THE SCOTS COLLEGE
Sydney

Extension One Mathematics

HSC Task 3

Weighting 20%

1st June 2009

Total marks – 33

- Attempt all questions.

General Instructions

- **Working time – 45 Minutes.**
- Write using black or blue pen.
- Start a new page for every question
- Board-approved calculators may be used.
- All necessary working should be shown in every question.

Table of Standard Integrals provided at the end of the paper.

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right) \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

QUESTION 1 (10 Marks)

a) The temperature of a cup of coffee varies according to the rate given by $\frac{dT}{dt} = -k(T - T_o)$, where T is the temperature in degrees after elapsed time, t (in minutes), and T_0 is the temperature of the environment.

The cup, initially at $120^\circ C$, is kept in a cold chamber at $-20^\circ C$. After 3 minutes, the temperature of the cup drops down to $80^\circ C$.

i. Show that $T = T_o + Ae^{-kt}$ is a possible function that represents the variation of temperature with time for the cup. [1]

ii. Find the values of A and k. [3]

iii. If the cup at $80^\circ C$ is now placed in a room whose temperature is $20^\circ C$, assuming that the value of k remains unchanged, find the temperature of the cup after a further 20 minutes. [2]

b) A kite flying at a *constant* height of 40 m above the ground, is being dragged along by wind at a rate of 10 m/s. The kite is initially vertically above the ground. At what rate is the length of the string, tied to the kite, being released from the ground, increasing after 3 seconds. (Assume that the string remains straight). [4]

QUESTION 2 (14 Marks) START A NEW PAGE

a) Prove the following by mathematical induction

$$2\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\ldots\ldots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{n}$$ for all positive integers $n \geq 2$. [4]

b) Consider the series $\sum_{r=1}^{\infty} (\log_e x)^r$, where $x > 0$. [6]

i. Write down the first term, common ratio and the sum of n terms of the series.

ii. Find the range of values of x, such that a limiting sum exists for this series.

iii. Find the limiting sum if $x = \sqrt{e}$.

c) Let T_n and S_n represent the n^{th} Term and the Sum of n terms respectively, of an Arithmetic Progression, with first term a and common difference d $(a, d \neq 0)$. If T_{10}, T_4 and T_6 form consecutive terms of a Geometric Progression,

i. Show that $S_{10} = 0$. [2]

ii. Show that $S_6 + S_{12} = 0$ [2]

QUESTION 3 (9 Marks) START A NEW PAGE

Gordon and Gabbie take a loan of $500,000 from Community Bank, to buy a new house. The period of the loan is 30 years and interest is charged at the rate of 6% p.a. on the amount owing. Repayment is through a fixed monthly instalment of $*M*, paid at the end of each month.

Let A_n be the amount owing at the end of the n^{th} month, after the payment of the monthly instalment.

i. Show that at the end of the third month, the amount owing is given by

$$A_3 = 500000(1.005)^3 - M(1 + 1.005 + 1.005^2)$$ [2]

ii. By first arriving at a general expression for A_n, find the value of M. [2]

iii. Find the amount owing to the bank at the end of 4 years. [2]

iv. At the end of the 4^{th} year, the bank raises the interest rate to 7.2%. At the same time, Gordon and Gabbie decide to make fixed monthly payments of $4200 to the bank. Find the time it would now take for the couple to completely pay off the loan. Express your answer in years and months. [3]

