The Scots College

Year 12 Mathematics Extension 2

Pre-Trial Assessment

April 2009

General Instructions

- All questions are of equal value
- Working time - 2 hours + 5 minutes reading time
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Standard Integrals Table is attached

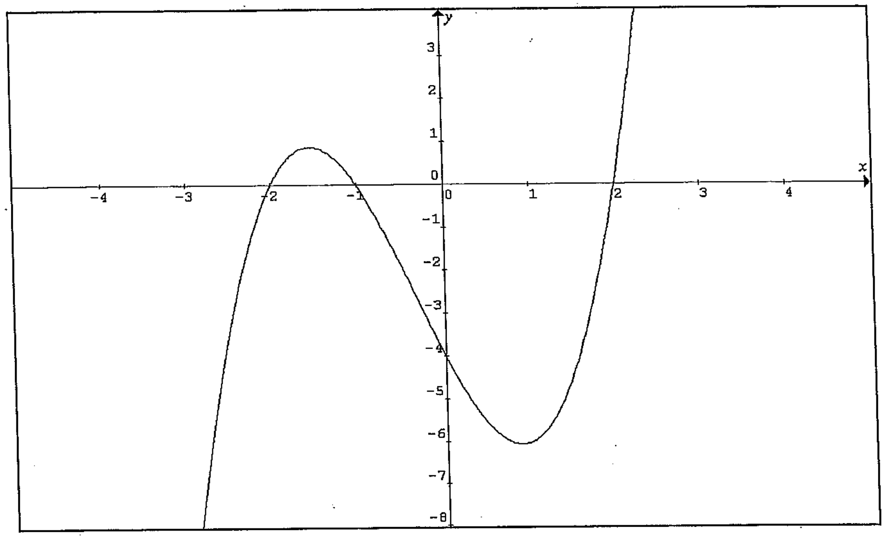
TOTAL MARKS: 75

WEIGHTING: 30 %

- **Start each question in a new booklet**

QUESTION 1 **[15 MARKS]**

(a) In the diagram below y = f(x) is given



On separate diagrams, and without the use of calculus, sketch the following graphs. Indicate clearly any asymptotes, any intercepts with the coordinate axes, and any other significant features.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = -\sqrt{f(x)}$ **2**

(iii) $y = e^{f(x)}$ **2**

(iv) $y = f(2-x)$ **2**

(v) $y = f'(x)$ **2**

(b) Given the function $f(x) = \sqrt{2-\sqrt{x}}$

(i) State the domain of this function **1**

(ii) Show that $f(x)$ is a decreasing function and hence find its range **1**

(iii) Draw a neat sketch of $f(x) = \sqrt{2-\sqrt{x}}$. ,showing all important points. **3**

QUESTION 2 (START A NEW BOOKLET) **[15 MARKS]**

(a) Find $\int 7x\sqrt{(4x^2-3)}\,dx$ **1**

(b) Find $\int \frac{x+1}{\sqrt{(x-1)}}dx$ **2**

(c) Evaluate

(i) $\int_0^{\frac{\pi}{3}} \tan x.\sec^4 x\,dx$ **2**

(ii) $\int_0^{\frac{\pi}{2}} \frac{dx}{\cos x+2}$, by using the substitution $t=\tan\frac{x}{2}$ **2**

(d) Find the values of A and B such that

$$\frac{e^x}{(e^x+2)(e^x+1)}=\frac{A}{(e^x+2)}+\frac{B}{(e^x+1)}$$ **1**

Hence find $\int \frac{e^x}{e^{2x}+3e^x+2}dx$. **3**

(e) $I_n=\int_0^{\frac{\pi}{2}} \cos^n x dx \quad n\geq 0$

(i) Show that $I_n=\frac{n-1}{n}I_{n-2}$ **2**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x dx$ **2**

QUESTION 3 (START A NEW BOOKLET) **[15 MARKS]**

(a) Find the equation of the circle such that the points A (3 , -1) and B (9 , 3) are at opposite ends of a diameter. **2**

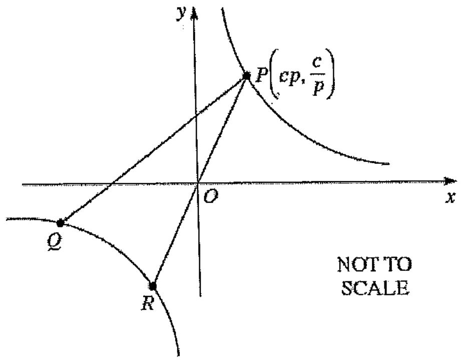
(b) State the foci, directrices and eccentricity of $4x^2+25y^2-100=0$ **2**

