HIGHER SCHOOL CERTIFICATE EXAMINATION TRIAL PAPER

2006

MATHEMATICS

EXTENSION 2

Time Allowed – Three Hours
(Plus 5 minutes reading time)

Directions to Candidates

- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Board-approved calculators may be used.

YEAR 12 – TRIAL 2006 – EXTENSION 2

QUESTION 1 — MARKS

a) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos x}{1+\sin^2 x}\,dx$ — 2

b) Use integration by parts to find $\displaystyle\int x^3 \ln x\,dx$ — 2

c) Use the substitution $x = 3\tan\theta$ to find — 2

$$\int_0^3 \frac{dx}{(9+x^2)^{\frac{3}{2}}}$$

d) i) Find real numbers a and b such that — 2

$$\frac{3x^2-4x+4}{(x-2)(x^2+4)} \equiv \frac{a}{x-2} + \frac{bx}{4+x^2}$$

ii) Hence find $\displaystyle\int \frac{3x^2-4x+4}{(x-2)(x^2+4)}\,dx$ — 2

e) i) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin\theta+\cos\theta}$ — 2

ii) Hence use the substitution $u = \frac{\pi}{2} - \theta$ to evaluate — 3

$$\int_0^{\frac{\pi}{2}} \frac{\theta}{1+\sin\theta+\cos\theta}\,d\theta$$

QUESTION 2 — MARKS

a) i) Express $1 + i\sqrt{3}$ and $1 - i\sqrt{3}$ in modulus argument form. 1

ii) Hence, simplify $(1 + i\sqrt{3})^6 + (1 - i\sqrt{3})^3$. 2

b) Let $\alpha = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

i) Show that α is a root of the equation $z^6 + 1 = 0$. 1

ii) Hence, or otherwise, factorise the polynomial $z^6 + 1$ over the the real numbers. 4

c) i) Shade the region in the Argand diagram containing all points representing the complex numbers z that satisfy all the following three inequations. 2

$0 \le \text{Arg}(z - 1 - i) \le \frac{2\pi}{3}$, $|z - 1| \le |z - 2|$ and $\text{Re}(z) \ge 0$.

ii) Find the range of Arg z when z lies in the region shaded in part (i). 1

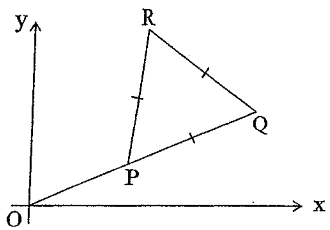
d) The two points P and Q in the Argand plane represent respectively the complex numbers w with modulus 1 and the complex number $\frac{k}{w}$, where k is a positive real number.

i) Show that the origin O, P and Q are collinear points. 1

ii) For what value of k do P and Q coincide? 1

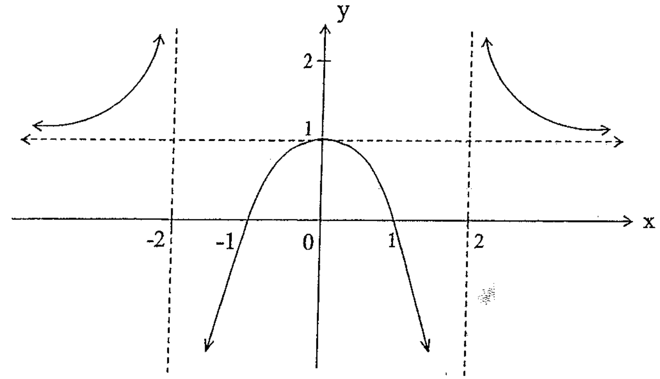
iii) Let R be the point representing the complex number z such that ΔPQR is equilateral as shown in the diagram. 2

Find z in terms of w.



QUESTION 3 — MARKS

a)



The above diagram shows the graph of $y = f(x)$.
Sketch on separate diagrams the following curves, indicating clearly any turning points and asymptotes.

i) $y = \frac{1}{f(x)}$ 1

ii) $y = f(x) + f(-x)$ 2

iii) $y = f(\sqrt{x})$ 2

iv) $y = f(x - 1)$ 2

v) $y = (f(x))^3$ 2

vi) $y = \ln(f(x))$ 2

b) The region bounded by $y = 2x(2 - x)$ and $y = x(2 - x)$ is rotated about the line $x = 2$ to form a solid. 4

Use the method of cylindrical shells to find the volume of this solid.

QUESTION 4 **MARKS**

a) A particle P of mass m is attached by two equal light inextensible strings each of length ℓ to two points A and B on a smooth vertical rod.

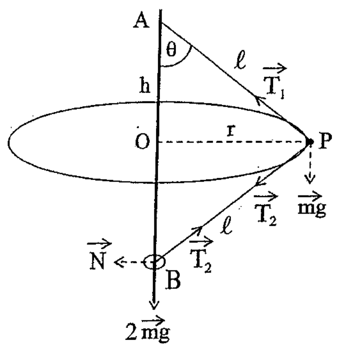
The end of the string at A is fixed, while the end of the string at point B holds a small ring of mass 2m which can slide along the rod.

Particle P is set to move in a uniform circular motion with centre O, radius r and angular velocity w.

Let h be the distance between centre O and point A, and let θ be the angle between the string AP and the rod AB.

i) Show that the tensions T_1 and T_2 in the strings AP and BP are respectively 3

$$T_1 = \frac{3mg}{\cos\theta} \quad \text{and} \quad T_2 = \frac{2mg}{\cos\theta}$$



ii) Show that $w^2 h = 5g$. 2

iii) Given that the string AP will break if the tension in it exceeds 6mg, show that $w^2 < \frac{10g}{\ell}$ 2

b) Given that the roots of $x^3 - x^2 + 2x - 1 = 0$ are α, β and γ, find the value of $\frac{\alpha}{\beta\gamma} + \frac{\beta}{\alpha\gamma} + \frac{\gamma}{\alpha\beta}$ 3

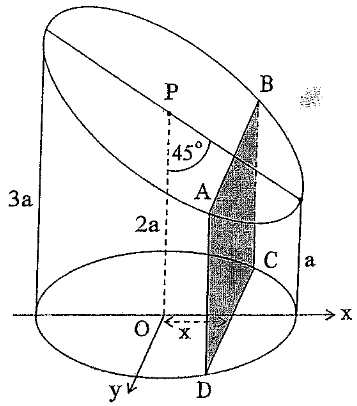
c) Let α, β and γ be the roots of the equation $x^3 - 12x + k = 0$

i) Show that if $k \neq \pm 16$, no two of the roots α, β and γ are equal. 2

ii) If $\alpha = 1$, evaluate $1 + \beta^4 + \gamma^4$ 3

QUESTION 5 **MARKS**

a) A solid is obtained by removing the top part of a cylinder with circular base centred at O and with radius a. The top face of the solid makes an angle of 45° with its vertical axis OP. The height of the solid formed is 3a at its highest point, 2a at P and a at its lowest point.



