NAME: ____________________

TEACHER: ____________________

THE SCOTS COLLEGE

YEAR 12

TRIAL HSC EXAMINATION 2007

MATHEMATICS

EXTENSION 1

TIME ALLOWED: **TWO HOURS**
[plus 5 minutes reading time]

INSTRUCTIONS:

- Attempt **ALL** questions.
- **ALL** questions are of equal value.
- **ALL** necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are attached.
- Board approved calculators **ONLY** may be used.
- Answer each question in a **SEPARATE** Writing Booklet.
- Additional Writing Booklets are available if you require them.

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\, x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

QUESTION 1 [12 MARKS]

(a) The interval CD has end points C(−2, 3) and D (10,11). Find the co-ordinates of the point P which divides the interval CD internally in the ratio of 3:1. [2]

(b) Solve the inequality $\frac{1}{x^2-1}<0$ algebraically and show your solution on a number line. [2]

(c) The graphs $y=x$ and $y=x^5$ intersect at $x=1$. Find the size of the acute angle between these curves at this point of intersection (to the nearest degree). [3]

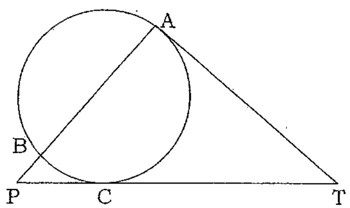
(d) Find the exact value of $\int_0^{\sqrt{5}} \frac{1}{\sqrt{5-x^2}}dx$ [2]

(e) By using the value of $x=0.5$ as a first approximation, find a root of the equation $x+\ln x=0$ using Newton's method to find a second approximation. Give your answer correct to 2 decimal places. [3]

QUESTION 2 [12 MARKS] **START A NEW WRITING BOOKLET**

(a) Prove the identify $\frac{\sin A}{\sin A+\cos A}-\frac{\sin A}{\sin A-\cos A}=\tan 2A$. [3]

(b) [5]



AB is the diameter of a circle ABC. The tangents at A and C meet at T. The lines TC and AB are produced to meet at P.

(i) Copy the diagram into your examination booklet. Join AC and CB.

(ii) Prove that $\angle CAT=90^\circ-\angle BCP$

(iii) Hence, or otherwise, prove that $\angle ATC=2\angle BCP$

(c) **(i)** State the domain and range of the function $y=\cos^{-1}\frac{x}{2}$

(ii) Sketch the graph of the function given by $y=\cos^{-1}\frac{x}{2}$

(iii) Find the equation of the tangent to the curve at the point where it cuts the y axis. [4]

QUESTION 3 [12 MARKS] **START A NEW WRITING BOOKLET**

(a) Consider the circle with equation $x^2+y^2-2x-14y+25=0$. [6]

(i) Determine the co-ordinates of its centre and find the length of the radius.

(ii) Show that if the line $y=kx$ intersects the circle at two distinct points, then:

$$(1+7k)^2>25(1+k^2)$$

(iii) Find all values of k for which the line $y=kx$ is a tangent to the circle.

(b) Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 3x\,dx$. [2]

(c) It is known that the polynomial $p(x)=x^3+ax^2+bx+c$ has a relative minimum at $x=\beta$ and a relative maximum at $x=\alpha$. [4]

(i) Prove $\alpha+\beta=-\frac{2}{3}a$

(ii) Show that the point of inflexion occurs at $x=\frac{\alpha+\beta}{2}$

QUESTION 4 [12 MARKS] **START A NEW WRITING BOOKLET**

(a) Using the substitution $u=2x+1$, or otherwise, find the exact value of $\int_0^1 \frac{4x}{2x+1}dx$ [3]

(b) The function $f(x)$ is given by $f(x)=\sin^{-1}x+\cos^{-1}x,\ 0\le x\le 1$. [3]

(i) Find $f'(x)$

(ii) Sketch the graph $y=f(x)$

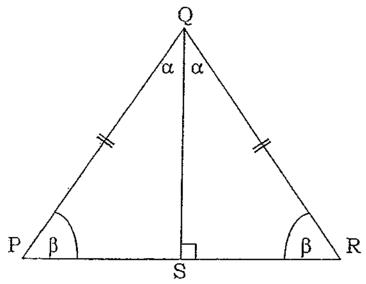
(c) **(i)** By equating the coefficients of $\sin x$ and $\cos x$, or otherwise, find the values of the constants A and B which satisfy the identity [6]

$$A(2\sin x+\cos x)+B(2\cos x-\sin x)\equiv \sin x+8\cos x$$

(ii) Using **(i)**, find $\int \frac{\sin x+8\cos x}{2\sin x+\cos x}dx$

QUESTION 5 [12 MARKS] START A NEW WRITING BOOKLET

(a) [5]



The triangle PQR is isosceles with PQ = RQ and QS is perpendicular to PR.