(c) Prove that for any point P on the Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the **difference** of its distances from the foci, S *and* S' is constant. **2**

(d) Find the equation of the tangent to the Ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from the point (x_1,y_1). **1**

Hence explain why the equation of the chord of contact from the exterior point (x_0,y_0) is $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$ **1**

(e)



The point $P\left(cp,\frac{c}{p}\right)$, where $p\neq\pm1$, is a point on the hyperbola $xy=c^2$. The normal to the hyperbola at P intersects the second branch at Q. The line through P and the origin O intersects the second branch at R.

(i) Show that the equation of the normal at P is $py-c=p^3(x-cp)$ **2**

(ii) Show that the x coordinates of P and Q satisfy the equation **2**

$$x^2-c\left(p-\frac{1}{p^3}\right)x-\frac{c^2}{p^2}=0$$

(iii) Find the co-ordinates of Q and deduce that the $\angle QRP$ is a right angle. **3**

QUESTION 4 (START A NEW BOOKLET) **[15 MARKS]**

(a) Express the complex number $z = \dfrac{5+i}{3-2i}$ in polar form. 2

(b) Find the square roots of the complex number $z = 5 - 12i$. 2

(c) If $z = -2 - i$ and $w = 3 + 4i$ show on separate Argand diagrams the points represented by

(i) $z + w$ 1

(ii) $z.\bar{z} + w^2$ 2

(d) P represents z on an Argand Diagram. z satisfies $|z - 2i| = 1$

(i) Sketch the locus of P as z varies. 1

ok (ii) Find the maximum and minimum values of Arg(z) where $-\pi < Arg(z) \leq \pi$ 2

ok (iii) Find the complex number z_0, in polar form value, when Arg(z) takes a minimum value. 1
Mark on your sketch the position z_0.

ok (e) Given that $z^6 = -i$,

(i) Find the 6 roots of unity in Polar Form. 2

(ii) Sketch the roots on an Argand Diagram 1

(iii) Explain why there are no conjugate complex numbers in your answer 1

QUESTION 5 (START A NEW BOOKLET) **[15 MARKS]**

(a) Using Integration by parts, to show $\int_1^e \sin(\ln x)dx = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2}$. 4

(b) Without the use of calculus, sketch the following graph. Indicate clearly any asymptotes, any intercepts with the coordinate axes, and any other significant features.

$$|y| = (x-1)(x-2)(x-3)$$ 3

(c) Normals to the ellipse $4x^2 + 9y^2 = 36$, at the points $P(3\cos\alpha, 2\sin\alpha)$ *and* $Q(3\cos\beta, 2\sin\beta)$ are at right angles to each other.

Show that:

(i) the gradient of the normal at P is $\dfrac{3\sin\alpha}{2\cos\alpha}$ 2

(ii) $4\cot\alpha\cot\beta = -9$ 2

(d) The complex number z is represented in the Argand diagram by a point P.

P moves so that $|z + 5| - |z - 5| = 4$.

(i) Explain why the locus of P is the right branch of a hyperbola. 1

(ii) State the foci and find the Cartesian equation of this hyperbola. 2

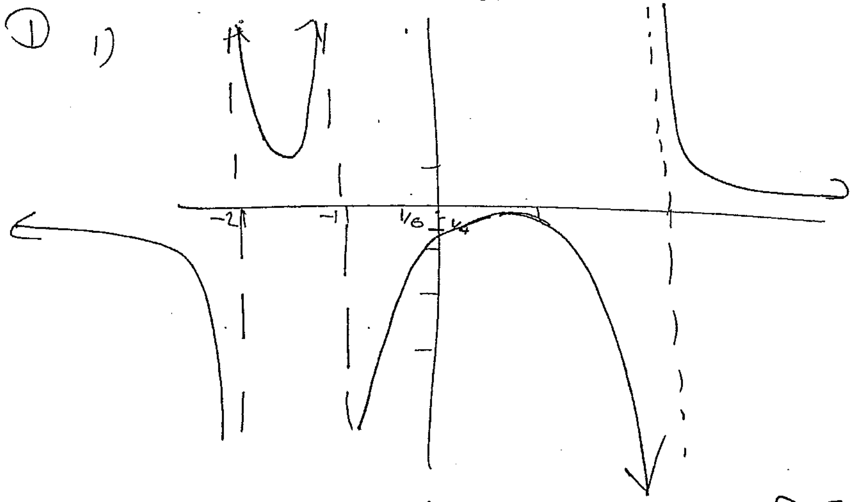
(iii) Find the length of the latus rectum. 1