END OF PAPER

Yr 12 Maths Ext 1 Task 3

SOLUTIONS

Question 1

(a) (i) $T = T_0 + Ae^{-kt}$

$$\frac{dT}{dt} = -kAe^{-kt}$$

$$= -k(T - T_0) \quad (\because Ae^{-kt} = T - T_0)$$

Hence $T = T_0 + Ae^{-kt}$ is a possible solution

(ii) When $t = 0$, $T = 120°C$, $T_0 = -20°C$

$$120 = -20 + Ae^0$$

$$\therefore \underline{A = 140°C}$$

When $t = 3$ minutes, $T = 80°C$

$$80 = -20 + 140e^{-3k}$$

$$100 = 140e^{-3k}$$

$$e^{-3k} = \frac{10}{14}$$

$$= \frac{5}{7}$$

$$\therefore -3k = \log_e \frac{5}{7}$$

$$k = -\frac{1}{3}\log_e \frac{5}{7} = 0.112157\ldots$$

$$\simeq \underline{0.112}$$

(iii) $T_0 = 20°C$

at $t = 0$, $T = 80°C$

$$80 = 20 + Ae^0 \quad \therefore A = 60$$

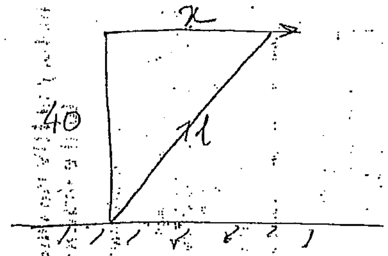
When $t = 15$ min. $T = 20 + 60e^{-15 \times k}$

$$= 31.156\ldots$$

$$= 31.2°C$$

Question 1

(b)



$$\frac{dx}{dt} = 10 \text{ m/s}$$

$$\frac{dl}{dt} = ?$$

$$l^2 = x^2 + 40^2$$

$$l = \sqrt{x^2 + 1600} \quad (\because l > 0)$$

$$\frac{dl}{dx} = \frac{1}{2}(x^2 + 1600)^{-\frac{1}{2}} \cdot 2x$$

$$= \frac{x}{\sqrt{x^2 + 1600}}$$

$$\frac{dl}{dt} = \frac{dx}{dt} \times \frac{dl}{dx}$$

$$= 10 \times \frac{x}{\sqrt{x^2 + 1600}}$$

When $t = 3$ seconds $x = 30$

$$\frac{dl}{dt} = \frac{10 \times 30}{\sqrt{900 + 1600}}$$

$$= \frac{10 \times 3\cancel{0}}{5\cancel{0}}$$

$$= \underline{6 \text{ m/s}}$$

Question 2

(a) Step 1 : prove true for $n = 2$

LHS : $2\left(1-\frac{1}{4}\right) = 2 \times \frac{3}{4} = \frac{3}{2}$

RHS = $\frac{2+1}{2} = \frac{3}{2}$

Hence true for $n = 1$

Step 2 : Assume true for $n = k$

ie $2\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right) \cdots \left(1-\frac{1}{k^2}\right) = \frac{k+1}{k}$, $k \geq 2$

Step 3 : Prove true for $n = k+1$

ie $2\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right) \cdots \left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{k+1}$

LHS = $\left(\frac{k+1}{k}\right)\left(1-\frac{1}{(k+1)^2}\right)$ (from Step 2)

$= \frac{k+1}{k} - \frac{1}{k(k+1)}$

$= \frac{1}{k(k+1)}\left[(k+1)^2 - 1\right]$

$= \frac{1}{k(k+1)}\left(k^2+2k+\cancel{1}-\cancel{1}\right)$

$= \frac{k(k+2)}{k(k+1)}$

$= \frac{k+2}{k+1}$

$=$ RHS

Hence true for $n = k+1$

Therefore by the Principle of Mathematical Induction, the statement is true for all $n \geq 2$

(b) Series (i) $a = \log_e x$, $r = \log_e x$, $S_n = \frac{\log_e x\left[(\log_e x)^n - 1\right]}{\log_e x - 1}$

(ii) $S_\infty = \log_e x + (\log_e x)^2 + (\log_e x)^3 + \cdots$

$|\log_e x| < 1$

$\log_e x < 1$ or $\log_e x > -1$

$x < e^1$ $\quad x > e^{-1}$

$\underline{\frac{1}{e} < x < e}$, $\left(\begin{array}{l} x \neq 1 \\ \because \log_e 1 = 0 \end{array}\right)$