Let $\angle PQS = \angle RQS = \alpha$ and $\angle QPS = \angle QRS = \beta$

(i) Show that $\cos\alpha = \sin\beta$.

(ii) Using the sine rule for triangle PQR, show that $\sin 2\alpha = 2\sin\alpha\cos\alpha$.

(iii) Given that $0 < \alpha < \dfrac{\pi}{2}$, show that the limiting sum of the geometric series $\sin 2\alpha + \sin 2\alpha\cos^2\alpha + \sin 2\alpha\cos^4\alpha + \sin 2\alpha\cos^6\alpha + \dots\dots\dots$ is equal to $2\cot\alpha$.

(b) Find the value of the term that is independent of x in the expansion of $\left(x^2 - \dfrac{2}{x}\right)^6$. [3]

(c) Find all angles θ, where $0 \le \theta \le 2\pi$, for which $\sqrt{3}\cos\theta - \sin\theta = 1$. [4]

QUESTION 6 [12 MARKS] START A NEW WRITING BOOKLET

(a) Consider the parabola $x^2 = 4ay$, where $a > 0$, and let the tangents at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ intersect at the point T. Let the focus of the parabola be S, with co-ordinates $(0, a)$. [6]

(i) Show that the equation of the tangent at P is $y = px - ap^2$.

(ii) Find the co-ordinates of T.

(iii) Show that $SP = a(p^2 + 1)$.

(iv) Suppose that P and Q move on the parabola $x^2 = 4ay$ in such a manner that $SP + SQ = 4a$.

Show that the locus of the point T is a parabola and write down the co-ordinates of the vertex and the focus of this parabola.

(b) A particle is moving in simple harmonic motion about a fixed point 0. Its period is 4π seconds and its amplitude is 3cm.

Find its speed at the point 0. [2]

(c) Let T be the temperature inside a building at time t and let T_0 be the constant air temperature outside the building. Newton's law of cooling states that the rate of change of the temperature t is proportional to $T - T_0$. [4]

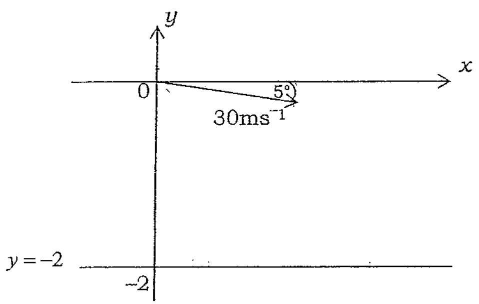
(i) Show that $T = T_0 + Ce^{kt}$ (where C and k are constants) satisfies Newton's law of cooling.

(ii) The outside air temperature is 5°C and a break down in the heating system causes the temperature inside the building to drop from 20°C to 17°C in half an hour. After how many hours, correct to 2 decimal places, is the temperature inside the building equal to 10°C?

QUESTION 7 [12 MARKS] **START A NEW WRITING BOOKLET**

(a) A particle moves along the x axis. Its velocity v at position x is given by $v=\sqrt{10x-x^2}$. Find the acceleration of the particle when $x=2$. [2]

(b) [6]



A tennis ball leaves the player's racquet 2 metres above the ground with a velocity of 30ms^{-1} at an angle of 5° below the horizontal. The equations of motion for the ball are $\ddot{x}=0$ and $\ddot{y}=-10$.

Take the origin to be the point where the ball leaves the player's racquet.

(i) Using calculus, show that the co-ordinates of the ball at time t are given by:
$$x=30t\cos 5^\circ$$
$$y=-30t\sin 5^\circ-5t^2$$

(ii) Find the time at which the ball strikes the ground, correct to 2 decimal places.

(iii) Calculate the angle, to the nearest degree, at which the ball strikes the ground.

(c) (i) For positive integers n and r, with $n > r$, show that [4]

$$^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1}$$

where $^nC_r=\dfrac{n!}{r!(n-r)!}$. Do **NOT** use induction.