Let O be the origin of the x and y axes and ABCD be a rectangular slice perpendicular to the x-axis at a distance x from O, as shown in the diagram.

i) Show that the area A of the slice ABCD is given by 3

$$A = 2\sqrt{a^2 - x^2}\,(2a - x)$$

ii) Find the volume of the solid. 3

b) From a point $T(x_0, y_0)$ that lies outside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b$, two tangents are drawn to the ellipse.
Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the points of contact.

i) Show that the equation of tangent TP is 2

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

ii) Show that the equation of the chord of contact PQ is 2

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$$

iii) Find the locus of T if PQ is a focal chord. 2

iv) Find the locus of T if PQ is tangent to the circle $x^2 + y^2 = r^2$ where $r < b$ 3

QUESTION 6 **MARKS**

a) Given that $u_{n+2} = u_{n+1} + u_n$ and $u_1 = u_2 = 1$, use mathematical induction to prove that $u_n < \left(\frac{7}{4}\right)^n$ for all integers $n \geq 1$. 3

b) Given that a, b and c are positive real numbers, 2

show that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c}$

c) i) Show that $\int_0^{\frac{\pi}{4}} (\tan x)^{2k} \sec^2 x \, dx = \frac{1}{2k+1}$ 1

ii) Show that $\int_0^{\frac{\pi}{4}} (\sec x)^{2n+2} dx = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2k+1}$ 3

d) A particle of unit mass is thrown vertically upward from ground level with initial velocity of V_o. It experiences a vertical resistance force of magnitude kv^2, where v is its velocity at any time. It returns to the ground with a velocity U.

i) Show that the maximum height reached by the particle is 2

$$h = \frac{1}{2k} \ln\left(\frac{g + kV_o^2}{g}\right)$$

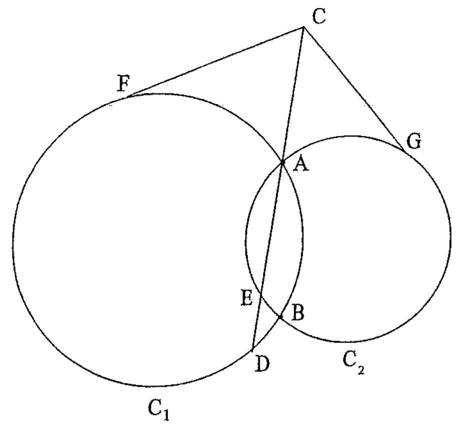
ii) Show that the terminal velocity V of the particle is 1

$$V = \sqrt{\frac{g}{k}}$$

iii) Show that $U^2 = \frac{V_o^2 \; V^2}{V_o^2 + V^2}$ 3

QUESTION 7 **MARKS**

a) Two circles C_1 and C_2 intersect at A and B. From a point C outside both circles the line CA is produced to meet C_1 at D and C_2 at E. Points F and G are on C_1 and C_2 respectively.



i) If CF is tangent to C_1, Show that $CF^2 = CA \times CD$. 2

ii) If CF is tangent to C_1, CG is tangent to C_2 and CF = CG, show that the points B, D and E coincide. 2

iii) If B, D and E coincide and $CF^2 = CA \times CB$, show that CF is a tangent to the C_1. 3

b) Consider the function $f(x) = x^n e^{-x}$ where $x > 0$ and n is a positive integer.

i) Find the maximum value of f(x). Hence, show that 3

$x^n e^{-x} < n^n e^{-n}$ for $x \neq n$.

ii) Deduce from part (i), or otherwise, that $\left(1+\frac{1}{n}\right)^n < e$. 2

iii) By considering the function $g(x) = x^{n+1} e^{-x}, x > 0$, 3

show that $\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$

QUESTION 8 MARKS

a) Consider $f(x) = x^{a+1} - 1 - (a+1)(x-1)$ where $x > 0$ and $a > 0$. 2

i) Show that $f(x)$ has only one turning point at $x = 1$, and that this is a minimum turning point.

ii) Show that $x^{a+1} \geq 1 + (a+1)(x-1)$ 1

iii) Show that $(a+1)p^a < \dfrac{q^{a+1} - p^{a+1}}{q-p} < (a+1)q^a$, 3

where p, q are integer and $q > p \geq 0$.

iv) Show that $1^a + 2^a + \ldots + (n-1)^a < \dfrac{n^{a+1}}{a+1} < 1^a + 2^a + \ldots + n^a$ 3

where n is positive integer

b) A game is played using a barrel containing balls numbered 1 to 15. The game consists of drawing 3 balls, without replacement, from the barrel.

i) What is the probability that in at least four of five games the three numbers drawn in each game are all divisible by 3? 3

ii) What is the probability that the sum of the numbers on the balls drawn in any game is divisible by 3? 3

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq 1; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

QUESTION 1

a) Let $u=\sin x$

$\therefore \frac{du}{dx}=\cos x \quad \therefore du=\cos x\,dx$

when $x=\frac{\pi}{4},\ u=\frac{1}{\sqrt{2}}$

$x=0,\quad u=0$

$$\int_0^{\pi/4}\frac{\cos x\,dx}{1+\sin^2 x}=\int_0^{\frac{1}{\sqrt{2}}}\frac{du}{1+u^2}$$

$$=\left[\tan^{-1}u\right]_0^{1/\sqrt{2}}=\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

(2marks)

b) Let $u=\ln x$, $dv=x^3dx$

$du=\frac{1}{x}dx \quad v=\frac{1}{4}x^4$

$$\int x^3\ln x\,dx=\frac{x^4}{4}\ln x-\int\frac{x^4}{4}\times\frac{1}{x}\,dx$$

$$=\frac{x^4}{4}\ln x-\frac{x^4}{16}+c$$

(2 marks)

c) Let $x=3\tan\theta$

$dx=3\sec^2\theta\,d\theta$

$(9+x^2)^{3/2}=(9+9\tan^2\theta)^{3/2}$

$=(9\sec^2\theta)^{3/2}$

$=27\sec^3\theta$

for $x=0,\ \theta=0$

$x=3,\ \theta=\frac{\pi}{4}$

$$\therefore\int_0^3\frac{dx}{(9+x^2)^{3/2}}=\frac{3}{27}\int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{\sec^3\theta}$$

$$=\frac{3}{27}\int_0^{\pi/4}\frac{1}{\sec\theta}d\theta$$

$$=\frac{1}{9}\int_0^{\pi/4}\cos\theta\,d\theta=\frac{1}{9}\left[\sin\theta\right]_0^{\pi/4}$$

$$=\frac{1}{9}\times\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{18}$$

(2 marks)

d) i) Let $\frac{3x^2-4x+4}{(x-2)(x^2+4)}=\frac{a}{(x-2)}+\frac{bx}{(x^2+4)}$

$\therefore 3x^2-4x+4=a(4+x^2)+bx(x-2)$

$=x^2(a+b)+x(-2b)+4a$

Equating coefficients of like terms:

$\left.\begin{array}{l}a+b=3\\-4=-2b\\4=4a\end{array}\right\}\therefore a=1,\ b=2$

(2marks)

ii) $$\int\frac{3x^2-4x+4}{(x-2)(x^2+4)}dx=\int\frac{dx}{x-2}+\int\frac{2x\,dx}{4+x^2}$$

$=\ln|x-2|+\ln(x^2+4)+c$

(2marks)

e) i) Let $u=\tan\frac{\theta}{2}$.