(iii) $S_\infty = \frac{a}{1-r}$

$a = \log_e \sqrt{e} = \frac{1}{2}$

$r = \log_e \sqrt{e} = \frac{1}{2}$

$S_\infty = \frac{1/2}{1-1/2} = 1$

(c) (i) $T_{10} = a + 9d$, a = first term, d = common difference

$T_4 = a + 3d$

$T_6 = a + 5d$

$\frac{a+3d}{a+9d} = \frac{a+5d}{a+3d}$

or $(a+3d)^2 = (a+9d)(a+5d)$

$a^2 + 6ad + 9d^2 = a^2 + 14ad + 45d^2$

$8ad + 36d^2 = 0$ or $\cancel{4}d(2a+9d) = 0$

$d \neq 0$ $\therefore 2a + 9d = 0$

$$S_{10} = \frac{10}{2}(2a+9d)$$
$$= 5(2a+9d)$$
$$= 0 \quad (\because 2a+9d = 0)$$

(ii) $S_6 + S_{12} = 0$

$$LHS = \frac{6}{2}(2a+5d) + \frac{12}{2}(2a+11d)$$
$$= 6a + 15d + 12a + 66d$$
$$= 18a + 81d$$
$$= 9(2a+9d)$$
$$= 0 \quad (\because 2a+9d = 0)$$

Yr 12 Extension 1 Mathematics

Assessment Task 3 - SOLUTIONS

Question 3

(a) Amt Borrowed = \$500,000

Interest Rate: 6% p.a = 0.5% p.month = 0.005

Period: 30 years = 360 months

Repayment: \$M per month.

(i) A_n = Amt owing at the end of n^{th} month.

$$A_1 = 500\,000(1.005) - M$$
$$A_2 = [500000(1.005) - M]\,1.005 - M$$
$$= 500000(1.005)^2 - 1.005M - M$$
$$A_3 = 500000(1.005)^3 - 1.005^2M - 1.005M - M$$
$$= 500000(1.005)^3 - M(1 + 1.005 + 1.005^2)$$

(ii) $A_n = 500000(1.005)^n - M(1 + 1.005 + 1.005^2 + \cdots + 1.0$

$$0 = A_{360} = 500000(1.005)^{360} - M(1 + 1.005 + 1.005^2 + \cdots + 1.00$$

$$M\left[\frac{1(1.005^{360} - 1)}{1.005 - 1}\right] = 500000(1.005)^{360}$$

$$\therefore M = \frac{0.005 \times 500000(1.005)^{360}}{1.005^{360} - 1}$$

$$= 2997.7526\ldots$$
$$= \$2997.75$$

(iii) $A_{48} = 500000(1.005)^{48} - 2997.75\,(1 + 1.005 + 1.005^2 + \dots + 1.005^{4}$

$= 500000\,(1.005)^{48} - 2997.75\,\left(\dfrac{1.005^{48} - 1}{1.005 - 1}\right)$

$= 473072.807\dots$

$\simeq \$\,473073$

(iv) $r = 7.2\%$ p.a $= 0.006$

$M = \$4200$, $P = \$\,473073$

$A_n = 473073\,(1.006)^n - 4200\,(1 + 1.006 + 1.006^2 + \dots + 1.006^{n-}$

$0 = 473073\,(1.006)^n - 4200\,\left(\dfrac{1.006^n - 1}{0.006}\right)$

$= 473073\,(1.006)^n - 700000\,(1.006^n - 1)$

$= 473073\,(1.006)^n - 700000\,(1.006^n) + 700000$

$226927\,(1.006)^n = 700000$

$1.006^n = \dfrac{700000}{226927}$

$n \log 1.006 = \log \dfrac{700000}{226887}$

$n = \dfrac{\log \dfrac{700000}{226927}}{\log 1.006}$

$= 188.3$ months

$= 15.7$ years i.e 15 years & 9^{th} month

Therefore, they repay the loan in a further 15 yrs & 9^{th} month

or total of 19 years & 9^{th} month.