END OF PAPER

~~EXT 2~~ - Pre Trial 2004 Question 1

SCOTS COLLEGE (Suggested Ans)

① i)



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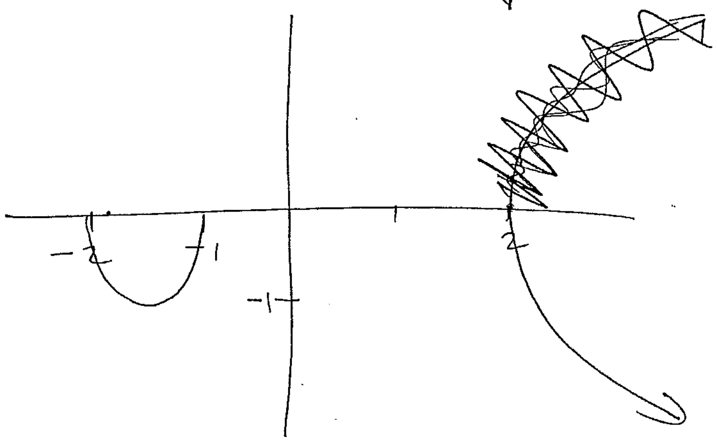
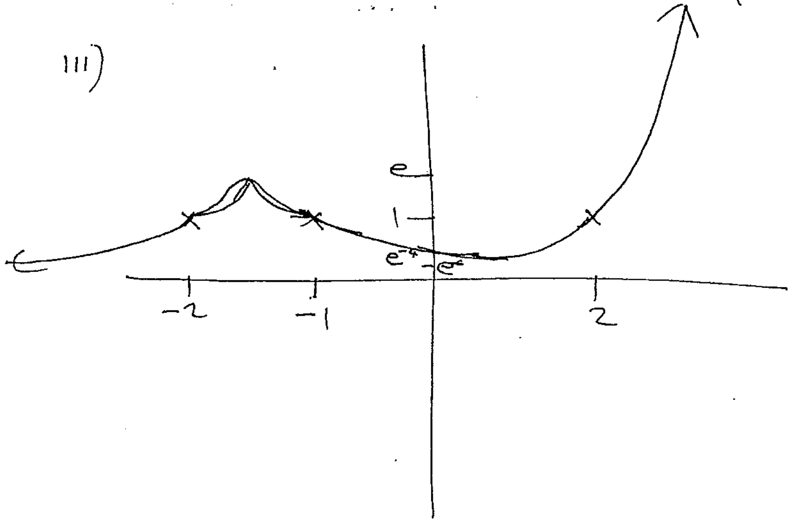


iii)

Ext 2 Pre Trial 2004 Question 1

iii)



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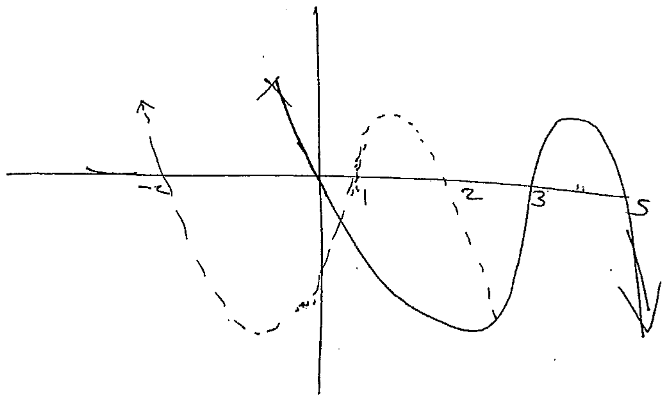

