SCOTS COLLEGE – 2005 TRIAL HSC EXAM – EXT. 2

QUESTION 1 (15 marks)

Scots 2005 Ext 2 trial

a) Evaluate $|3+2i|$ 1

b) i) If $v=\frac{1+i\sqrt{3}}{2}$ show that $v^3=-1$. 2

ii) Hence calculate v^{10}. 2

c) If z is a complex number so that $|z|=2$ and $\arg z=\frac{\pi}{6}$, mark clearly on the same Argand diagram the points representing the complex numbers:

i) z ii) iz iii) $\bar{z}$ iv) $\frac{1}{z}$ v) $z\bar{z}$ vi) z^2 vii) z^2+z viii) z^2-z 10

QUESTION 2 (15 marks)

a) Find $\int \frac{dx}{x^2-6x+13}$ 2

b) Find $\int \tan x \sec^2 x\, dx$ 2

c) i) Show that $f(x)=\sin^{-1}x$ is an odd function. 2

ii) Hence or otherwise find $\int_{-1}^{1}(\sin^{-1}x)^3 dx$ 1

d) $\int_0^{\sqrt{2}}\sqrt{4-x^2}\, dx$ 4

e) $\int e^x \cos x\, dx$ 4

QUESTION 3 (10 marks)

a) Use the method of cylindrical shells to find the volume of the solid (paraboloid) obtained when the region between the curve $y=\frac{1}{2}\sqrt{x-2}$, the x-axis and the line $x=6$ is rotated about the x axis. 4

b) Prove $\frac{d^2x}{dt^2}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$, where x denotes displacement, and v denotes velocity. 2

c) The acceleration of a particle moving in a straight line is given by $\ddot{x}=xe^x$, where x is the displacement from 0. The particle is initially at rest. The particle starts at $x=0$.

i) Prove that $v^2=2e^x(x-1)+2$ 3

ii) Describe the subsequent motion of the particle after it leaves the origin and explain why the particle can only move in one direction 1

QUESTION 4 (18 marks)

a) The equation $x^3-x^2-3x+2=0$ has roots α,β,γ. Find the monic polynomial equation with roots $\alpha^2,\beta^2,\gamma^2$. 4

b) If $x=\alpha$ is a double root of the equation $P(x)=0$, show that $x=\alpha$ is a root of the equation $P'(x)=0$. 4

c) i) Show that $1+i$ is a root of the polynomial $Q(x)=x^3+x^2-4x+6$ 2

ii) hence resolve $Q(x)$ into irreducible factors over the complex number field. 3

d) If α,β,γ are the roots of the cubic equation $x^3+qx+r=0$, prove that $(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2=-6q$. 5

QUESTION 5 (18 marks)

The ellipse $\mathcal{E}$ has the cartesian equation $\frac{x^2}{25}+\frac{y^2}{16}=1$.

i) Write down the eccentricity **1**

ii) Write down the coordinates of the foci S and S' **1**

iii) Write down the equations of the directrices. **1**

iv) Sketch the ellipse $\mathcal{E}$. **1**

v) Show that any point P on $\mathcal{E}$ can be represented by the coordinates $(5\cos\theta, 4\sin\theta)$ **1**

vi) Prove that $PS + PS'$ is independent of the position of P on the ellipse $\mathcal{E}$. **3**

vii) Show that the equation of the normal N at the point P on the ellipse $\mathcal{E}$ is $5\sin\theta\, x - 4\cos\theta\, y = 9\sin\theta\cos\theta$ **2**

viii) If this normal meets the major axis of the ellipse in M and the minor axis in N, prove that $\frac{PM}{PN}=\frac{16}{25}$. **3**

ix) Also show that the line PN bisects the angle $S'PS$. **5**

QUESTION 6 (14 marks)

i) By considering the curve $g(x) = x^6 - 4x^5 + 4x^4$, sketch the graph of $f(x) = x^6 - 4x^5 + 4x^4 - 1$ showing that it has 4 real zeroes. **4**

On different diagrams sketch the curves:

ii) $y = |f(x)|$ **2**

iii) $y = f(|x|)$ **2**

iv) $y^2 = f(x)$ **3**

v) Calculate the slope of the curve $y^2 = f(x)$ at any point x and describe the nature of the curve at a zero of $f(x)$. **3**