$\frac{\theta}{2}=\tan^{-1}u$

$\theta=2\tan^{-1}u$

$\frac{d\theta}{du}=\frac{2}{1+u^2}$

$\therefore d\theta=\frac{2\,du}{1+u^2}$

when $\theta=\pi/2,\ u=1$

$\theta=0,\ u=0$

$\sin\theta=\frac{2u}{1+u^2}$, $\cos\theta=\frac{1-u^2}{1+u^2}$

$$\int_0^{\pi/2}\frac{1}{1+\sin\theta+\cos\theta}d\theta$$

$$=\int_0^1\frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\times\frac{2du}{1+u^2}$$

$$=\int_0^1\frac{1+u^2}{1+u^2+2u+1-u^2}\times\frac{2du}{1+u^2}$$

$$=\int_0^1\frac{du}{1+u}=\left[\ln|1+u|\right]_0^1=\ln 2$$

(2marks)

ii) Let $u=\frac{\pi}{2}-\theta$

$\frac{du}{d\theta}=-1 \quad \therefore du=-d\theta$

When $\theta=0,\ u=\pi/2$

$\theta=\pi/2,\ u=0$

Let $$I=\int_0^{\pi/2}\frac{\theta\,d\theta}{1+\sin\theta+\cos\theta}$$

$$=\int_{\pi/2}^0\frac{(\frac{\pi}{2}-u)(-du)}{1+\sin(\frac{\pi}{2}-u)+\cos(\frac{\pi}{2}-u)}$$

$$=-\int_{\pi/2}^0\frac{(\frac{\pi}{2}-u)\,du}{1+\cos u+\sin u}$$

$$=\int_0^{\pi/2}\frac{\pi/2\,du}{1+\cos u+\sin u}-\int_0^{\pi/2}\frac{u\,du}{1+\cos u+\sin u}$$

$\therefore I=\frac{\pi}{2}\ln 2-I$ (by using the result in (i))

$\therefore 2I=\frac{\pi}{2}\ln 2$

$\therefore I=\frac{\pi}{4}\ln 2$

(3 marks)

QUESTION 2

a) i) let $z=1+i\sqrt{3}$ & $w=1-i\sqrt{3}$

$|z|=2 \qquad |w|=2$

$\tan\theta=\frac{\sqrt{3}}{1} \qquad \tan\theta=\frac{-\sqrt{3}}{1}$

$\therefore\theta=\frac{\pi}{3} \qquad \therefore\theta=-\frac{\pi}{3}$

$\therefore \text{Arg}\,z=\frac{\pi}{3} \qquad \therefore \text{Arg}\,w=-\frac{\pi}{3}$

$\therefore z=2\text{cis}\frac{\pi}{3}$, $w=2\text{cis}(-\frac{\pi}{3})$

1 mark

ii) $(1+i\sqrt{3})^6+(1-i\sqrt{3})^3$

$=(2\text{cis}\frac{\pi}{3})^6+(2\text{cis}(-\frac{\pi}{3}))^3$ (from i)

$=64\text{cis}\,2\pi+8\text{cis}(-\pi)$ (by de Moivre's theorem)

$=64-8$

$=56$

(2 marks)

b) i) $\alpha=\frac{\sqrt{3}}{2}+\frac{i}{2}$

$|\alpha|=\sqrt{\frac{3}{4}+\frac{1}{4}}=1$

$\tan\theta=\frac{1}{\sqrt{3}} \quad \therefore\theta=\frac{\pi}{6}$ i.e. $\text{Arg}\,\alpha=\frac{\pi}{6}$

$\therefore\alpha=\text{cis}\frac{\pi}{6}$

$\alpha^6=(\text{cis}\frac{\pi}{6})^6=\text{cis}\,\pi=-1$

$\therefore\alpha^6=-1$

$\therefore\alpha^6+1=0$

Clearly, α verifies the equation $z^6+1=0$

$\therefore\alpha$ is a root of this equation.

(1mark)

ii) $z^6=-1=\text{cis}\,\pi$

let $z=r\,\text{cis}\,\theta$

Using de Moivre's theorem,

$r^6\text{cis}\,6\theta=\text{cis}\,\pi$

$r^6=1 \qquad \text{cis}\,6\theta=\text{cis}\,\pi$

$r^6=1 \qquad 6\theta=\pi+2k\pi$

$\therefore r=1 \qquad \therefore\theta=\frac{\pi}{6}+\frac{k\pi}{3}$

for $k=0,\ z_1=\text{cis}\frac{\pi}{6}$

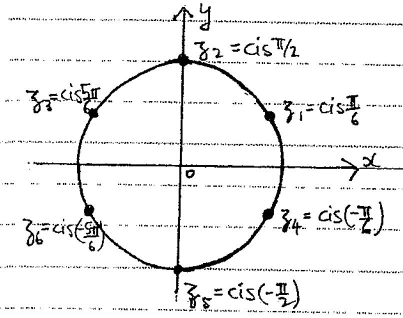
$k=1,\ z_2=\text{cis}\frac{\pi}{2}$

$k=2,\ z_3=\text{cis}\frac{5\pi}{6}$

$k=3,\ z_4=\text{cis}(-\frac{\pi}{6})$

$k=4,\ z_5=\text{cis}(-\frac{\pi}{2})$

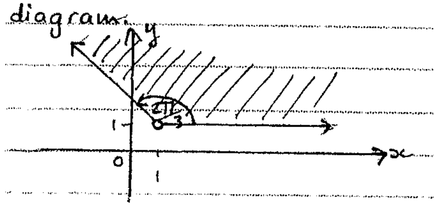
$k=5,\ z_6=\text{cis}(-\frac{5\pi}{6})$



$z^6+1 = (z-z_1)(z-z_2)(z-z_3)(z-z_4)$
$\times (z-z_5)(z-z_6)$
$= (z-cis\frac{\pi}{6})(z-cis(-\frac{\pi}{6}))(z-cis\frac{\pi}{2})(z-cis(-\frac{\pi}{2}))$
$\times (z-cis\frac{5\pi}{6})(z-cis(-\frac{5\pi}{6}))$
$= (z^2-(cis\frac{\pi}{6}+cis(-\frac{\pi}{6}))z+cis\frac{\pi}{6}cis(-\frac{\pi}{6}))$
$\times (z^2-(cis\frac{\pi}{2}+cis(-\frac{\pi}{2}))z+cis\frac{\pi}{2}.cis(-\frac{\pi}{2}))$
$\times (z^2-(cis\frac{5\pi}{6}+cis(-\frac{5\pi}{6}))z+cis\frac{5\pi}{6}cis(-\frac{5\pi}{6}))$

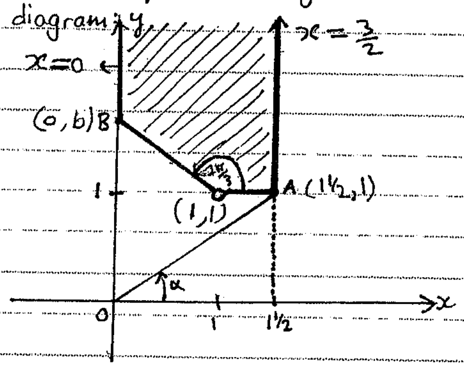
Now, the addition $cis\,\theta + cis(-\theta) = 2\cos\theta$, a real number.
Also, $cis\,\theta . cis(-\theta) = |cis\,\theta|^2 = 1$
$\therefore z^6+1 =$
$(z^2-2\cos\frac{\pi}{6}z+1).(z^2-2\cos\frac{\pi}{2}z+1).(z^2-2\cos\frac{5\pi}{6}z+1)$
$= (z^2-\sqrt{3}z+1)(z^2+1)(z^2+\sqrt{3}z+1)$

c) i) The inequality $0 \le Arg(z-1-i) \le \frac{2\pi}{3}$ shades the region shown in this diagram.