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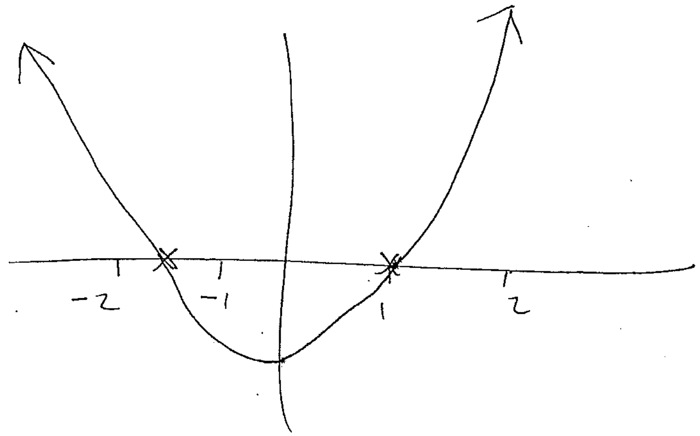


b) $f(x) = \sqrt{2-\sqrt{x}}$

i) Domain $\sqrt{X} \cancel{\leq} 2$

$0 \leq X \cancel{\leq} 4$

ii) $f(x) = (2-x^{1/2})^{1/2}$

$\therefore f'(x) = +\frac{1}{2}(2-x^{1/2})^{-1/2} \times -\frac{1}{2}x^{-1/2}$

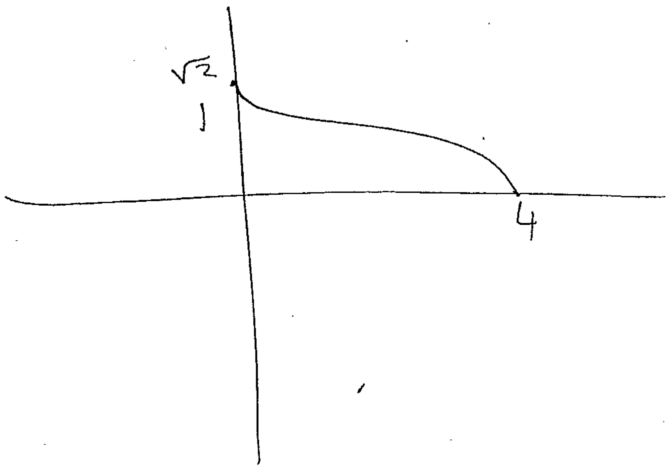
$= \frac{-1}{4(2-x^{1/2})^{1/2}x^{1/2}}$ ← Positive for Domain

$\therefore f'(x)$ always $-$ve $\therefore$ Alway decrea

$\therefore f(0) = \sqrt{2}$ $f(4) = 0$

$\therefore$ Range $0 \leq y \leq \sqrt{2}$

iii)



a) $\int 7x\sqrt{4x^2-3}$

by observation $7x(4x^2-3)^{1/2}$

$\frac{7}{12}(4x^2-3)^{3/2} + C$

b) $\int \frac{x+1}{\sqrt{x-1}}\,dx$ by sub $u = x-1$

$\frac{du}{dx} = 1$

$du = dx$

$\int \frac{u+2}{\sqrt{u}}\,du$

$\int u^{1/2}\,du + \int 2u^{-1/2}\,dx$

$= \frac{2}{3}u^{3/2} + 4u^{1/2} + C$

Can tidy up

$\frac{2}{3}(x-1)^{3/2} + 4(x-1)^{1/2}$

$(x-1)^{1/2}\left[\frac{2}{3}(x-1)+4\right]$

$\frac{1}{3}(x-1)^{1/2}(2x-10) + c$

Ext 2 Trial 2004 Question 2

c

$$\int_0^{\pi/3} \tan x \sec^4 x \, dx$$

$\tan x \sec^2 x \sec^2 x$

$\tan x (\tan^2 x + 1) \sec^2 x$

$\sec^2 x \tan^3 x + \sec^2 x \tan x$

$$\left[\tfrac{1}{4}\tan^4 x + \tfrac{1}{2}\tan^2 x\right]_0^{\pi/3}$$

$\tfrac{1}{4}(\sqrt{3})^4 + \tfrac{1}{2}(\sqrt{3})^2$

$9/4 + 3/2 \quad = 3\tfrac{3}{4}$

$$\int_0^{\pi/2} \frac{dx}{\cos x + 2}$$

$t = \tan \frac{x}{2}$

$\frac{dt}{dx} = \tfrac{1}{2}\sec^2 \frac{x}{2}$

$\frac{dx}{dt} = \frac{2}{\sec^2 \frac{x}{2}}$

$dx = \frac{2}{1+t^2}\,dt$

$x = \frac{\pi}{2} \quad t = \tan \frac{\pi}{4} = 1$

$x = 0 \quad t = \tan 0 = 0$

$$\int_0^1 \frac{1}{\frac{1-t^2}{1+t^2} + 2} \times \frac{2}{1+t^2}$$

$$\int_0^1 \frac{1}{\frac{1-t^2+2+2t^2}{1+t^2}}$$

$$\int_0^1 \frac{\cancel{1+t^2}}{3+t^2} \times \frac{2}{\cancel{1+t^2}}\,dt$$

$$\frac{2}{\sqrt{3}}\int_0^1 \frac{1}{3+t^2} \Rightarrow \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$$