QUESTION 7 (15 marks)

a) A parachutist of M kilograms is dropped from a stationary helicopter of height H metres above the ground. The parachutist experiences air resistance during its fall equal to MkV^2, where V is its velocity in metres per second and k is a positive constant. Let x be the distance in metres of the parachutist from the helicopter, measured positively as it falls.

i) Show that the equation of motion of the parachutist is $\ddot{x} = g - kV^2$, where g is the acceleration due to gravity. **1**

ii) Find V^2 as a function of x. **4**

iii) Find the velocity U of the parachutist as he hits the ground in terms of g, k and H. **1**

iv) Find the velocity of the parachutist as he hits the ground if air resistance is neglected. **2**

b)

i) Prove the identity $\cos 3A = 4\cos^3 A - 3\cos A$ **2**

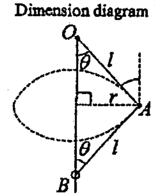
ii) Show that $x = 2\sqrt{2}\cos A$ is a root of the equation $x^3 - 6x + 2 = 0$ provided that $\cos 3A = -\frac{1}{2\sqrt{2}}$ **2**

iii) Find the three roots of the equation $x^3 - 6x + 2 = 0$, using the results from part (ii) above. Give your answer to three decimal places. **3**

QUESTION 8 (15 marks)

a) A particle A of mass $2m$ is attached by a light inextensible string of length l to a fixed point O and is also attached by another light inextensible string of the same length to a small ring B of mass $3m$ which can slide on a fixed smooth vertical wire passing through O. The particle A describes a horizontal circle of radius r, and OA is inclined at an angle $\theta = \frac{\pi}{3}$ with the downward vertical.

Dimension diagram



$\theta = \frac{\pi}{3}$

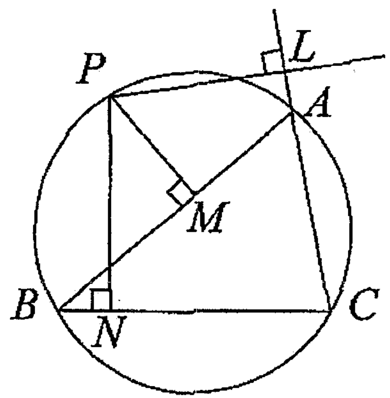
i) Find the tension in the strings OA and AB **5**

ii) Find the angular velocity of A. **3**

iii) Describe what happens to the system as the angular velocity increases. **1**

b) **ABC** is a triangle inscribed in the circle. ***P*** is a point on the minor arc ***AB***. The points ***L***, ***M***, and ***N*** are the feet of the perpendiculars from ***P*** to ***CA*** produced, ***AB***, and ***BC*** respectively.

Copy the diagram into your answer booklet and show that ***L***, ***M*** and ***N*** are collinear.



END OF EXAM

Q1 (a) $|3+2i|$

$= \sqrt{13}$

(b)(i) $v = \frac{1+i\sqrt{3}}{2}$ $\quad |v| = 1$, $\arg(z) = \pi/3$.

$v = \text{cis}\,\pi/3$

$v^3 = \text{cis}\,\pi$ (De Moivre's)

$= \cos\pi + i\sin\pi$

$= -1$

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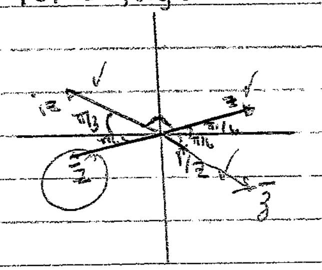
(ii) $v^{10} = \text{cis}\,\frac{10\pi}{3}$

