THE SCOTS COLLEGE

YEAR 12 EXTENSION 1 MATHEMATICS

ASSESSMENT TASK 3

TUESDAY, 10TH JUNE 2008

Weighting: 20%

Time Allowed: 45 minutes

Instructions to Students:

- Attempt all three questions
- Start a new page for every question
- Board approved calculators may be used
- All working out must be shown
- A standard table of integrals is attached

Outcomes:

HE2 Uses inductive reasoning in construction of proofs

HE3 Uses a variety of strategies to investigate mathematical models of situations involving binomial probability, projectiles, simple harmonic motion or exponential growth and decay

HE5 Applies the chain rule to problems including those involving velocity and acceleration as functions of displacement

Question 1

(a) [4 marks]

Show by mathematical induction that, if n is an integer and $n > 1$, $7^n - 6n - 1$ is divisible by 36.

(b) [4 marks]

Prove by mathematical induction, for all positive integers n:

$$1\times 2 + 2\times 3 + 3\times 4 + \dots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$$

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Question 2

(a)

A particle moves so that $\ddot{x} = \dfrac{1}{36 + x^2}$ and is initially at rest at $x = 0$.

(i) Find v^2 as a function of x. [2 marks]

(ii) Explain why v is always positive for $t > 0$. [1 mark]

(iii) Find the velocity at $x = 6$. [2 marks]

(iv) Find the limiting velocity of the particle. [1 mark]

(b)

A projectile is fired from a point P on horizontal ground with initial speed $50m/s$ at an angle of elevation to the ground of θ, where $\tan\theta = \frac{4}{3}$. (Take $g = 10m/s^2$)

(i) Prove that $x = 30t$ and $y = 40t - 5t^2$ **[3 marks]**

(ii) Find the time of flight **[2 marks]**

(iii) Find the greatest height attained **[2 marks]**

(iv) Find the position of the particle after 3 seconds **[1 mark]**

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Question 3

(a)

A particle, moving in a straight line, has after t seconds a position given by $x = 2\cos(3t + \frac{\pi}{6})$, where x is in metres.

(i) Prove that the motion is simple harmonic **[2 marks]**

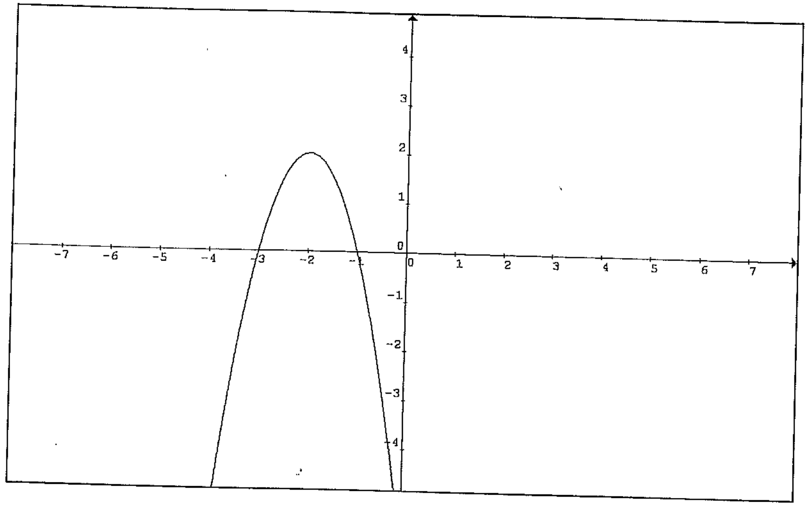
(ii) Find the amplitude and the period of the motion **[1 mark]**

(iii) What was the particle's initial position and in which direction did it first move? **[3 marks]**

(iv) What time elapsed before the particle was next at its initial position? **[2 marks]**

(b)

The graph of $v^2 = f(x)$ representing a particle moving in simple harmonic motion is given below. v m/s represents the velocity of the particle with respect to x m, where x is the displacement of the particle.



Find:

(i) between which two points the particle oscillates **[1 mark]**

(ii) where maximum acceleration occurs **[1 mark]**

(iii) the centre of motion **[1 mark]**

(iv) the amplitude **[1 mark]**

(v) the period **[1 mark]**

Solutions
Ext 1 Mathematics, Task 3, June 2008

Question 1

(a)
Step 1: Need to prove that $n = 2$ is true

$7^2 - 6(2) - 1 = 49 - 12 - 1 = 36$ which is divisible by 36
$\therefore$ $n = 2$ is true

Step 2: Assume that $n = k$ is true ie $7^k - 6k - 1 = 36M$, where M is any integer

Need to prove that $n = k+1$ is also true ie $7^{k+1} - 6(k+1) - 1$ is divisible by 36

$$7^{k+1} - 6(k+1) - 1 = 7^k.7 - 6k - 6 - 1$$
$$= 7(7^k - 6k - 1) + 6.6k$$
$$= 7.36M + 36k$$
$$= 36(7M + k) \text{ which is divisible by 36}$$

$\therefore$ $n = k+1$ is true

Step 3: Since $n = 2$, $n = k$ and $n = k+1$ are all true, then $n = 3$, $n = 4$ ...are all true
$\therefore$ $7^n - 6n - 1$ is divisible by 36 for $n > 1$.