Now, consider the second inequation $|z-1| \le |z-2|$.
let $z = x+iy$.
$|(x-1)+iy| \le |(x-2)+iy|$
$\sqrt{(x-1)^2+y^2} \le \sqrt{(x-2)^2+y^2}$
Since both sides of the inequality are positive, the inequality will hold when they are squared.
$(x-1)^2+y^2 \le (x-2)^2+y^2$
$x^2-2x+1 \le x^2-4x+4$
$2x \le 3$
$x \le \frac{3}{2}$

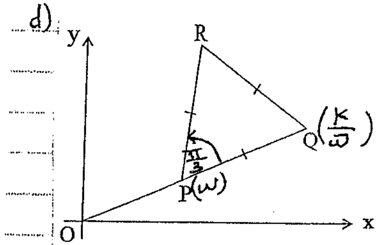
Now, consider the third inequation, $Re(z) \ge 0$. This means $x \ge 0$ for all x.
By satisfying the conditions of all three inequations, we get the following diagram:



ii) z is a complex number in the shaded region. The angle that z makes with the x-axis at the origin is $Arg\,z$.
From the above diagram, the minimum value of $Arg\,z$ is obtained

when z is at A. We can see that $\tan\alpha = \frac{1}{\frac{3}{2}}$
But $Arg\,z = \alpha$
$\therefore Arg\,z = \tan^{-1}(\frac{1}{\frac{3}{2}}) = \tan^{-1}\frac{2}{3}$.
The greatest value of $Arg\,z$ is obtained when $z = 0+yi$ where $y \ge b$ (b is the y coordinate of B)
ie. $Arg\,z = \frac{\pi}{2}$.
$\therefore$ The range of $Arg\,z$ is:
$\tan^{-1}(\frac{2}{3}) \le Arg\,z \le \frac{\pi}{2}$

d)



i) The vectors $\vec{OP}$ and $\vec{OQ}$ represent, respectively, the complex numbers $w, \frac{k}{\bar{w}}$.

Since $Arg(\frac{k}{\bar{w}}) = Arg\,k - Arg\,\bar{w}$
$= 0 - (-Arg\,w)$
$= Arg\,w$,
$\therefore$ the vectors $\vec{OP}$ and $\vec{OQ}$ coincide as they have the point O in common and the same argument.
$\therefore$ O, P, Q are collinear.

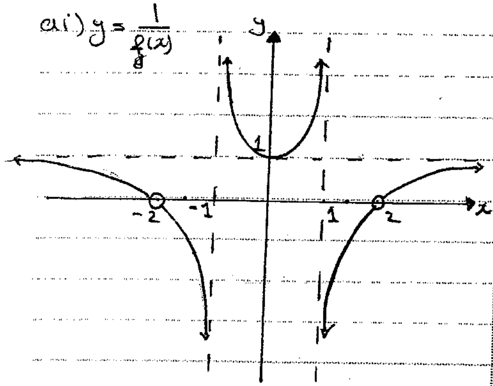
ii) For P & Q to coincide, $|\vec{OP}| = |\vec{OQ}|$
ie. $|\frac{k}{\bar{w}}| = |w|$
but $|w| = 1$
$\therefore |\frac{k}{\bar{w}}| = 1$. But $|\bar{w}| = |w| = 1$
$\therefore |k| = 1$, ie. $k = \pm 1$. But $k > 0$. $\therefore k = 1$

iii) The vector $\vec{PQ}$ represents the complex number
$\frac{k}{\bar{w}} - w = \frac{k}{w\bar{w}}w - w$
$= \frac{k}{|w|^2}w - w$
but $|w| = 1$
$\therefore \vec{PQ}$ is $(k-1)w$.
$\vec{PR}$ is the vector obtained by rotating $\vec{PQ}$ by an angle of $\frac{\pi}{3}$ in an anti clockwise direction.
$\therefore \vec{PR}$ represents the complex number $(k-1)w\,cis\frac{\pi}{3}$
$= (k-1)(\frac{1}{2}+i\frac{\sqrt{3}}{2})w$
But $\vec{OR} = \vec{OP} + \vec{PR}$
$\therefore z = w + (k-1)(\frac{1}{2}+i\frac{\sqrt{3}}{2})w$
$= w[1+(k-1)(\frac{1}{2}+i\frac{\sqrt{3}}{2})]$
$= w[1+(k-1)\frac{1}{2}+i(k-1)\frac{\sqrt{3}}{2}]$
$= w[\frac{k}{2}+\frac{1}{2}+i(k-1)\frac{\sqrt{3}}{2}]$
$= \frac{1}{2}w[(k+1)+i(k-1)\sqrt{3}]$

(2 marks)

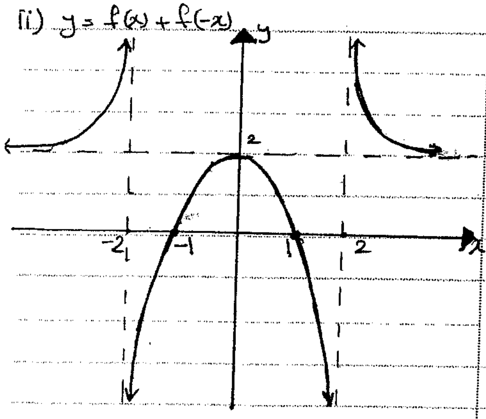
Question 3

a i) $y = \frac{1}{f(x)}$



The x-intercepts of $y=f(x)$ becomes asymptotes, and the maximum at $(0,1)$ becomes the minimum at $(0,1)$.
The asymptotes become points of discontinuity.

