Series and Sequences

① Consider the series $3 + 8 + 13 + 18 + \ldots + 488$:

(i) how many terms are in this series?

(ii) find the sum of all the terms in this series.

② (i) Express $0 \cdot 1\dot{5}$ as an infinite series.

(ii) Hence express $0 \cdot 1\dot{5}$ as a fraction with no common factors.

③ A timber worker is stacking logs. The logs are stacked in layers, where each layer contains one log less than the layer below. There are five logs in the top layer, six logs in the next layer, and so on. There are n layers altogether.

(i) Write down the number of logs in the bottom layer.

(ii) Show that there are $\frac{1}{2}n(n+9)$ logs in the stack.

④ Can there by an infinite geometric series with a limiting sum of 5/8 and a first term of 2? (All working and reasoning must be shown.)

⑤ Evaluate :

(i) $\sum_{n=1}^{6}(5+4n)$

(ii) $\sum_{r=2}^{\infty}(0.5)^r$

⑥ The third term of a G.P. is $\frac{27}{4}$ and the fifth term is $\frac{234}{16}$.
Find the sequence(s).

⑦ The sum of the first two terms of a geometric progression is 25, and its limiting sum is $26\frac{2}{3}$. Find the first three terms of the sequence if it is known that the common ratio is greater than zero.

⑧ The sponsors of a golf tournament have provided \$232 500 for the total prizes for the first 15 places. The prize for the winner is to be \$26 000 and, from there down, each prize decreases by a constant amount. Find

(i) the prize for finishing 15th;

(ii) the prize for finishing 2nd.

⑨ Find the sum of the multiples of 3 between 100 and 250.

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(i) For what values of r does the geometric series

$$a + ar + ar^2 + \cdots$$

have a limiting sum?

For these values of r write down the limiting sum.

(ii) Find a geometric series with common ratio $\frac{1}{w}$ that has limiting sum $\frac{1}{1-w}$.

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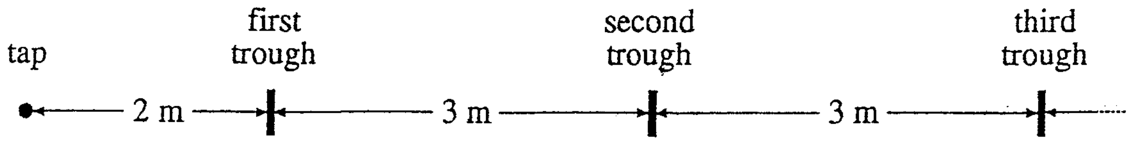

A tap and n water troughs are in a straight line. The tap is first in line, 2 metres from the first trough, and there is 3 metres between consecutive troughs. A stable hand fills the troughs by carrying a bucket of water from the tap to each trough and then returning to the tap. Thus she walks $2 + 2 = 4$ metres to fill the first trough, 10 metres to fill the second trough, and so on.

(i) How far does the stable hand walk to fill the kth trough?

(ii) How far does the stable hand walk to fill all n troughs?

(iii) The stable hand walks 1220 metres to fill all the troughs. How many water troughs are there?

12

An expedition sets out to travel from Mawson Base to the South Pole, a distance of 2420 km. On the first day, they travel 312 km. Subsequently, the distance travelled each day is 85% of that travelled the previous day.

(i) How far (to the nearest km) will they travel on the seventh day?

(ii) How far (to the nearest km) will they be from Mawson Base after a week's travelling?

(iii) Show that the expedition cannot reach the South Pole.

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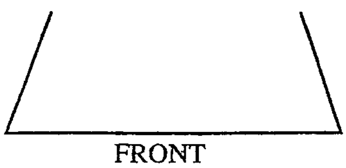

The diagram above shows part of the floor plan for a proposed concert hall. The floor narrows from front to back so that each row of seats behind the first has two less seats than the row in front of it. The first row has fifty-seven seats.

(i) How many seats are in the nth row?

(ii) What is the greatest value n can take?

(iii) The hall is planned to seat 720 people. How many rows of seats will there be?

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Given the arithmetic series 180 + 174 + 168 + ...

a) Find the value of the twenty-third term.
Find the least number of terms such that the sum should be negative.