(ii) Use mathematical induction to prove that, for $n\geq 3$,

$$\sum_{j=3}^{n}{}^{j-1}C_2={}^nC_3$$

END OF PAPER

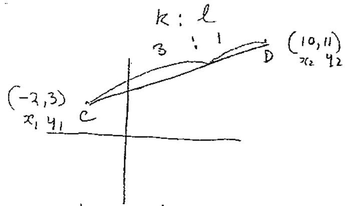
TRIAL SOLUTION. 2007

Y12 EXT 1

(1)

Q1

(a)



HE1

$x_p = \dfrac{kx_2 + lx_1}{k+l}$

$= \dfrac{3(10)+1(-2)}{3+1}$

$= \dfrac{28}{4}$

$= 7$

$y_p = \dfrac{ky_2 + ly_1}{k+l}$

$= \dfrac{3(11)+1(3)}{3+1}$

$= \dfrac{36}{4}$

$= 9.$

(1,1)

(2)

$\therefore$ P has co-ordinates (7, 9)

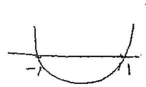
(b) $\dfrac{1}{x^2-1} < 0 \rightarrow \dfrac{1}{(x+1)(x-1)} < 0$

$\dfrac{(x+1)^2(x-1)^2}{(x+1)(x-1)} < 0 \left[(x+1)^2(x-1)^2\right]$ } 1

PE3

$(x+1)(x-1) < 0$

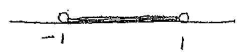
$\therefore \{-1 < x < 1\}$



} 1

(3)

Graphical solution :



1

(c) l_1 ; $y = x$

gradient $m_1 = 1$

l_2 ; $y = x^5$

$y' = 5x^4$

$\therefore m_2 = 5$ at $x = 1.$ } 1

acute angle $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$

HE1

$= \dfrac{5-1}{1+5(1)}$

$= \dfrac{4}{6}$

$= \dfrac{2}{3}$ } 1

(2)

$\theta = \tan^{-1}\left(\dfrac{2}{3}\right)$

$= 33.69$

$= 34°$ (nearest minute) } 1

(2)

(d) $\displaystyle\int_0^{\sqrt{5}} \frac{1}{\sqrt{(\sqrt{5})^2 - x^2}}$

$= \left[\sin^{-1}\dfrac{x}{\sqrt{5}}\right]_0^{\sqrt{5}}$ } 1

HE4

$= \sin^{-1}\dfrac{\sqrt{5}}{\sqrt{5}} - \sin^{-1}\dfrac{0}{\sqrt{5}}$

$= \sin^{-1}1 - \sin^{-1}0$

$= \dfrac{\pi}{2}$ } 1

(2)

(e) Let $f(x) = x + \ln x$

$f'(x) = 1 + \dfrac{1}{x}$

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ } 1

using $x_1 = 0.5$

PE1

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 0.5 - \dfrac{0.5 + \ln 0.5}{1 + \frac{1}{0.5}}$

$= 0.5 - \dfrac{0.5 + \ln(0.5)}{3}$

$= 0.5643.$

$= 0.56$ (2dp) } 1

(2)

③

Q2

(a) $\dfrac{\sin A}{\sin A+\cos A}-\dfrac{\sin A}{\sin A-\cos A}$

$=\sin A\left(\dfrac{(\sin A-\cos A)-(\sin A+\cos A)}{\sin^2 A-\cos^2 A}\right)$ 1

$=\dfrac{\sin A(-2\cos A)}{\sin^2 A-\cos^2 A}$

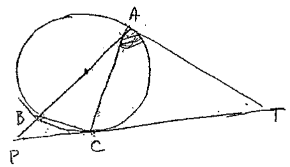
PE2 $=\dfrac{2\sin A\cos A}{\cos^2 A-\sin^2 A}$ 1

$=\dfrac{\sin 2A}{\cos 2A}$

$=\tan 2A$ 1

③

(b) (i)



1 ①

(ii) $\angle BCP=\angle BAC$ (alt. segment theorem)

PE3 $\angle BAT=90^\circ$ (angle between radius/tangent)

$\angle BAT=\angle BAC+\angle CAT=\angle BCP+\angle CAT$ 1

$\angle CAT=\angle BAT-\angle BCP$

$\angle CAT=90^\circ-\angle BCP$ 1

②

(iii) $\angle ACT=\angle CAT$ (base angles of tangents from an ext. pt)

$\angle ATC+\angle ACT+\angle CAT=180^\circ$ (angle sum of Δ)

$\angle ATC+(90^\circ-\angle BCP)+(90^\circ-\angle BCP)=180^\circ$

$\angle ATC-2\angle BCP=0$ ①

$\angle ATC=2\angle BCP$

④

④

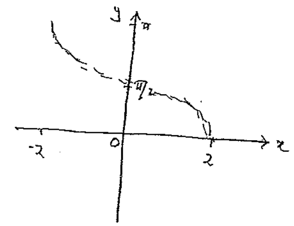
(c) (i) $y=\cos^{-1}\dfrac{x}{2}$

$D:\{-1\le \frac{x}{2}\le 1\}$

$D=\{-2\le x\le 2\}$

$R:\{0\le y\le \pi\}$ ②

(ii)



①

(iii) The curve cuts the y axis at $x=0$.