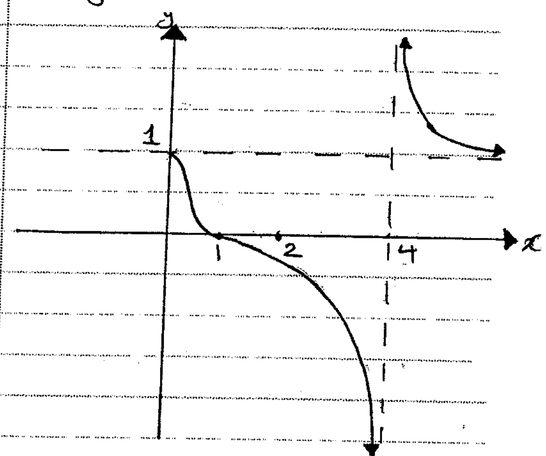
ii) $y = f(x) + f(-x)$



NB: $f(x)$ is even
ie $f(-x) = f(x)$
$\therefore$ $f(x) + f(-x) = f(x) + f(x) = 2f(x)$

This indicates that the y-values of $f(x)$ are multiplied by 2, hence the horizontal asymptote moves to $y=2$, while the vertical asymptotes remain in the same position.

iii) $y = f(\sqrt{x})$



The curve is not defined for $x<0$, so it is restricted to the 1st and 4th quadrants where $x \geq 0$.

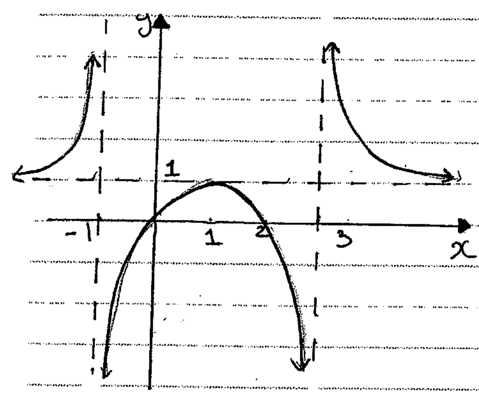
- For $f(\sqrt{0}) = f(0)$ $\therefore$ the curve starts at $(0,1)$
- For $f(\sqrt{1}) = f(1) = 0$ $\therefore$ the curve crosses the x-axis at $x=1$
- For $f(\sqrt{4}) = f(2) \longrightarrow \pm\infty$ $\therefore$ the vertical asymptote is shifted to

$x=4$

note:

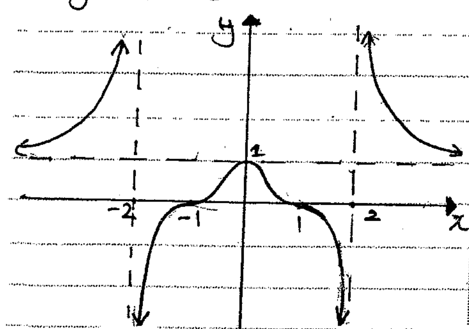
- For $0 < x < 1$ $f(0) < f(\sqrt{x}) < f(1)$
  $\therefore$ $f(x) > 0$
- For $1 < x < 4$ $f(1) \leq f(\sqrt{x}) \leq f(2)$
  $\therefore$ $f(x) < 0$
- For $x > 4$ $f(\sqrt{x}) > f(2) > 1$

iv) $y = f(x-1)$



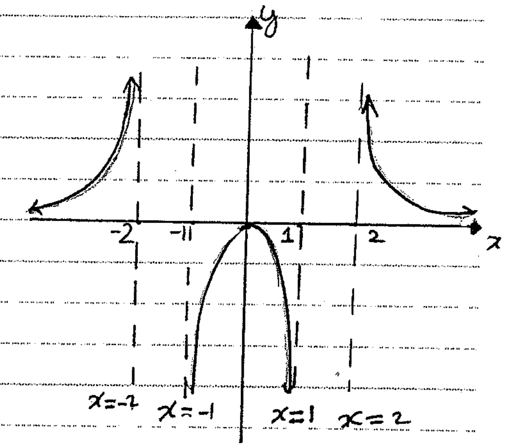
The curve $f(x)$ is shifted one unit to the right.

v) $y = (f(x))^3$



The y-values keep the same sign, hence the x intercepts become horizontal point of inflections.

vi) $y = \ln(f(x))$



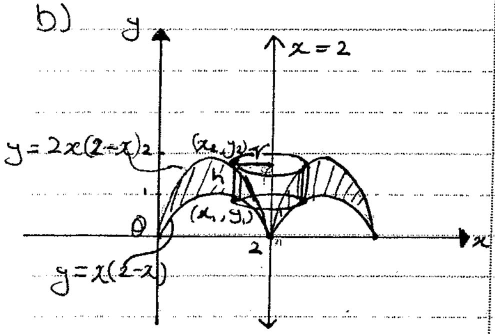
$y = \ln(f(x))$ is defined only when $f(x) > 0$. ie not defined for $-2 \leq x \leq -1$ and $1 \leq x \leq 2$.

x intercepts of $y=f(x)$ become vertical asymptotes for $y = \ln(f(x))$.

note: when $0 < f(x) < 1$
$\ln(f(x)) \leq 0$

As $f(x) \rightarrow 0$, $\ln(f(x)) \longrightarrow -\infty$

b)



Volume of the cylindrical shell obtained by rotating a thin vertical section of thickness δx, with a radius r about $x = 2$.

$\therefore \delta V = 2\pi r h\, \delta x$

but $r = 2 - x$ and $h = y_2 - y_1$

$h = 2x(2-x) - x(2-x)$

$= x(2-x)$

$= 2x - x^2$

$\therefore \delta V = 2\pi(2-x)[2x - x^2]\delta x$

$V = \lim_{\delta x \to 0} \sum_{x=0}^{x=2} \delta V$

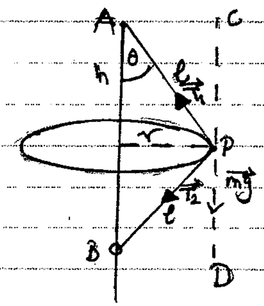
$\therefore V = 2\pi \int_0^2 (2-x)(2x-x^2)\,dx$

$= 2\pi \int_0^2 (4x - 4x^2 + x^3)\,dx$

$= 2\pi \left[2x^2 - \frac{4}{3}x^3 + \frac{x^4}{4}\right]_0^2$

$= \frac{8\pi}{3}$ units^3

(4 marks)



i)

Forces acting on P are $\vec{T_1}$, $\vec{mg}$ and $\vec{T_2}$ as shown.

now. $\angle OAP = \angle CPA = \theta$ (alternate $\angle$'s, AB // CD, vertical lines).

AP = PB (given)

$\therefore \triangle APB$ is isosceles

$\therefore \angle PBA = \theta$ (base angles of isosceles $\triangle$ are equal)

$\therefore \angle BPD = \theta$ (alternate $\angle$'s, AB // CD vertical lines)

Projecting the force at P

Vertically ($\uparrow +$): $T_1 \cos\theta - T_2 \cos\theta - mg = 0$ —(1)

(no acceleration vertically).

Radially: $T_1 \sin\theta + T_2 \sin\theta = m\omega^2 r$ —(2)

(

The forces acting on P are $2\vec{mg}$, $\vec{T_2}$ and the normal reaction $\vec{N}$ of rod AB.