Series & Sequences. (/55)

① $3+8+13+18+\ldots+488$

AP $a=3$
$d=5$

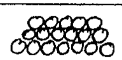
$T_n = a+(n-1)d$
$488 = 3+(n-1)\times 5$
$485 = 5n-5$
$5n = 490$
$n = 98$

ii) $S_n = \frac{n}{2}[a+l]$
$S_{98} = \frac{98}{2}[3+488]$
$= 24059$

② i) $0.1\dot{5}$
$= 0.1 + (0.05 + 0.005 + 0.0005 + \ldots$
GP $a=0.05$
$r=0.1$

ii) $S = \frac{a}{1-r}$
$= \frac{0.05}{1-0.1}$
$= \frac{0.05}{0.9}$
$= \frac{5}{90}$

$\therefore 0.1\dot{5} = \frac{1}{10}+\frac{5}{90}$
$= \frac{7}{45}$

③  $5+6+7+8+\ldots+T_n$

i)
AP $a=5$
$d=1$
$T_n = a+(n-1)d$
$= 5+(n-1)\times 1$
$= n+4$

There are $(n+4)$ logs in the bottom layer.

ii) $S_n = \frac{n}{2}[2a+(n-1)d]$
Easier to use.
$S_n = \frac{n}{2}[a+l]$
$= \frac{n}{2}[5+n+4]$
$= \frac{1}{2}n(n+9)$

④ $S = \frac{a}{1-r}$ $S = 5/8,\ a=2$
$\frac{5}{8} = \frac{2}{1-r}$
cross multiply
$5-5r = 16$
$-5r = 11$
$r = -11/5$
$r = -2\,1/5$

It is not possible that this series can have a limiting sum because it has common ratio in the range $-1<r<1$.

⑤ i) $\sum_{n=1}^{6}(5+4n)$
$= 9+13+17+21+25+29$
AP $a=9$
$d=4$
$n=6$
$S_n = \frac{n}{2}[a+l]$
$S_6 = \frac{6}{2}[9+29]$
$= 3\times 38$
$= 114$

ii) $\sum_{r=2}^{\infty}(0.5)^r$
$= 0.5^2+0.5^3+0.5^4+\ldots$
GP $a=0.25$
$r=0.5$
Limiting sum
$S = \frac{a}{1-r}$
$= \frac{0.25}{1-0.5}$
$= \frac{0.25}{0.5} = \frac{25}{50} = 1/2$

$T_3 = ar^2 = \frac{27}{4}$ ①
$T_5 = ar^4 = \frac{234}{16}$ ②

Divide ② by ①
$\frac{ar^4}{ar^2} = \frac{234/16}{27/4}$
$r^2 = 2\,1/6$
$r^2 = \frac{13}{6}$
$r = \pm\sqrt{\frac{13}{6}}$

There are two sequences.
$r = \pm\sqrt{\frac{13}{6}}$
sub. in ①
$a\times\frac{13}{6} = \frac{27}{4}$
$a = \frac{27/4}{13/6}$
$a = 3\,3/26$.

Sequence 1
$a = 3\,3/26,\ r = \sqrt{\frac{13}{6}}$

Sequence 2
$a = 3\,3/26,\ r = -\sqrt{13/6}$

⑦ GP
$T_1 + T_2 = 25$
$a + ar = 25$ ①
$S = \frac{a}{1-r} = 26\frac{2}{3}$
$\frac{a}{1-r} = \frac{80}{3}$
$a = \frac{80}{3}(1-r)$ ②

Substitute ② into ①
$\frac{80}{3}(1-r) + \frac{80}{3}(1-r)r = 25$
$80 - \cancel{80r} + \cancel{80r} - 80r^2 = 75$
$80r^2 = 5$
$r^2 = \frac{5}{80}$
$r^2 = \frac{1}{16}$
$r = \pm\frac{1}{4}$

Given that $r>0$ $\therefore r = \frac{1}{4}$
Find a subst. into ②
$a = \frac{80}{3}(1-\frac{1}{4})$
$= \frac{80}{3}\times\frac{3}{4}$
$= 20$.

$\therefore$ GP $a=20,\ r=1/4$
$20, 5, \frac{5}{4}, \ldots$

⑧ $\$26000, -, -, -, \ldots$
AP $n=15$
$S_n = \$232500$
$a = 26000$

i) $S_n = \frac{n}{2}[a+l]$
$232500 = \frac{15}{2}[26000+l]$
$31000 = 26000 + l$
$l = 5000$

The prize for 15^{th} place is \$5000.

Find d
ii) $l = a+(n-1)d$
$5000 = 26000 + (15-1)\times d$
$-21000 = 14d$
$d = -1500$

