{dt^2} = 0$ and initially $x = 0$, $y = 10$, find expressions for x and y. [3]

(ii) If $V = 13ms^{-1}$ and $\theta = \tan^{-1}\left(\frac{5}{12}\right)$ find the range of the projectile. [2]

(b) (i) Use the Chain Rule to show that
$$\frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$ [1]

(ii) The acceleration due to gravity is inversely proportional to the square of the distance x from the centre of the earth.

This can be written as $\frac{dv}{dt} = \frac{-k}{x^2}$. Find k if $\frac{dv}{dt} = -g$ when $x = R$. [1]

(iii) Hence show that $v^2 = \frac{2R^2g}{x} + u^2 - 2gR$ where the initial velocity of a rocket is u ms^{-1}, g is the acceleration due to gravity and R is the radius of the earth. [2]

(iv) Find the maximum distance that the rocket will travel from the centre of the earth. (Answer in terms of g, R and u). [2]

(v) Taking $g = 9.8\text{ms}^{-2}$, $R = 6400\text{km}$ find the value of u in ms^{-1} for which the rocket will escape the gravity of the earth. [1]

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}+C,\quad n\neq -1;\ x\neq 0,\text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x + C,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}+C,\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax + C,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C,\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}+C,\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right)+C,\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)+C$$

NOTE : $\ln x \equiv \log_e x,\quad x>0$

2003 Ext 1 Trial Paper.

Question 1

a) $m_1 = 2,\quad m_2 = -\frac{1}{3}$

$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$= \left|\frac{2+\frac{1}{3}}{1-2(\frac{1}{3})}\right|$

$= |7|$

$\therefore \theta = 81^\circ 52'$

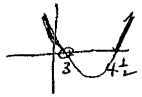
b) $\frac{x}{x-3} \le 3$

$x(x-3) \le 3(x-3)^2$

$x^2 - 3x \le 3x^2 - 18x + 27$

$2x^2 - 15x + 27 \ge 0$

$(2x-9)(x-3) \ge 0$



$\therefore x < 3$ or $x \ge \frac{9}{2}$

c) $x^3 - 4x + 1 = 0.$

$u+v+w = 0$

$uv+uw+vw = -4$

$uvw = -1$

$\frac{1}{u}+\frac{1}{v}+\frac{1}{w} = \frac{vw+uw+uv}{uvw}$

$= \frac{-4}{-1}$

$= 4.$

d) $\sin 2x = \tan x \qquad 0 \le x \le \pi$

$2\sin x\cos x = \frac{\sin x}{\cos x}$

$2\sin x\cos^2 x - \sin x = 0$

$\sin x(2\cos^2 x - 1) = 0$

$\therefore \sin x = 0$ or $\cos x = \pm\frac{1}{\sqrt{2}}$

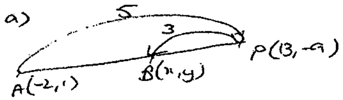
$x = 0, \pi \qquad x = \frac{\pi}{4},\ \pi - \frac{\pi}{4}$

$x = \frac{\pi}{4},\ \frac{3\pi}{4}$

Question 2

$m : n$
$5 : -3$

a)



$x_4 = \frac{nx_1+mx_2}{m+n} \qquad y = \frac{ny_1+my_2}{m+n}$

$13 = \frac{-3(-2)+5x}{5-3} \qquad -9 = \frac{-3(1)+5y}{5-3}$

$26 = 6+5x \qquad -18 = -3+5y$

$5x = 20 \qquad 5y = -15$

$x = 4 \qquad y = -3$

b) $x^2 = 4ay$

(i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

At $x = 2at$,

$\frac{dy}{dx} = t.$

$\therefore$ grad of normal $= -\frac{1}{t}$

$\therefore$ eqn of normal

$y - at^2 = -\frac{1}{t}(x-2at)$

$ty - at^3 = -x + 2at$

$x + ty = 2at + at^3$

(ii) $S(0,a)$

$at = 2at + at^3$

$at + at^3 = 0$

$at(1+t^2) = 0$

$\therefore t = 0$ or $t^2 = -1$ no solution.

c) (i) $\tan 15^\circ = \frac{h}{OY}$

$\therefore OY = \frac{h}{\tan 15^\circ}$

(ii) $\tan 10^\circ = \frac{h}{OX}$

$OX = \frac{h}{\tan 10^\circ}$

$\therefore OX^2 + OY^2 = 400^2$