Projecting the force at B

Vertically ($\uparrow +$): $T_2 \cos\theta - 2mg = 0$

$T_2 \cos\theta = 2mg$ —(3)

$\therefore T_2 = \frac{2mg}{\cos\theta}$

Radially with positive towards centre: $N - T_2 \sin\theta = 0$

$\therefore T_2 \sin\theta = N$ By substituting (3) into (1)

$T_1 \cos\theta - 2mg - mg = 0$

$\therefore T_1 \cos\theta = 3mg$ $\therefore T_1 = \frac{3mg}{\cos\theta}$

(3 marks)

ii) Substituting the values of T_1 and T_2 obtained from part (i) into (2).

$\frac{3mg}{\cos\theta} \times \sin\theta + \frac{2mg}{\cos\theta} \times \sin\theta = m\omega^2 r$

$3mg\tan\theta + 2mg\tan\theta = m\omega^2 r$

$5mg\tan\theta = m\omega^2 r$ $\therefore \tan\theta = \frac{m\omega^2 r}{5mg}$

$\tan\theta = \frac{\omega^2 r}{5g}$

from $\triangle OAP$ $\tan\theta = \frac{r}{h}$

$\therefore \frac{r}{h} = \frac{\omega^2 r}{5g}$ $\therefore \omega^2 h = 5g$

(2 marks)

iii) The string will not break if T_1 is less than $6mg$, that is $\frac{3mg}{\cos\theta} < 6mg$

$\therefore \frac{1}{\cos\theta} < 2$

as θ is acute, $\cos\theta > 0$

$\therefore$ both sides are positive. By taking reciprocal of both sides the inequation would hold & we get

$\cos\theta > \frac{1}{2}$. But from $\triangle OAP$

$\cos\theta = \frac{h}{\ell}$ $\therefore \frac{h}{\ell} > \frac{1}{2}$

$\therefore h > \frac{\ell}{2}$ But $h = \frac{5g}{\omega^2}$ (from (ii))

$\therefore \frac{5g}{\omega^2} > \frac{\ell}{2}$

As both sides are positive by talking the reciprocal the inequation would still hold.

$\frac{\omega^2}{5g} < \frac{2}{\ell}$ $\therefore \omega^2 < \frac{10g}{\ell}$

(2 marks)

b) $x^3 - x^2 + 2x - 1 = 0$

S_1 $\alpha + \beta + \gamma = 1$ S_2 $\alpha\beta + \alpha\gamma + \beta\gamma = 2$

S_3 $\alpha\beta\gamma = 1$

now $\frac{\alpha}{\beta\gamma} + \frac{\beta}{\alpha\beta} + \frac{\gamma}{\alpha\beta} = \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha\beta\gamma}$

$= \frac{(\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)}{\alpha\beta\gamma}$

from S_1, S_2 and S_3

$= \frac{(1)^2 - 2(2)}{1} = -3$

(3 marks)

c) i) let $P(x) = x^3 - 12x + k$

$\therefore P'(x) = 3x^2 - 12$ if two roots of the equation are equal, then $P(x) = P'(x) = 0$

$\therefore 3x^2 - 12 = 0$ $\therefore x = \pm 2$

now: $P(2) = 8 - 24 + k = 0$ $\therefore k = 16$

and $P(-2) = -8 + 24 + k = 0$ $\therefore k = -16$

In order for the equation to have two roots $k = \pm 16$

$\therefore$ If $k \neq \pm 16$ no two roots α, β, γ are equal (2 marks)

ii) $P(x) = x^3 - 12x + k$

α is a root $\therefore P(\alpha) = 0$

$\therefore P(1) = 0 \quad \therefore 1 - 12 + k = 0$

$\therefore k = 11$

The equation can be written as $x^3 - 12x + 11 = 0$ where α, β, γ are the roots

$S_1 = \alpha + \beta + \gamma = 0$

$S_2 = \alpha\beta + \alpha\gamma + \beta\gamma = -12$

$S_3 = \alpha\beta\gamma = -11$

$\therefore \alpha^4 - 12\alpha^2 + 11\alpha = 0$ that is $\alpha^4 = 12\alpha^2 - 11\alpha$

similarly $\beta^4 = 12\beta^2 - 11\beta$

and $\gamma^4 = 12\gamma^2 - 11\gamma$

$(\alpha^4 + \beta^4 + \gamma^4) = 12(\alpha^2 + \beta^2 + \gamma^2) - 11(\alpha + \beta + \gamma)$

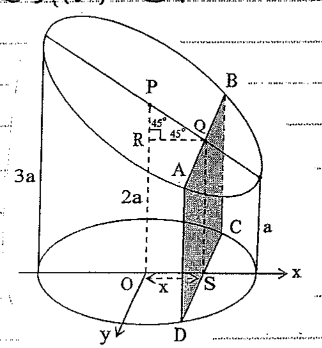
$= 12[(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)] - 11(\alpha + \beta + \gamma)$

$= 12[0 - 2(-12)] + 0 \times -11$

$= 288$

(3 marks)

Question 5.



i) let $S(x, 0)$ be the point of intersection of CD with the x-axis. let Q the point on AB which is vertically above S.

The equation of the circular base is $x^2 + y^2 = a^2$. Since C & D have x coordinates x, $\therefore$ their y coordinates would be $y = \pm\sqrt{a^2 - x^2}$

$\therefore CD = 2y = 2\sqrt{a^2 - x^2}$

Let R be the foot of the perpendicular from Q to OP in ΔPQR

$\angle PQR = 180 - 45 - 90 = 45$

$\therefore \Delta PQR$ is an isosceles triangle. $\therefore PR = RQ = x$ (equal sides of isosceles triangle).

$\therefore SQ = OP - PR = 2a - x$

now: area of slice ABCD $= 2\sqrt{a^2 - x^2}\,(2a - x)$.

(3 marks)

ii) Let δV be the volume of the rectangular slice with thickness δx. $\therefore \delta V = A\,\delta x$

$\delta V = 2\sqrt{a^2 - x^2}\,(2a - x)\,\delta x$

$$V = \lim_{\delta x \to 0} \sum_{x=-a}^{x=a} \delta V$$

$$= \int_{-a}^{a} 2\sqrt{a^2 - x^2}\,(2a - x)\,dx$$

$$= 4a\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx - 2\int_{-a}^{a} x\sqrt{a^2 - x^2}\,dx$$

let $I = \int_{-a}^{a} \sqrt{a^2 - x^2}\,dx$

note: $\sqrt{a^2 - x^2}$ is the equation of a semi-circle with radius a. $\therefore$ Area of the semi-circle is $A = \frac{\pi a^2}{2}$ $\therefore I = \frac{\pi a^2}{2}$

let $J = \int_{-a}^{a} x\sqrt{a^2 - x^2}\,dx$

let $u = a^2 - x^2$

$-\frac{1}{2}du = x\,dx \qquad J = -\frac{1}{2}\int_0^0 \sqrt{u}\,du$

$J = 0$

Note: (J is the integral of an odd function on symmetric interval must equal 0)

$\therefore V = 4a\left(\frac{\pi a^2}{2}\right) - 2 \times 0$

$= 2\pi a^3$ units^3

(3 marks)

b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ using implicit differentiation

$\frac{2x}{a^2} + \frac{2y}{a^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x}{a^2} \times \frac{b^2}{y} = \frac{-b^2x}{a^2y}$

at $P(x, y)$ $m_{tangent} = \frac{-b^2x_1}{a^2y_1}$

$\therefore$ Equation of tangent at P is $y - y_1 = \frac{-b^2x_1}{a^2y_1}(x - x_1)$

$a^2y_1y - a^2y_1^2 = -b^2x_1x + b^2x_1^2$

$b^2x_1x + a^2y_1y = b^2x_1^2 + a^2y_1^2$ —(1)

As $P(x_1, y_1)$ lies on the ellipse, its coordinates verify the ellipse.