$T_2 = a+d$
$= 26000 - 1500$
$= \$24500$

The second prize is \$24500

(9) $3 \times \underline{34} = 102$

$3 \times \underline{83} = 249$

AP

$102, 105, 108, 111, \ldots, 249$

$a = 102$
$d = 3$

$n = 83 - 34 + 1$
$= 50$

$S_n = \frac{n}{2}[a + l]$

$S_{50} = \frac{50}{2}[102 + 249]$

$= 25 \times 351$

$= 8775$

(10) $a + ar + ar^2 + \ldots$

i) This G.P. has a limiting sum when $-1 < r < 1$

The limiting sum is $S = \frac{a}{1-r}$

ii) $r = \frac{1}{w}$ $\quad S = \frac{1}{1-w}$

$S = \frac{a}{1-r}$

$\frac{1}{1-w} = \frac{a}{1 - \frac{1}{w}}$

$\frac{1}{1-w} = \frac{a}{\frac{w-1}{w}}$

$a = \frac{1}{1-w} \times \frac{w-1}{w}$

$= \frac{1}{1-w} \times \frac{-(1-w)}{w}$

$= -\frac{1}{w}$

GP $a = -\frac{1}{w}, \; r = \frac{1}{w}$

$-\frac{1}{w} + \frac{1}{w^2} - \frac{1}{w^3} + \frac{1}{w^4} - \ldots$

(11) Walks there and back to fill trough

$4, 10, 16, \ldots$

AP $a = 4$
$d = 6$

i) $T_n = a + (n-1)d$

$T_k = 4 + (k-1)6$

$= 4 + 6k - 6$

$= 6k - 2$

ii) $S_n = \frac{n}{2}[2a + (n-1)d]$

$= \frac{n}{2}[8 + (n-1).6]$

$= \frac{n}{2}[8 + 6n - 6]$

$= \frac{n}{2}[2 + 6n]$

$= n + 3n^2$

iii) $S_n = 1220$

$3n^2 + n = 1220$

$3n^2 + n - 1220 = 0$

$n = \frac{-1 \pm \sqrt{1^2 - 4.3.-1220}}{2 \times 3}$

$= \frac{-1 \pm \sqrt{1 + 14640}}{6}$

$= \frac{-1 \pm \sqrt{14641}}{6}$

$= \frac{-1 \pm 121}{6}$

$n = \frac{-1 - 121}{6}$ $\quad n = \frac{-1 + 121}{6}$

$= -20\frac{1}{3}$ $\quad = 20$

Not valid

$\therefore$ There are 20 water troughs.

(12)

$312, \; 0.85 \times 312, \; 0.85^2 \times 312, \ldots$

GP $a = 312$
$r = 0.85$

i) $T_n = ar^{n-1}$

$T_7 = 312 \times 0.85^6$

$\doteqdot 117.7$ km

ii) $S_n = \frac{a(1 - r^n)}{1 - r}$

$S_7 = \frac{312(1 - 0.85^7)}{1 - 0.85}$

$= \frac{312(1 - 0.85^7)}{0.15}$

$\doteqdot 1413.2$ km

iii) Find limiting sum

$S = \frac{a}{1-r}$

$= \frac{312}{1 - 0.85}$

$= \frac{312}{0.15}$

$= 2080$

The limiting sum is 2080 km so the expedition won't reach the South Pole distance of 2420 km. (Let's hope they turned back!)

(13) $57, 55, 53, 51, \ldots$

AP $a = 57$
$d = -2$

i) $T_n = a + (n-1)d$

$= 57 + (n-1) \times -2$

$= 57 - 2n + 2$

$= 59 - 2n$

ii) the number of seats in a row must be greater than zero

$T_n > 0$

$\therefore$

$59 - 2n > 0$

$-2n > -59$

$n < 29\frac{1}{2}$

$\therefore n = 29$ rows

iii) $S_n = \frac{n}{2}[2a + (n-1)d]$

$720 = \frac{n}{2}[2 \times 57 + (n-1) \times -2]$

$1440 = n[114 - 2n + 2]$

$1440 = n[116 - 2n]$

$1440 = 116n - 2n^2$

$2n^2 - 116n + 1440 = 0$

$n^2 - 58n + 720 = 0$

$n = \frac{58 \pm \sqrt{58^2 - 4 \times 1 \times 720}}{2 \times 1}$

$= \frac{58 \pm \sqrt{3364 - 2880}}{2}$

$= \frac{58 \pm \sqrt{484}}{2}$

$= \frac{58 \pm 22}{2}$

$n = \frac{58 + 22}{2}$ $\quad n = \frac{58 - 22}{2}$

$= \frac{80}{2}$ $\quad = \frac{36}{2}$

$= 40$ $\quad = 18$

Since the greatest value of $n = 29$

$\therefore n = 18$ rows.

(14) $180 + 174 + 168 + \ldots$

AP $a = 180$
$d = -6$

a) $T_n = a + (n-1)d$

$T_{23} = 180 + 22 \times -6$

$= 48$

b) $S_n = \frac{n}{2}[2a + (n-1)d]$

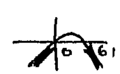
Solve $S_n < 0$

$\frac{n}{2}[360 + (n-1) \times -6] < 0$

$n[360 - 6n + 6] < 0$

$n(366 - 6n) < 0$

$6n(61 - n) < 0$



$n < 0$, $\quad n > 61$

not valid

$\therefore n = 62$ terms