$\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{1}{\sqrt{3}}\right] - 0 \qquad \frac{2\pi}{3\sqrt{3}}$

d)

$e^x = A(e^x+1) + B(e^x+2)$

$e^x = Ae^x + Be^x + A + 2B$

$A + B = 1 \qquad A + 2B = 0$

$\therefore A = 1 - B$

$1 - B + 2B = 0$

$B = -1$

$A = 2$

$$\int \frac{2}{e^x+2}\,dx - \int \frac{1}{e^x+1}\,dx$$

let $e^x + 2 = U$

$\therefore \frac{dU}{dx} = e^x$

$dx = \frac{dU}{e^x}$

let $e^x + 1 = V$

$\frac{dV}{dx} = e^x$

$dx = \frac{dV}{e^x}$

$$\int \frac{2}{U} \times \frac{dU}{U-2}$$

$$\int \frac{1}{V} \times \frac{dV}{V-1}$$

$$\int \cancel{\frac{2}{U^2-}} \quad \frac{2}{U(U-2)} = \frac{A}{U} + \frac{B}{(U-2)}$$

$2 = A(U-2) + BU$

$U=0 \quad A = -1$

$U=2 \quad B = 1$

$$\int -\frac{1}{U} + \frac{1}{U-2}$$

$\frac{1}{V(V-1)} = \frac{A}{V} + \frac{B}{V-1}$

$1 = A(V-1) + BV$

$V=0 \quad A = -1$

$V=1 \quad B = 1$

$$\int -\frac{1}{V} + \frac{1}{V-1}$$

$-\ln U + \ln U-2$

$\ln \frac{U-2}{U} - \ln U$

$\ln e^x - \ln e^x+2$

$1 - \ln(e^x+2)$

$-\ln V + \ln V-1$

$\ln V-1 - \ln V$

$\ln e^x - \ln e^x+1$

$1 - \ln(e^x+1)$

$\ln(e^x+1) - \ln(e^x+2)$

$$\frac{e^x}{(e^x+\frac{3}{2})^2 - \frac{1}{4}}$$

$$\frac{e^x}{e^x+1} - \frac{e^x}{e^x+2}$$

$$\frac{\cancel{e^{2x}}+2e^x - \cancel{e^{2x}} - e^x}{(\quad)(\quad)}$$

$$\frac{e^x}{(\quad)(\quad)}$$

✓

Ext 2 trelrial 2004 - question 2

$I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx \qquad n \geq 0$

$\int_0^{\frac{\pi}{2}} \cos x \cos^{n-1} x\,dx$

$dv = \cos x \qquad U = \cos^{n-1} x$

$V = \sin x \qquad du = n-1 \cos^{n-2} x \cdot -\sin x$

$I_n = \left[\sin x \cos^{n-1} x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^2 x \cos^{(n-2)} x\,dx$

$= (n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cos^{n-2} x\,dx$

$= n-1 \int_0^{\frac{\pi}{2}} \cos^{n-2} x\,dx - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x\,dx$

$= n-1\, I_{(n-2)} - (n-1) I_n$

$n I_n = n-1\, I_{(n-2)}$

$I_n = \frac{(n-1)}{n} I_{(n-2)}$

$I_4 = \frac{3}{4} I_2$

$I_2 = \frac{1}{2} I_0$

$I_0 = \int_0^{\frac{\pi}{2}} \cos^0 x$

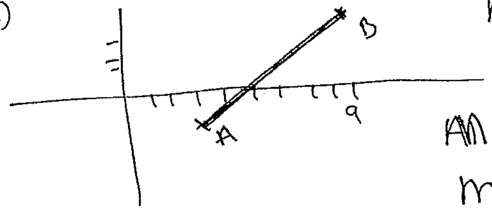
$= [x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}$

$I_2 = \frac{\pi}{4}$

$I_4 = \frac{3\pi}{16}$ //

a)



Mid Point = Centre

$(6, 1)$

$AMB^2 = \sqrt{3^2 + 2^2}$

$MB^2 = 13$ = Radius^2.