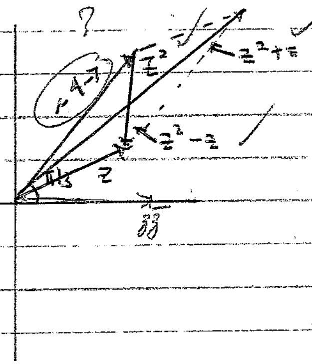
$= \text{cis}\,\frac{-2\pi}{3}$

$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

(c). $|z| = 2$, $\arg = \pi/6$



$z\bar{z} = |z|^2$
$= 4$



2.(a) $\int \frac{dx}{x^2-6x+13}$

$= \int \frac{dx}{x^2-6x+9+4}$

$= \int \frac{dx}{(x-3)^2+4}$

$= \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + c.$

(b). $\int \tan x \cdot \sec^2 x\, dx$

$= \frac{\tan^2 x}{2} + c.$

(c)(i) $f(x) = \sin^{-1}x$

$f(-x) = \sin^{-1}(-x)$

$= -\sin^{-1}(x)$

$= -f(x)$

∴ odd.

(ii) $\int_{-1}^{1} (\sin^{-1}x)^3\, dx$

$= 0.$

(d). $\int_0^{\sqrt{2}} \sqrt{4-x^2}\, dx$

Let $x = 2\sin\theta$ $\quad \frac{dx}{d\theta} = 2\cos\theta\, d\theta$

At $x = \sqrt{2}$, $\theta = \pi/4$.

$x = 0$, $\theta = 0$.

$I = \int_0^{\pi/4} \sqrt{4-4\sin^2\theta}\; 2\cos\theta\, d\theta$

$= \int_0^{\pi/4} 2\sqrt{(1-\sin^2\theta)} \cdot 2\cos\theta \cdot d\theta$

$= \int_0^{\pi/4} 4\cos^2\theta\, d\theta$

$\cos^2\theta = \frac{\cos 2\theta + 1}{2}.$

$I = \frac{1}{2}\int_0^{\pi/4} \cos 2\theta + 1\, d\theta$

$= \frac{1}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\pi/4}$

$= \frac{1}{2}\left[\frac{1}{2} + \pi/4\right]$

$= \frac{1}{4} + \pi/8.$

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(e). $\int e^x \cdot \cos x\, dx$

$u = \cos x \quad v = e^x$
$u' = -\sin x \quad v' = e^x$

$I = [e^x \cos x] + \int e^x \sin x\, dx$

$=$ $\quad u = \sin x \quad v = e^x$
$u' = \cos x \quad v = e^x$

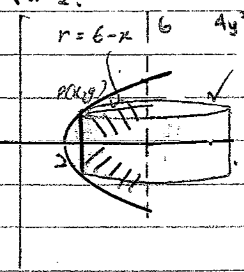
$I = [e^x\cos x] + [e^x \sin x] - \int e^x \cos x\, dx$

$2I = e^x[\cos x + \sin x]$

3. (a). $y = \frac{1}{2}\sqrt{x-2}$. $\quad 2y = \sqrt{x-2}$

$4y^2 = x-2$

$x = 2+4y^2$



$r = 6-2-4y^2$
$= 4-4y^2$

$V = 2\pi \int r h \cdot dy$ try again.

$= 2\pi \int \cancel{x}\, y \cdot \cancel{(2+4y^2)}\,(4-4y^2) \cdot dy$

$= 2\pi \int_2^6 4y^3 + 2y \cdot dy$

$= 2\pi[4y^4 + 2y^2]_2^6$

$= 2\pi[1312]$

$= 2624\pi.$

(b). $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}\frac{d}{dx}(v^2)$

LHS $= \frac{1}{2} 2v \cdot \frac{dv}{dx}$

$= \frac{2}{2} \cdot \frac{dx}{dt} \times \frac{dv}{dx}$

$= \frac{dv}{dt}$

$= \frac{d^2x}{dt^2}$

(c)(i) $\ddot{x} = xe^x$

$\frac{d}{dx}(\frac{1}{2}v^2) = xe^x$

$\frac{1}{2}v^2 = \int xe^x \, dx$

$u = x \quad v = e^x$

$u' = 1 \quad v' = e^x$

$\frac{1}{2}v^2 = [xe^x] - \int e^x$

$\frac{1}{2}v^2 = xe^x - e^x + c$

$v^2 = 2xe^x - 2e^x + c$

At $x = 0, v = 0$,

$v^2 = 2e^x(x-1)$

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(ii). Only can only move one direction since $e^x > 0$. as $x \to \infty$

Moves to the Right.