$y=\cos^{-1}\dfrac{x}{2}$

HE5 $y'=-\dfrac{1}{\sqrt{1-(\frac{x}{2})^2}}\times\dfrac{1}{2}$

$=-\dfrac{1}{2\sqrt{1-0}}$

$=-\dfrac{1}{2}$ 1

5

eq of tangent $y-y_1=m(x-x_1)$

eq cuts y axis at $(0,\frac{\pi}{2})$

$\therefore$ eq: $y-\frac{\pi}{2}=-\frac{1}{2}(x-0)$

$y=-\frac{1}{2}x+\pi/2$ 1

(5)

Q3

(a) (i) $x^2 - 2x + y^2 - 14y = -25$

$x^2 - 2x + 1 + y^2 - 14y + 49 = -25 + 1 + 49$

$(x-1)^2 + (y-7)^2 = 25$ } 1

$\therefore$ centre $(1, 7)$ and radius $= 5$ } 1

[2]

(ii) Solving $y = kx$ and $x^2 - 2x + y^2 - 14y + 25 = 0$.

$x^2 - 2x + (kx)^2 - 14kx + 25 = 0$

$x^2 + k^2x^2 - x(2 + 14k) + 25 = 0$

$(1+k^2)x^2 - (2+14k)x + 25 = 0$ } 1

Two distinct real roots if $\Delta > 0$

[2]

$\Delta = [2(1+7k)]^2 - 4(1+k^2)25 > 0$

$4(1+7k)^2 - 4.25(1+k^2) > 0$

$(1+7k)^2 - 25(1+k^2) > 0$

$(1+7k)^2 > 25(1+k^2)$ } 1

HE1

(iii) The line $y = kx$ is tangential if $(1+7k)^2 = 25(1+k^2)$, $\Delta = 0$

$(1+7k)^2 = 25 + 25k^2$

$1 + 14k + 49k^2 = 25 + 25k^2$

$24k^2 + 14k - 24 = 0$

$12k^2 + 7k - 12 = 0$

$(4k-3)(3k+4) = 0$

$\therefore k = \frac{3}{4}, -\frac{4}{3}$ } [2]

(6)

(b) $\int_0^{\pi/2} \sin^2 3x \, dx$

$= \int_0^{\pi/2} \frac{1}{2}(1 - \cos 6x)\,dx$

$= \frac{1}{2}\left[x - \frac{\sin 6x}{6}\right]_0^{\pi/2}$

$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin 3\pi}{6}\right) - 0\right]$ } 1

HE6

$= \frac{1}{2}\left[\left(\frac{\pi}{2} - 0\right) - 0\right]$

$= \frac{1}{2}\left[\frac{\pi}{2}\right]$

$= \frac{\pi}{4}$ } 1

(2)

(6)

Q3 (c) (i) $p(x) = x^3 + ax^2 + bx + c$.

$p'(x) = 3x^2 + 2ax + b$

turning points at $x = \alpha, \beta$ when $p'(x) = 0$

$p'(\alpha) = 3\alpha^2 + 2a\alpha + b = 0$

$p'(\beta) = 3\beta^2 + 2a\beta + b = 0$

$\therefore 3\alpha^2 + 2a\alpha + b = 3\beta^2 + 2a\beta + b$

$3\alpha^2 - 3\beta^2 = 2a\beta - 2a\alpha$

$3(\alpha-\beta)(\alpha+\beta) = 2a(\beta-\alpha)$

$3(\alpha+\beta) = -2a$.

$\alpha + \beta = -\frac{2}{3}a$

(2)

HE1

(4)

(ii) Inflexion occurs when $p''(x) = 0$

$p'' = 6x + 2a = 0$.

$\therefore 2a = -6x$

$a = -3x$.