$\therefore \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

$\therefore b^2x_1^2 + a^2y_1^2 = a^2b^2$. By substituting into (1)

$\therefore b^2x_1x + a^2y_1y = a^2b^2$

$\therefore \frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$ (2 marks)

ii) By similar work the equation of tangent at Q is

$\frac{x_2x}{a^2} + \frac{y_2y}{b^2} = 1$

now: consider the equation of a line which is in the form $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$ — (2)

from the equation TP $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ we can see that $P(x_1, y_1)$ lie on (2).

Similarly from the equation TQ $\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1$ we can see that $Q(x_2, y_2)$ lie on the line $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$.

Hence that line is the chord of contact.

(2marks)

iii) If PQ is the focal chord it must pass through the focus $(ae, 0)$ or the focus $(-ae, 0)$. By substituting $(ae, 0)$ into the equation of chord of contact PQ, we get

$$\frac{a e x_0}{a^2} + \frac{0 y_0}{b^2} = 1 \quad \therefore x = \frac{a}{e}$$

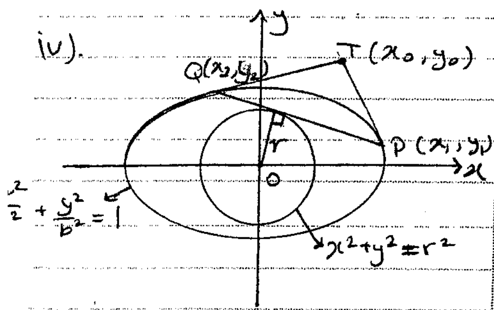
The point T moves on the directrix $x = \frac{a}{e}$

similarly by substituting $(-ae, 0)$ we get $x = -\frac{a}{e}$ that is the point T moves on the directrix $x = -\frac{a}{e}$

Hence if PQ is the focal chord, the point T must move on either directrices $x = \pm\frac{a}{e}$.

(2 marks).

iv).



If PQ is a tangent to the circle $x^2+y^2=r^2$ that means the distance from the origin $(0,0)$, which is the centre of the circle to the chord of contact PQ is r.

Using the perpendicular distance formula

$$\frac{\left|0\times\frac{x_0}{a^2} + \frac{0\, y_0}{b^2} - 1\right|}{\sqrt{\left(\frac{x_0}{a^2}\right)^2 + \left(\frac{y_0}{b^2}\right)^2}} = r$$

by squaring both sides.

$$\frac{1}{\left(\frac{x_0}{a^2}\right)^2+\left(\frac{y_0}{b^2}\right)^2} = r^2$$

$$r^2\left[\left(\frac{x_0}{a^2}\right)^2 + \left(\frac{y_0}{b^2}\right)^2\right] = 1$$

$$\frac{x_0^2}{\left(\frac{a^2}{r}\right)^2} + \frac{y_0^2}{\left(\frac{b^2}{r}\right)^2} = 1$$

This is the equation of an ellipse. Hence if PQ is tangent to the circle $x^2+y^2=r^2$, the point T would be moving on an ellipse.

(3 marks).

Question 6 step 1

a) For $n=1$, $u_1 = 1 < \left(\frac{7}{4}\right)^1$

For $n=2$, $u_2 = 1 < \left(\frac{7}{4}\right)^2$

For $n=1$, from $u_{n+2} = u_{n+1} + u_n$, by substituting $n=1$

$u_3 = u_2 + u_1 = 1+1 = 2$ but $2 < \left(\frac{7}{4}\right)^3$

$\therefore u_3 < \left(\frac{7}{4}\right)^3$. $\therefore$ The statement is true for $n=1$, $n=2$, & $n=3$.

Step 2: Assume that the statement is true for $n=k-1$ & $n=k$.

ie $u_{k-1} < \left(\frac{7}{4}\right)^{k-1}$ & $u_k < \left(\frac{7}{4}\right)^k$

Our aim is to prove true for $n=k+1$. ie $u_{k+1} < \left(\frac{7}{4}\right)^{k+1}$

By substituting $n=k-1$ into the formula we obtain

$u_{k+1} = u_k + u_{k-1}$

but from the assumptions

$u_k < \left(\frac{7}{4}\right)^k$ & $u_{k-1} < \left(\frac{7}{4}\right)^{k-1}$

$\therefore u_{k+1} < \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1}$

$u_{k+1} < \left(\frac{7}{4}\right)^{k-1}\left[\frac{7}{4} + 1\right]$

$u_{k+1} < \left(\frac{7}{4}\right)^{k-1}\left[\frac{11}{4}\right]$

but $\frac{11}{4} < \frac{49}{16}$

$\therefore u_{k+1} < \left(\frac{7}{4}\right)^{k-1} \frac{49}{16}$

$\therefore u_{k+1} < \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}\right)^2$

$u_{k+1} < \left(\frac{7}{4}\right)^{k+1}$

Hence if the statement is true for $n=k-1$ and $n=k$, it must also be true for $n=k+1$.

Step 3: The statement is true for $n=1$, $n=2$, & $n=3$, by induction it is also true for $n=4, 5, 6\ldots$ & so on. Hence the statement is true for integers $n \geq 1$.

(3 marks)

b) $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 1 + \frac{a}{b} + \frac{a}{c} + 1 + \frac{b}{a} + \frac{b}{c} + 1 + \frac{c}{a} + \frac{c}{b}$

$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 3 + \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{c}{b} + \frac{b}{c}\right) + \left(\frac{a}{c} + \frac{c}{a}\right)$.

note: $(a-b)^2 \geq 0 \quad \therefore a^2 - 2ab + b^2 \geq 0$

$a^2 + b^2 \geq 2ab$ (divide by ab)

$\frac{a}{b} + \frac{b}{a} \geq 2$ similarly $\frac{b}{c} + \frac{c}{b} \geq 2$ & $\frac{a}{c} + \frac{c}{a} \geq 2$ $\therefore$ By substituting into the above we get

$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 3+2+2+2$

$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$

$\therefore \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c}$ (2 marks)

ci) $I = \int_0^{\frac{\pi}{4}} (\tan x)^{2k} \sec^2 x \, dx$

let $u = \tan x$ $du = \sec^2 x \, dx$

for $x = \frac{\pi}{4}$ $u = \tan\frac{\pi}{4} = 1$

for $x = 0$, $u = \tan 0 = 0$

$\therefore I = \int_0^1 u^{2k} du = \left[\frac{u^{2k+1}}{2k+1}\right]_0^1$

$= \frac{1}{2k+1}$

(1 mark).