$\therefore (x-6)^2 + (y-1)^2 = 13.$

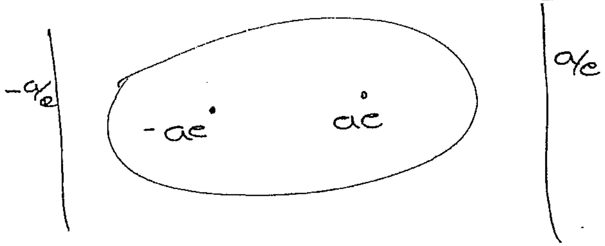
b) $\dfrac{x^2}{5^2} + \dfrac{y^2}{2^2} = 1$ [Ellipse]

$b^2 = a^2(1-e^2)$

$\dfrac{4}{25} = 1 - e^2 \quad \therefore e^2 = 21/25$

$\therefore e = \sqrt{21}/5$

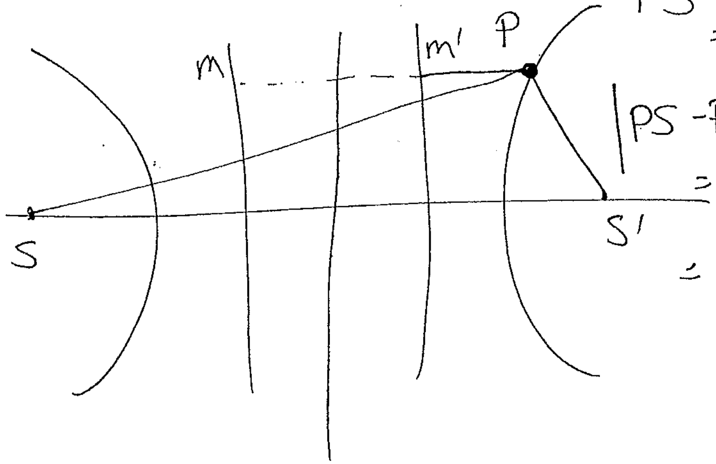
$\cancel{c)}$



Foci $[\pm\sqrt{21}, 0]$

Direct. $\left[x = \pm \dfrac{25}{\sqrt{21}}\right]$

c)



$PS' = ePM'$

$PS = ePM$

$= e[PM' + 2a/e]$

$|PS - PS'|$

$= e[PM' - PM' + 2a/e]$

$= 2a.$

d) i) $y = \dfrac{c^2}{x}$

$\dfrac{dy}{dx} = -\dfrac{c^2}{x^2}$

at $P = \dfrac{-c^2}{c^2p^2}$

$M_{tan} = -\dfrac{1}{p^2}$

$M_{NORMAL} = p^2$

$y - \dfrac{c}{p} = p^2(\cancel{x} - cp)$

$yp - c = p^3(x - cp)$

ii) For this to be true

$xy = c^2 \quad -①$

$py - c = p^3(x - cp)$

Rearrange ①

$y = \dfrac{c^2}{x}$

$py - c = \dfrac{pc^2}{x} - c$

$\therefore \dfrac{pc^2}{x} - c = p^3(x - cp)$

$\dfrac{c^2}{p^2} - \dfrac{cx}{p^3} = x^2 - cpx$

$x^2 + \dfrac{c}{p^3}x - cpx - \dfrac{c^2}{p^2} = 0$

$x^2 - c\left[p - \dfrac{1}{p^3}\right]x - \dfrac{c^2}{p^2} = 0$

$\therefore$ P and Q statisfy the equat

Roots of Quadratic are x_Q, x_P

$\therefore$ Products of Roots $= \frac{c^2}{p^2}$

$x_P = cp$

$\therefore$

$x_Q = c/p^3$

Sub in $xy = c^2$

$y_Q = \frac{c^2}{c/p^3}$

$y_Q = cp^3$

$Q\left(-c/p^3, -cp^3\right)$

$xy = c^2$ is odd function

$\therefore R\left(-cp, -c/p\right)$

$m_{PR} = \left[\frac{\frac{2c}{p}}{2cp}\right]$

$= 1/p^2$

$m_{QR} \left[\frac{-cp^3 - c/p}{-c/p^3 - cp}\right]$

$= \frac{c}{c}\left[\frac{\frac{p^4-1}{p}}{\frac{1-p^4}{p^3}}\right]$

$= -p^2$

$1/p^2 \times -p^2 = -1$

$\therefore \perp$

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a) $\frac{5+i}{3-2i} \times \frac{3+2i}{3+2i}$