(2)

$\therefore \alpha + \beta = -\frac{2}{3} a = -\frac{2}{3} \cdot (-3x) = 2x$. using (i)

$\therefore 2x = \alpha + \beta$

$x = \frac{\alpha+\beta}{2}$

(7)

Q.4

(a) $u = 2x+1$ $\Rightarrow$ $2x = u-1$

$du = 2dx$ $\qquad 4x = 2(u-1)$

at $x=0,\ u=1$ $\qquad$ and $dx = \frac{1}{2}du$

at $x=1,\ u=3$

$$\int_0^1 \frac{4x\,dx}{2x+1}$$

$$= \int_1^3 \frac{2(u-1)}{u}\cdot\frac{1}{2}\,du$$

$$= \int_1^3 \left(\frac{u-1}{u}\right) du$$

$$= \int_1^3 \left(1-\frac{1}{u}\right) du \qquad 1$$

HE6 $\qquad$ (3)

$$= \Big(u - \ln u\Big]_1^3$$

$$= (3-\ln 3) - (1-\ln 1) \qquad 1$$

$$= 2 - \ln 3 + 0.$$

$$= 2 - \ln 3.$$

Alternative

$$\int_0^1 \frac{4x}{2x+1}\,dx$$

$$= \int_0^1 \frac{4x+2}{2x+1} - \frac{2}{2x+1}\,dx$$

$$= \int_0^1 2 - \frac{2}{2x+1}\,dx$$

$$= \int_0^1 2 - \frac{d(2x+1)}{2x+1}\,dx.$$

$$= \Big[2x - \ln(2x+1)\Big]_0^1$$

$$= (2-\ln 3) - 0$$

$$= 2 - \ln 3$$

(b) (i) $f(x) = \sin^{-1} x + \cos^{-1} x$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \qquad 1$$

$$= 0$$

(ii) Since $f'(x) = 0$, $f(x)$ is a constant

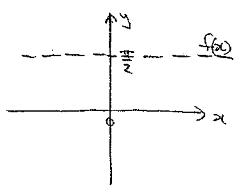
at $x = 0$, $\quad f(x) = \sin^{-1} 0 + \cos^{-1} 0$

$$= \frac{\pi}{2} + 0 \qquad 1$$

$$f(x) = \frac{\pi}{2}$$

(3)

HE4



1

(8)

4(c)

(i) $A(2\sin x + \cos x) + B(2\cos x - \sin x) \equiv \sin x + 8\cos x$

$2A\sin x + A\cos x + 2B\cos x - B\sin x$ $\qquad 1$

$= (2A-B)\sin x + (A+2B)\cos x$

$\therefore\ 2A - B = 1 \quad \Rightarrow \quad 4A - 2B = 2$

$A + 2B = 8 \qquad\qquad A + 2B = 8$

$\qquad\qquad 5A = 10 \qquad 1$

$\qquad\qquad A = 2$

(3)

$2A - B = 1$

$2(2) - B = 1$

$4 - 1 = B$ $\qquad 1$

$\therefore B = 3$ and $A = 2$.

Alternative

$A(2\sin x + \cos x) + B(2\cos x - \sin x) \equiv \sin x + 8\cos x.$

Let $x = 0$, $\quad A(0+1) + B(2-0) = 8 \quad \to A + 2B = 8$

Let $x = \pi/2$, $\quad A(2+0) + B(0-1) = 1 \quad \to 2A - B = 1$

$\therefore A = 2,\ B = 3$ (see above)

(6)

(ii) $\sin x + 8\cos x = 2(2\sin x + \cos x) + 3(2\cos x - \sin x)$ from (i)

$$\therefore \frac{\sin x + 8\cos x}{2\sin x + \cos x} = \frac{2(2\sin x + \cos x)}{2\sin x + \cos x} + \frac{3(2\cos x - \sin x)}{2\sin x + \cos x}. \qquad 1$$

$$\therefore \int \frac{\sin x + 8\cos x}{2\sin x + \cos x}\,dx = \int \left[2 + 3\left(\frac{2\cos x - \sin x}{2\sin x + \cos x}\right)\right] dx$$

$$= \int \left[2 + 3\,\frac{d(2\sin x + \cos x)}{2\sin x + \cos x}\right] dx \qquad 1$$

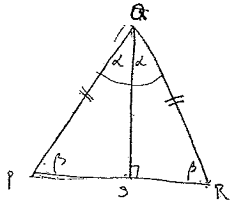
(3)

HE1

$$= 2x + 3\ln(2\sin x + \cos x) + C \qquad 1$$

Q5

(a)



(i) $\frac{QS}{QP} = \sin\beta = \cos\alpha$ 1

(ii) $\frac{PR}{\sin 2\alpha} = \frac{QP}{\sin\beta}$

$\frac{2PS}{\sin 2\alpha} = \frac{QP}{\sin\beta}$

$\frac{2PS}{QP} = \frac{\sin 2\alpha}{\sin\beta}$

$2\sin\alpha = \frac{\sin 2\alpha}{\sin\beta}$

$\sin 2\alpha = 2\sin\alpha\sin\beta$

$\therefore \sin 2\alpha = 2\sin\alpha\cos\alpha$ 2

(iii) $\sin 2\alpha + \sin 2\alpha\cos^2\alpha + \sin 2\alpha\cos^4\alpha + \ldots$