Q4.(a). $x^3 - x^2 - 3x + 2 = 0$.

For $\alpha^2, \beta^2, \gamma^2$ $\quad x^2 = y$

$x = \sqrt{y}$

$y\sqrt{y} - y - 3\sqrt{y} + 2 = 0$.

$\sqrt{y}(y-3) = y-2$

$y(y-3)^2 = (y-2)^2$

$y(y^2 - 6y + 9) = y^2 - 4y + 4$.

$y^3 - 6y^2 + 9y = y^2 - 4y + 4$.

$y^3 - 7y^2 + 13y - 4 = 0$.

$x^3 - 7x^2 + 13x - 4 = 0$

(b). $P(x) = Q(x)(x-\alpha)^2$

Let $u = Q(x) \qquad v = (x-\alpha)^2$

$u' = Q'(x) \qquad v' = 2(x-\alpha)$

$P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x)$

$= (x-\alpha)[2Q(x) + (x-\alpha)Q'(x)]$

$\therefore P(\alpha) = 0$

$\therefore$ $P'(x)$ has a root at $x = \alpha$.

(c)(i) $Q(x) = x^3 + x^2 - 4x + 6$

$Q(1+i) = (1+i)^3 + (1+i)^2 - 4(1+i) + 6$

$= -2 + 2i + 2i - 4 - 4i + 6$

$= 0$.

$\therefore$ $(1+i)$ is a root.

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(ii). $(x-1-i)(x-1+i)$ is a factor $\quad \therefore (1-i)$ is also a root.

$x^2 - \sum\alpha\, x + \alpha\beta$

$x^2 - 2x + 2$

$\therefore$ By Inspection, the other is $(x+3)$

$Q(x) = (x-1-i)(x-1+i)(x+3)$

(d). $x^3 + qx + r = 0$.

$(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2$

$= \beta^2 - 2\beta\gamma + \gamma^2 + \gamma^2 - 2\gamma\alpha + \alpha^2 + \alpha^2 - 2\alpha\beta + \beta^2$

$= 2(\alpha^2+\beta^2+\gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\beta)$

$= 2((\sum\alpha)^2 - 2(\sum\alpha\beta)) - 2(\sum\alpha\beta)$

$= 2[0 - 2(q)] - 2q$

$= -4q - 2q$

$= -6q$

5. $\frac{x^2}{25} + \frac{y^2}{16} = 1 \qquad a = 5, \quad b = 4$

(a)(i) $b^2 = a^2(1 - e^2) \qquad a > b$.

$\frac{16}{25} = 1 - e^2$.

$e^2 = 1 - \frac{16}{25}$.

$= 9/25$

$e = 3/5$.

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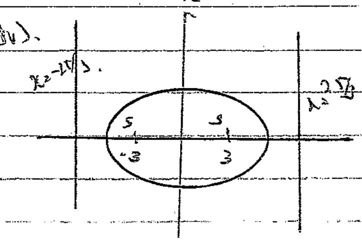
(ii) Foci, $= S, (\pm ae, 0)$

$= (\pm 3, 0)$

(iii) ~~Directrix~~ $= \cancel{y} = \frac{a}{e}$

Eqn. of the directrix $x = \pm \frac{25}{3}$.

(iv).



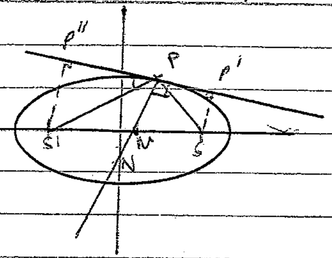
(v). $P(5\cos\theta, 4\sin\theta)$

$$\frac{25\cos^2\theta}{25} + \frac{16\sin^2\theta}{16} = 1.$$

$$\cos^2\theta + \sin^2\theta = 1$$

$\therefore$ True.