$\frac{15+10i+3i-2}{9+4}$

$\frac{13}{13} + \frac{13i}{13}$

$1+i$

$\sqrt{2}\text{ cis } \pi/4$

b) $w^2 = 5-12i$

$(a+bi)^2 = 5-12i$

$a^2-b^2+2abi = 5-12i$

$a^2-b^2 = 5 \quad 2ab = -12$

$a = \frac{-6}{b}$

$\frac{36}{b^2} - b^2 = 5$

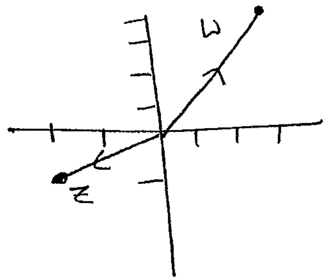
$b^4 + 5b^2 - 36 = 0$

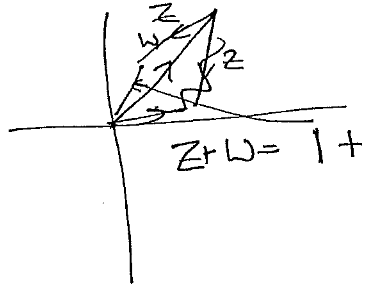
$[b^2+9][b^2-4] = 0$

Reject $\quad b = \pm 2$

$a = \mp 3$

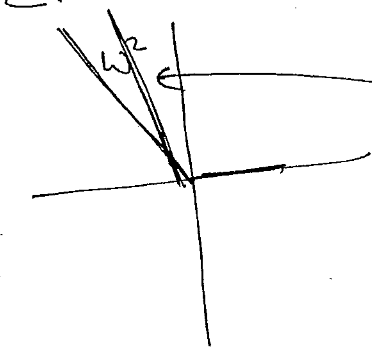
$-3+2i \quad 3-2i$





$z + w = 1+3i$

$z.\bar{z} + w^2$



$w^2 = 9-16+24i$

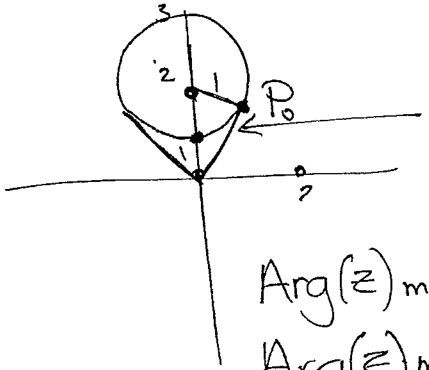
$\therefore -7+24i$

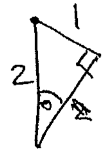
$z\bar{z} = (-2-i)(-2+i)$

$4+1 = 5$

$-2+24i$

d)





$\sin\theta = \frac{1}{2}$

$\frac{\pi}{6}$.

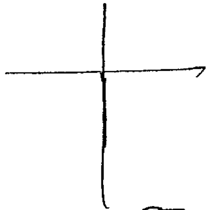
$\text{Arg}(z)_{\min}$ $\frac{\pi}{3}$

$\text{Arg}(z)_{\max}$ $\frac{2\pi}{3}$

$P_0 = \sqrt{3}\,\text{cis}\,\frac{\pi}{3}$

e)

$z^6 = -1$



$z^6 = 1\,\text{cis}\,\frac{-\pi}{2}$.

$\frac{2\pi}{6} = \frac{\pi}{3}$

$\therefore\ z_1 = 1\,\text{cis}\,\frac{-\pi}{12}$.

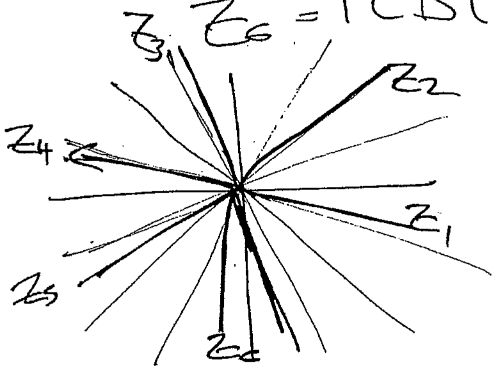
$z_2 = 1\,\text{cis}\left(\frac{-\pi}{12} + \frac{\pi}{3}\right) = \text{cis}\left(\frac{3\pi}{12}\right) = \text{cis}\,\frac{\pi}{4}\left(\frac{4\pi}{12}\right)$