$= \sin 2\alpha\,(1 + \cos^2\alpha + \cos^4\alpha + \ldots)$

$= \sin 2\alpha \left(\frac{1}{1-\cos^2\alpha}\right)$ using $S_\infty = \frac{a}{1-r}$

$= \sin 2\alpha \cdot \frac{1}{\sin^2\alpha}$

$= \frac{2\sin\alpha\cos\alpha}{\sin^2\alpha}$

$= \frac{2\cos\alpha}{\sin\alpha}$

$= 2\cot\alpha$ 2

PE2

5

5 (b) $\left(x^2 - \frac{2}{x}\right)^6 = \ldots + {}^6C_r (x^2)^{6-r}\left(\frac{2}{x}\right)^r + \ldots$ 1

$= \ldots + {}^6C_r (2^r)(x^2)^{6-r}(x^{-r}) + \ldots$

$= \ldots + {}^6C_r\, 2^r\, x^{12-2r} \cdot x^{-r}$

$= \ldots + {}^6C_r\, 2^r\, x^{12-3r} + \ldots$

The term is independent of x for x^0 ie $12 - 3r = 0$

$\therefore r = 4$ 1

$\therefore$ required term ${}^6C_4 \cdot 2^4$

$= \frac{6!}{4!\,2!} \cdot 16$

$= \frac{6.5.4!}{4!\,2!} \cdot 16$

$= 240$ 1

HE3

3

(c) $\sqrt{3}\cos\theta - \sin\theta = 1.$

$\therefore \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = \frac{1}{2}.$ 1

$\sin\alpha\cos\theta - \cos\alpha\sin\theta = \frac{1}{2}$ for $\sin\alpha = \frac{\sqrt{3}}{2}$, $\cos\alpha = \frac{1}{2}$ $\Rightarrow \alpha = \frac{\pi}{3}$ 1

$\sin(\alpha - \theta) = \frac{1}{2}.$

$\frac{\pi}{3} - \theta = \sin^{-1}\frac{1}{2}.$ 1

$\therefore \frac{\pi}{3} - \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\theta = \frac{\pi}{3} - \frac{\pi}{6}, \frac{\pi}{3} - \frac{5\pi}{6}$

$= \frac{\pi}{6}, -\frac{3\pi}{6}$

$= \frac{\pi}{6}, -\frac{\pi}{2}.$

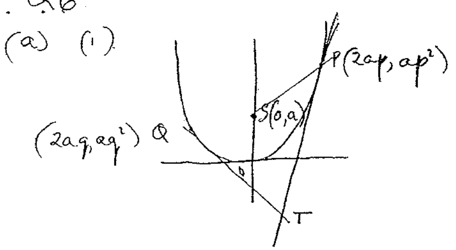
$= \frac{\pi}{6}, \frac{3\pi}{2}$ 1

HE7

4

Q6

(a) (i)