(vi).



$$\frac{SP}{PM} = e$$

$SP = ePM$ — ①

$S'P = ePM'$ — ②.

$$SP + S'P = e(PM + PM') = e\left(\frac{2a}{e}\right) = 2a.$$

(ix) In (vi) Since $\frac{S'P''}{PP''} = \frac{SP'}{PP'}$ (by perp. dist & trig ratio)

$\therefore \angle P''PS' = \angle P'PS$

$\therefore 90^\circ - \angle P''PS' = 90^\circ - \angle P'PS$

$\Rightarrow S'PN = SPN$

$\therefore PN$ bisects $S\hat{P}S'$

(vii). Normal: $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

At $P(5\cos\theta, 4\sin\theta)$

$$\frac{25x}{5\cos\theta} - \frac{16y}{4\sin\theta} = 25 - 16.$$

$$\frac{5x}{\cos\theta} - \frac{4y}{\sin\theta} = 9$$

$$5\sin\theta x - 4\cos\theta y = 9\sin\theta\cos\theta$$

(viii). M: $y = 0$. $5\cos\theta x = 9\sin\theta\cos\theta$

$x = \frac{9}{5}\sin\theta$ $\left(\frac{9}{5}\sin\theta, 0\right)$

N: $x = 0$, $y = -\frac{9}{4}\sin\theta$ $\left(0, -\frac{9}{4}\sin\theta\right)$

$$\therefore \frac{PM}{PN} = \frac{\sqrt{\left(\frac{9}{5} - 5\right)^2\cos^2\theta + (-4\sin\theta)^2}}{\sqrt{(-5\cos\theta)^2 + \left(-\frac{9}{4}\sin\theta - 4\sin\theta\right)^2}}$$

$$= \frac{\sqrt{\frac{256}{25}\cos^2\theta + 16\sin^2\theta}}{\sqrt{25\cos^2\theta + \frac{225}{16}\sin^2\theta}} = \frac{\frac{1}{5}\sqrt{256\cos^2\theta + 400\sin^2\theta}}{\frac{1}{4}\sqrt{400\cos^2\theta + 225\sin^2\theta}}$$

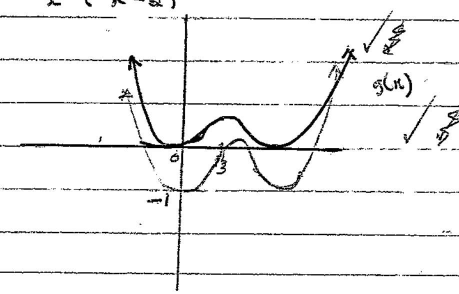
$$= \frac{\sqrt{\frac{16}{25}\left(16\cos^2\theta + 25\sin^2\theta\right)}}{\sqrt{\frac{25}{16}\left(16\cos^2\theta + 25\sin^2\theta\right)}} = \sqrt{\frac{16^2}{25^2}} = \frac{16}{25} \text{ as req}^d.$$

6. (i) $g(x) = x^6 - 4x^5 + 4x^4$

$x^2 - 4x + 4$

$x^4(x^2 - 4x + 4)$

$x^4(x-2)^2$



Use calculus to find the stationary pts

$$\frac{dy}{dx} = 6x^5 - 20x^4 + 16x^3$$
$$= 2x^3(3x^2 - 10x + 8)$$
$$= 2x^3(3x-4)(x-2) = 0$$

$x = 0$, $x = \frac{4}{3}$ or 2

$$\frac{d^2y}{dx^2} = 30x^4 - 80x^3 + 48x^2$$
$$= 6x^2(5x^2 - 15x + 8)$$

At $x = 0$, $f''(0) = 0$ probable pt of inflexion.

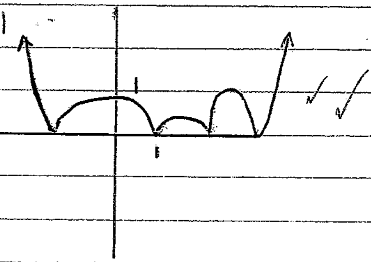
	$x = 0^-$	$x = 0$	$x = 0^+$
$f''(x)$	+	0	+

$\therefore$ min.

At $x = \frac{4}{3}$, $f''\left(\frac{4}{3}\right) < 0$ $\therefore$ Max at $\left(\frac{4}{3}, \ldots\right)$

At $x = 2$, $f''(2) > 0$ $\therefore$ Min.