(b)
Step 1: Need to prove that $n = 1$ is true

LHS $= 1 \times 2 = 2$

$$\text{RHS} = \frac{1}{3} \times 1 \times (1+1)(1+2)$$
$$= 2$$
$$= \text{LHS}$$

$\therefore$ $n = 1$ is true

Step 2: Assume that $n = k$ is true ie $1 \times 2 + 2 \times 3 + ...k(k+1) = \frac{1}{3}k(k+1)(k+2)$

Need to prove that $n = k+1$ is also true ie $1 \times 2 + 2 \times 3 + ...k(k+1) + (k+1)(k+2) = \frac{1}{3}(k+1)(k+2)(k+3)$

$$LHS = 1 \times 2 + 2 \times 3 + ...k(k+1) + (k+1)(k+2)$$
$$= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$$
$$= \frac{1}{3}(k+1)(k+2)(k+3)$$
$$= \text{RHS}$$

$\therefore$ $n = k+1$ is also true

Step 3: Since $n = 1$, $n = k$ and $n = k+1$ are all true, then $n = 2$, $n = 3$...are all true

$\therefore$ $1 \times 2 + 2 \times 3 + ...n(n+1) = \frac{1}{3}n(n+1)(n+2)$ for all integers

Question 2

(a)

(i)
$$\ddot{x} = \frac{1}{36 + x^2}$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{36 + x^2}$$
$$\frac{1}{2}v^2 = \int \frac{1}{36 + x^2}\,dx$$
$$\frac{1}{2}v^2 = \frac{1}{6}\tan^{-1}\frac{x}{6} + c$$

When $t = 0, x = 0, v = 0 \therefore c = 0$

$$\therefore v^2 = \frac{1}{3}\tan^{-1}\frac{x}{6}$$

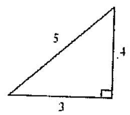
(ii) velocity is always positive since the acceleration is always positive and it starts from rest at $x = 0$.

(iii) sub $x = 6$
$$v^2 = \frac{1}{3}\tan^{-1}\frac{6}{6}$$
$$v^2 = \frac{1}{3} \times \frac{\pi}{4}$$
$$v^2 = \frac{\pi}{12}$$
$$\therefore v = \sqrt{\frac{\pi}{12}}$$

(iv) as $x \to \infty$, $\tan^{-1}\frac{x}{6} \to \frac{\pi}{2}$

$\therefore v^2 \to \frac{\pi}{6}$ $\qquad$ $\therefore v \to \sqrt{\frac{\pi}{6}}$

(b)
(i) initially,



$\dot{x} = V\cos\theta = 50\times\frac{3}{5} = 30$ and $\dot{y} = V\sin\theta = 50\times\frac{4}{5} = 40$

$\ddot{x} = 0$

$\dot{x} = 30$

$\ddot{y} = -10$

$\dot{y} = \int -10dt$

$\dot{y} = -10t + c_2$ when $t = 0, \dot{y} = 40 \therefore c_2 = 40$

$\dot{y} = 40 - 10t$

$x = \int 30\,dt$

$x = 30t + c_1$

when $t = 0, x = 0 \therefore c_1 = 0$

$\therefore x = 30t$

$y = \int 40 - 10t\,dt$

$y = 40t - 5t^2 + c_3$

when $t = 0, y = 0 \therefore c_3 = 0$

$\therefore y = 40t - 5t^2$

(ii) *sub* $y = 0$, $5t(t-8) = 0$

$\therefore t = 0$ or $t = 8$ time of flight was 8 seconds

(iii) greatest height occurs when $t = 4$

sub $t = 4$, $y = 40(4) - 5(4)^2 = 240$ greatest height was 240 metres

(iv) sub $t = 3$,

$x = 30(3) = 90$

$y = 40(3) - 5(3)^2 = 75$ the particle is $90m$ right and $75m$ up from its starting position

Question 3

(a)

(i) $x = 2\cos(3t + \frac{\pi}{6})$

$\dot{x} = -6\sin(3t + \frac{\pi}{6})$

$\ddot{x} = -18\cos(3t + \frac{\pi}{6})$

$\ddot{x} = -9x$ $\therefore$ the motion is simple harmonic

(ii)

Amplitude = 2 period = $\frac{2\pi}{3}$

(iii)

sub $t = 0$,

$x = 3\cos\frac{\pi}{6} = \sqrt{3}$ $\dot{x} = -6\sin\frac{\pi}{6} = -3$

$\therefore$ initially the particle was $\sqrt{3}m$ to the right of the origin moving in a negative direction with a speed of $3m/s$.

(iv)

$2\cos(3t + \frac{\pi}{6}) = \sqrt{3}$

$\cos(3t + \frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$3t + \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$

$3t = 0, \frac{2\pi}{3}, \ldots$

$\therefore t = 0, \frac{2\pi}{9}, \ldots$

$\therefore$ the particle returns to its original position after $\frac{2\pi}{9}$ seconds.

(b)

(i) where $v = 0$, $\therefore x = -3$ and $x = -1$

(ii) maximum acceleration occurs at $x = -3$ and $x = -1$

(iii) centre of motion, $x = -2$

(iv) amplitude = 1

(v) $v^2 = -2(x^2 + 4x + 3)$

$a = \frac{d}{dx}(\frac{1}{2}v^2)$

$a = \frac{d}{dx}(-(x^2 + 4x + 3))$

$a = -(2x + 4)$

$a = -2(x + 2)$,

$\therefore n = \sqrt{2}$ $\therefore$ period $= \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$