gradient PT : $y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a}$

$= \frac{2.2ap}{4a}$

$= p$

eq PT : $y - ap^2 = p(x - 2ap)$

$y = px - 2ap^2 + ap^2$

$y = px - ap^2$ 1

(ii) eq. QT : $y = qx - aq^2$

Pt. intersection : $y = px - ap^2$

$y = qx - aq^2$

$0 = (p-q)x - a(p^2 - q^2)$

$\therefore (p-q)x = a(p^2 - q^2)$

$\cancel{(p-q)}x = a\cancel{(p-q)}(p+q)$

$x = a(p+q)$

$y = q[a(p+q)] - aq^2$

$= apq + aq^2 - aq^2$

$y = apq$ 2

PE4

Co-ords of T ; $(a(p+q), apq)$

(iii) $SP^2 = (2ap - 0)^2 + (ap^2 - a)^2$

$= 4a^2p^2 + a^2(p^2-1)^2$

$= 4a^2p^2 + a^2(p^4 - 2p^2 + 1)$

$= 4a^2p^2 + a^2p^4 - 2ap^2 + a^2$

$= a^2(p^4 + 2p^2 + 1)$

$= a^2(p^2+1)^2$

$\therefore SP = a(p^2+1)$ 1

(iv) From (iii) $SQ = a(q^2+1)$

$SP + SQ = a(p^2+1) + a(q^2+1)$

$= a(p^2 + q^2 + 2)$

$= 4a$

$\therefore p^2 + q^2 + 2 = 4$

$p^2 + q^2 - 2 = 0 \rightarrow p^2 + q^2 = 2$

From (ii) $y = apq$ $x = a(p+q)$

$pq = \frac{y}{a}$ $p + q = \frac{x}{a}$

$(p+q)^2 = \frac{x^2}{a^2}$

$p^2 + q^2 + 2pq = \frac{x^2}{a^2}$

PE4 $\therefore 2 + 2\left(\frac{y}{a}\right) = \frac{x^2}{a^2}$

$2a^2 + 2ay = x^2$

$x^2 = 4a\left(\frac{y}{2} + a\right)$ which is the equation of a parabola 2

6

(b). For SHM. $x = a\cos(nt + \alpha)$

$x = 3\cos\left(\frac{1}{2}t + \alpha\right)$

$T = \frac{2\pi}{n}$ $a = 3$

$4\pi = \frac{2\pi}{n}$

$\therefore n = \frac{1}{2}$

Let O have displacement $x = 0$

Initially at $t = 0$, $x = 0$

$0 = 3\cos(0 + \alpha)$

$\cos\alpha = 0$. $\therefore \alpha = \frac{\pi}{2}$

HE3

$x = 3\cos\left(\frac{1}{2}t + \frac{\pi}{2}\right)$

$\dot{x} = -3\sin\left(\frac{1}{2}t + \frac{\pi}{2}\right)\frac{1}{2}$

$= -\frac{3}{2}\sin\left(\frac{1}{2}t + \frac{\pi}{2}\right)$

at $x = 0$, $\dot{x} = -\frac{3}{2}\sin\left(\frac{1}{2}\cdot 0 + \frac{\pi}{2}\right)$

$= -\frac{3}{2}\sin\frac{\pi}{2}$

$= -\frac{3}{2}$ cm s^{-1}

Speed is $\frac{3}{2}$ cm s^{-1}

Alternative

$T = \frac{2\pi}{n}$ $\therefore 4\pi = \frac{2\pi}{n} \rightarrow n = \frac{1}{2}$

$a = 3$

$v^2 = n^2(a^2 - x^2)$

$v^2 = \frac{1}{2}^2(3^2 - 0)$ at $x = 0$.

$v^2 = \left(\frac{3}{2}\right)^2$

$v = \pm\frac{3}{2}$ ms^{-1}

$\therefore$ speed $= \frac{3}{2}$ ms^{-1} about $x = 0$

Q6(c)

(i) $T = T_0 + Ce^{kt} \Rightarrow T - T_0 = Ce^{kt}$.

$\frac{dT}{dt} = kCe^{kt}$

$= k(T - T_0)$

$\therefore \frac{dT}{dt} \propto T - T_0$ satisfying the differential equation

(ii) Initially $t=0$, $T_0 = 5$, $T=20$

$20 = 5 + Ce^{k(0)} = 5 + C$

$\therefore C = 15$

In general $T = 5 + 15e^{kt}$.

at $T = 17$, $t = 0.5h$.

$17 = 5 + 15e^{\frac{1}{2}k}$.

$e^{\frac{k}{2}} = \frac{12}{15} = 0.8$

$\frac{k}{2} = \ln 0.8$

$\boxed{HE3}$

$k = 2\ln(0.8)$

$= -0.44629\ldots$ (4 dp)

$\textcircled{4}$

at $T = 10$, $T = 5 + 15e^{kt}$

$10 = 5 + 15e^{-0.4463t}$

$\frac{5}{15} = e^{-0.4463t}$

$-0.4463t = \ln\left(\frac{1}{3}\right)$

$t = \frac{\ln\left(\frac{1}{3}\right)}{-0.4463}$

$= 2.4616$

$= 2.46$ (2 dp)

Q7.

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(10x - x^2\right)^{1/2}$

$= \frac{1}{2}\left(10x - x^2\right)^{-\frac{1}{2}}(10 - 2x)$.

$= \frac{(10 - 2x)}{2\sqrt{10x - x^2}}$.