(ii) $y = |f(x)|$



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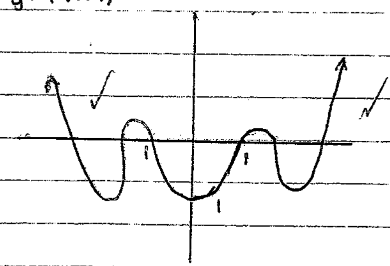
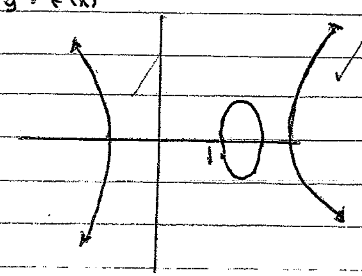
(iii) $y = f(|x|)$



(iv) $y^2 = f(x)$



(v). $y^2 = f(x)$

$$y^2 = x^6 - 4x^5 + 4x^4 - 1$$

$$f'(x) = 2y\frac{dy}{dx} = 6x^5 - 20x^4 + 16x^3$$

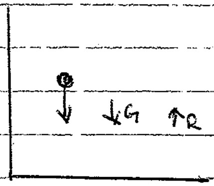
$$\frac{dy}{dx} = \frac{6x^5 - 20x^4 + 16x^3}{2y}$$

$$= \frac{3x^5 - 10x^4 + 8x^3}{2y}$$

$$= \frac{x^3(3x^2 - 10x + 8)}{2y}$$

$$= \frac{x^3(3x-4)(x-2)}{2y}$$

7.(a)



(i) $m\ddot{x} = mg - R$

$= mg - mkv^2$

$\ddot{x} = g - kv^2$

(ii) $v \cdot \frac{dv}{dx} = g - kv^2$ ... continue.

(b) (i) $\cos 3A = 4\cos^3 A - 3\cos A$

LHS $= \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A$

$= (2\cos^2 A - 1)\cos A - 2\sin^2 A \cos A$

$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$

$= 2\cos^3 A - \cos A - 2\cos A + 4\cos^3 A$

$= 4\cos^3 A - 3\cos A$

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(ii) $\cos 3A = \frac{-1}{2\sqrt{2}}$

$\cancel{4\cos^3 A - 3\cos A = \frac{-1}{2\sqrt{2}}}$ $\Longrightarrow$ $\cancel{8\sqrt{2}\cos^3 A - 6\sqrt{2}\cos A + 1 = 0.}$

$\cancel{\text{Let } x = 2\sqrt{2}\cos A}$

$\cancel{4x^3 - 3x + 1}$

(ii) $P(x) = x^3 - 6x + 2 = 0.$

$P(2\sqrt{2}\cos A) = 16\sqrt{2}\cos^3 A - 12\sqrt{2}\cos A + 2 =$

$16\sqrt{2}\cos^3 A - 12\sqrt{2}\cos A = -2$

Divide by $4\sqrt{2}$.

$4\cos^3 A - 3\cos A = \frac{-2}{4\sqrt{2}}$

$\cos 3A = \frac{-1}{2\sqrt{2}}$ (part (i))

(iii) $x^3 - 6x + 2 = 0.$

$\cos 3A = -\frac{1}{2\sqrt{2}}$

$3A = \cos^{-1}\left(\frac{-1}{2\sqrt{2}}\right)$

$= -69°18'$, $110°42'$, $249°18'$

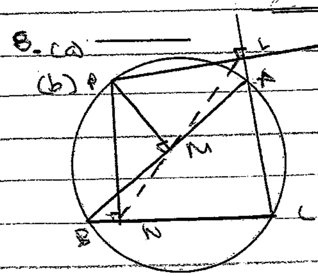
$A = -23°6'$, $36°54'$, $83°6'$

$\therefore x = 2\sqrt{2}\cos A$

$= -2.6026$, 2.2626, 0.340

8. (a) ———

(b)



BMA is a straight line.

$\therefore B\hat{M}P + P\hat{M}L + L\hat{M}A = \pi$

In $L\hat{M}N$: $B\hat{M}P + P\hat{M}L + B\hat{M}N$

But $B\hat{M}N = L\hat{M}A$ (vert opp)

$\therefore L\hat{M}A = \pi$

$\therefore$ L, M, N is collinear

This is only possib if LMN are collinear which is the question!