$z_3 = 1\,\text{cis}\left(\frac{7\pi}{12}\right)$

$z_4 = 1\,\text{cis}\left(\frac{11\pi}{12}\right)$

$z_5 = 1\,\text{cis}\left(\frac{15\pi}{12}\right) = 1\,\text{cis}\left(-\frac{9\pi}{12}\right) = \text{cis}\left(\frac{-3\pi}{4}\right)$

$z_6 = 1\,\text{cis}\left(-\frac{5\pi}{12}\right)$



Multiply to make an imaginary number
$\therefore$ no conjugate

a) $\int_1^e \sin(\ln x)\,dx$

$dv = 1 \quad u = \sin(\ln x)$
$v = x \quad du = \frac{1}{x}\cos(\ln x)$

$I = \left[x\sin(\ln x)\right]_1^e + \int_1^e -\cos(\ln x)\,dx$

$dv = 1 \quad u = -\cos(\ln x)$
$v = x \quad du = \frac{1}{x}\sin(\ln x)$

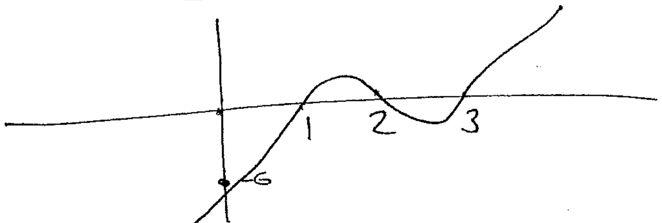
$I = \left[x\sin(\ln x)\right]_1^e + \left[-x\cos(\ln x)\right]_1^e - I$

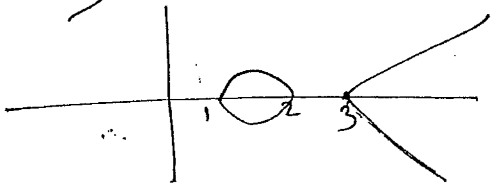
$2I = e\sin 1 - 0 + -e\cos 1 + 1$

$2I = e[\sin 1 - \cos 1] + 1$

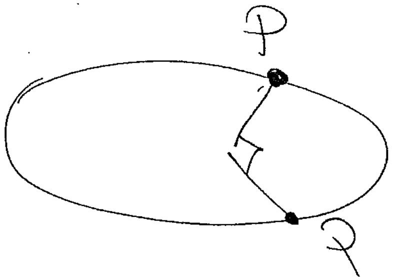
$I = \frac{e}{2}[\sin 1 - \cos 1] + \frac{1}{2}$

b)





c)



$$4x^2+9y^2=36$$

$$8x+18y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-8x}{18y}$$

at $P_{TAN}=\dfrac{-24\cos\alpha}{36\sin\alpha}$

$\therefore$ $P_{NORMAL}=+\dfrac{3\sin\alpha}{2\cos\alpha}$

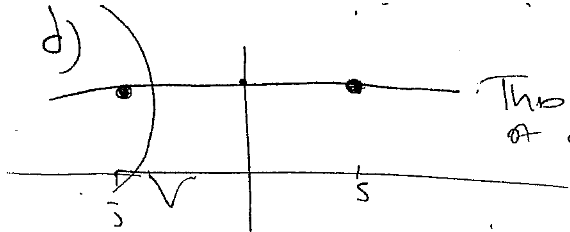
Similarly $Q_{NORMAL}=\dfrac{3\sin\beta}{2\cos\beta}$

we know $\dfrac{3\sin\alpha}{2\cos\alpha}\cdot\dfrac{3\sin\beta}{2\cos\beta}=-1$

$$\frac{9\tan\alpha\tan\beta}{4}=-1$$

$$\therefore 4\cot\alpha\cot\beta=-9$$

d)



This is the definition of a Hyperbola $= 2a$

ii) Foci $(5,0)$ $(-5,0)$

$2ae=10$ $\quad a=2$ $\quad \therefore e=\frac{5}{2}$ $\quad b^2=21$

$b^2=a^2(e^2-1)$

$b^2=4\left(\frac{25}{4}-1\right)$

$\frac{b^2}{4}=\frac{21}{4}$

$$\frac{x^2}{4}-\frac{y^2}{21}=1$$

Same shape rotated & show shift

$x=5$

$$\frac{25}{4}-\frac{y^2}{21}=1$$

$$\frac{21}{4}=\frac{y^2}{21}$$

$$\frac{21^2}{4}=y^2$$

$$\pm\frac{21}{2}$$

$\therefore$ latus $=\frac{42}{2}$

$=21$?