$= \frac{10 - 4}{2\sqrt{20 - 4}}$ at $x = 2$

$\textcircled{2}$

$\boxed{HE5}$

$= \frac{6}{2(\sqrt{16})}$

$= \frac{3}{4}$

$= 0.75\ \text{ms}^{-2}$

(b) (i) ${}^nC_r + {}^nC_{r+1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)![n-(r+1)]!}$

$= \frac{n!(r+1)}{(r+1)r!(n-r)!} + \frac{n!}{(r+1)![n-(r+1)]!}$

$= \frac{n!(r+1)}{(r+1)!(n-r)!} + \frac{n!}{(r+1)!} \frac{(n-r)}{[n-(r+1)](n-r)}$

$= \frac{n!(r+1)}{(r+1)!(n-r)!} + \frac{n!(n-r)}{(r+1)!(n-r)!}$

$\boxed{HE3}$

$= \frac{n!}{(r+1)!(n-r)!}[\cancel{r+1} + n - \cancel{r}]$

$\textcircled{2}$

$= \frac{(n+1)n!}{(r+1)!(n-r)!}$

$= \frac{(n+1)!}{(r+1)![(n+1)-(r+1)]!}$

$= {}^{n+1}C_{r+1}$

Q 7(b)

(ii). To prove $\sum_{d=3}^{n} {}^{d-1}C_2 = {}^{n}C_3$

ie ${}^2C_2 + {}^3C_2 + {}^4C_2 + \dots {}^{n-1}C_2 = {}^nC_3$ for $n \geq 3$

at $n=3$, LHS $= {}^{3-1}C_2 = {}^2C_2 = 1$

RHS $= {}^3C_3 = 1$.

$\therefore$ true for $n=3$.

2

assume ${}^2C_2 + {}^3C_2 + {}^4C_2 + \dots {}^{n-1}C_2 = {}^nC_3$ is true for $n=k$, $k>3$

$\therefore$ ${}^2C_2 + {}^3C_2 + {}^4C_2 + \quad + {}^{k-1}C_2 = {}^kC_3$

For $n=k+1$

$${}^2C_2 + {}^3C_2 + {}^4C_2 + \dots + {}^{k-1}C_2 + {}^kC_2 = {}^kC_3 + {}^kC_2$$
$$= {}^kC_2 + {}^kC_3$$
$$= {}^{k+1}C_3 \quad \text{from (1)}$$

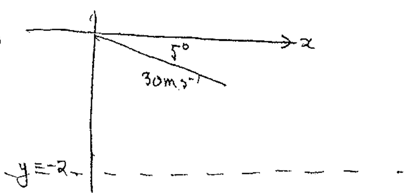
$\boxed{HE2}$

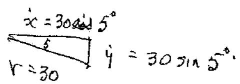
$\therefore$ ${}^2C_2 + {}^3C_2 + \dots {}^{k-1}C_2 = {}^kC_2$ is true for $n = k+1$

Since it is true for $n=3$, it is true for $n = 3+1 = 4$, since it is true for $n=4$, it is true for $n = 4+1 = 5$ and so on.

$\therefore$ $\sum_{d=3}^{n} {}^{d-1}C_2 = {}^nC_3$ is true for all $n \geq 3$.

(b) (i)





$\boxed{HE3}$

6

Initially $\ddot{x} = 0$

$\dot{x} = c$

at $t=0$, $\dot{x} = 30\cos 5° = c$

$x = \int 30\cos 5° \, dt$

$= 30t\cos 5°$.

(i) (cont) $\ddot{y} = -10$

$\dot{y} = \int -10 \, dt$

$\dot{y} = -10t + c$

at $t=0$, $\dot{y} = 30\sin 5°$ $\therefore$ $30\sin 5° = 0 + c$ $\rightarrow c = 30\sin 5°$.

$\dot{y} = -10t + 30\sin 5°$

$y = \int(-10t + 30\sin 5°)\,dt$.

$y = -5t^2 - 30t\sin 5° + c_2$

at $t=0$, $y=0$ $\therefore$ $c_2 = 0$

$y = -5t^2 - 30t\sin 5°$

(ii) The ball hits the ground at $y = -2$.

$\therefore$ $-2 = -5t^2 - 30t\sin 5°$

$5t^2 + (30\sin 5)t - 2 = 0$.

$$t = \frac{-30\sin 5° \pm \sqrt{30\sin 5 - 4.5(-2)}}{2(5)}$$

$$= \frac{-30\sin 5 \pm \sqrt{(30\sin 5)^2 + 40}}{10}$$

$$= \frac{-2.6146 \pm \sqrt{46.8365}}{10}$$

$$= \frac{-2.6146 + 6.8437}{10}, \quad t > 0.$$

$= 0.4229$

$= 0.423$ sec.

(iii)

at $t = 0.423$s, $\dot{x} = 30\cos 5° = 29.8858$

$\dot{y} = -10(0.423) + 30\sin 5°$

$= -6.8447$

$\tan\theta = \left|\frac{\dot{y}}{\dot{x}}\right| = \left|\frac{-6.8447}{29.8858}\right|$

$\theta = 13.23°$

$\approx 13°$ (nearest degree)