cii) $J = \int_0^{\frac{\pi}{4}} (\sec x)^{2n+2} dx$

note: $(\sec x)^{2n+2} = (\sec x)^{2n} \cdot \sec^2 x$

$= (\sec^2 x)^n \cdot \sec^2 x = (1+\tan^2 x)^n \cdot \sec^2 x$

From binomial theorem the expansion of $(1+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k$ let $y = \tan^2 x$

we get: $(1+\tan^2 x)^n = \sum_{k=0}^{n} \binom{n}{k} \tan^{2k} x$

$\therefore (\sec x)^{2n+2} = \sum_{k=0}^{n} \binom{n}{k} \tan^{2k} x \cdot \sec^2 x$

$\therefore J = \int_0^{\frac{\pi}{2}} \sum_{k=0}^{n} \binom{n}{k} \tan^{2k}x \cdot \sec^2 x\, dx$

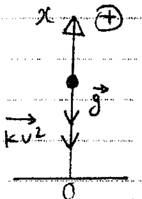
$J = \sum_{k=0}^{n} \binom{n}{k} \int_0^{\frac{\pi}{2}} \tan^{2k} x \cdot \sec^2 x\, dx$

$\therefore J = \sum_{k=0}^{n} \binom{n}{k} I$ (from c) $I = \frac{1}{2k+1}$)

$\therefore J = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2k+1}$

(3 marks).

d)



Forces acting on the particle, when in its upwards motion are $\vec{g}$ and $\vec{kv^2}$.

Projecting $\uparrow \oplus$: $-kv^2 - g = \frac{d^2x}{dt^2}$

As $\frac{d^2x}{dt^2} = v\frac{dv}{dx}$

$\therefore v\frac{dv}{dx} = -(kv^2+g)$

now: $-\int dx = \int \frac{v}{g+kv^2}\, dv$

$-x = \frac{1}{2k}\int \frac{2kv}{g+kv^2}\, dv$

$\therefore -2kx = \ln(g+kv^2) + c$

But $t=0$, $x=0$, $v=V_0$

$\therefore 0 = \ln(g+kV_0^2) + c$

$\therefore c = -\ln(g+kV_0^2)$

$\therefore -2kx = \ln(g+kv^2) - \ln(g+kV_0^2)$

$\therefore 2kx = \ln(g+kV_0^2) - \ln(g+kv^2)$

$\therefore 2kx = \ln\left(\frac{g+kV_0^2}{g+kv^2}\right)$

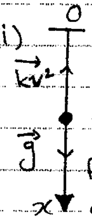
At maximum height $V=0$, when $x=h$, $v=0$

$\therefore 2kh = \ln\left(\frac{g+kV_0^2}{g+0}\right)$

$\therefore h = \frac{1}{2k}\cdot\ln\left(\frac{g+kV_0^2}{g}\right)$

(2 marks).

ii)



consider a new origin O at the maximum height, that is $x=0$ and as the particle is falling from O, at $t=0$, $V=0$.

The forces acting on the particle in its downwards motion are $\vec{g}$ and $\vec{kv^2}$.

projecting $\oplus\downarrow$: $g - kv^2 = \frac{d^2x}{dt^2}$

let $\frac{d^2x}{dt^2} = 0$ (as terminal velocity V occurs when $\frac{d^2x}{dt^2}=0$).

ie $g - kV^2 = 0$ $\therefore V = \sqrt{\frac{g}{k}}$

$V > 0$ as the particle is moving in a positive direction.

(1 mark).

iii) The equation of the downwards motion is $g - kv^2 = \frac{d^2x}{dt^2}$

let $\frac{d^2x}{dt^2} = v\frac{dv}{dx}$ $\therefore g - kv^2 = v\frac{dv}{dx}$

$\int dx = \int \frac{v\,dv}{g+kv^2}$ $\therefore x = -\frac{1}{2k}\int \frac{-2kv}{g+kv^2}\,dv$

$\therefore -2kx = \ln(g+kv^2) + c$

but when $x=0$, $v=0$

$\therefore 0 = \ln g + c$ $\therefore c = -\ln g$

$\therefore -2kx = \ln(g+kv^2) - \ln g$

$2kx = \ln g - \ln(g+kv^2)$

$\therefore 2kx = \ln\left(\frac{g}{g+kv^2}\right)$

now, when $x=h$ the particle reaches the ground ie $v=U$

$\therefore 2kh = \ln\left(\frac{g}{g+kU^2}\right)$

$h = \frac{1}{2k}\cdot\ln\left(\frac{g}{g+kU^2}\right)$

but from part i) $h = \frac{1}{2k}\ln\left(\frac{g+kV_0^2}{g}\right)$

$\therefore \frac{1}{2k}\ln\left(\frac{g}{g-kU^2}\right) = \frac{1}{2k}\ln\left(\frac{g+kV_0^2}{g}\right)$

$\therefore \ln\left(\frac{g}{g+kU^2}\right) = \ln\left(\frac{g+kV_0^2}{g}\right)$

ie $\frac{g}{g+kU^2} = \frac{g+kV_0^2}{g}$

$1 + \frac{k}{g}V_0^2 = \frac{1}{1+\frac{k}{g}U^2}$ (By dividing both sides by g)

But $V = \sqrt{\frac{g}{k}}$ $\therefore V^2 = \frac{g}{k}$

$\therefore \frac{1}{V^2} = \frac{k}{g}$ $\therefore 1 + \frac{V_0^2}{V^2} = \frac{1}{1+\frac{U^2}{V^2}}$

$\left(1+\frac{V_0^2}{V^2}\right)\left(1+\frac{U^2}{V^2}\right) = 1$

$1 + \frac{V_0^2}{V^2} - \frac{U^2}{V^2} - \frac{U^2V_0^2}{V^4} = 1$

$V_0^2 - U^2 - \frac{V_0^2U^2}{V^2} = 0$ $\therefore V_0^2 = U^2 + \frac{V_0^2U^2}{V^2}$

$\therefore V_0^2 = U^2\left(1+\frac{V_0^2}{V^2}\right)$

$U^2 = \frac{V_0^2}{1+\frac{V_0^2}{V^2}}$ (by multipling by V^2)

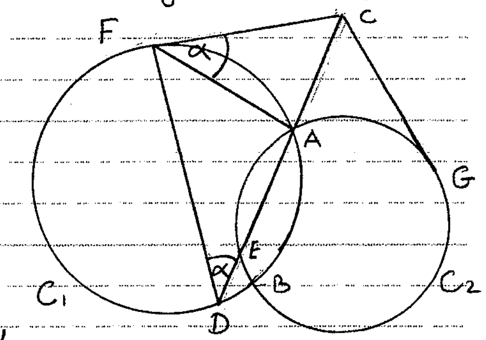
$U^2 = \frac{V_0^2V^2}{V^2+V_0^2}$ (3 marks).

Question 7.

Data: C_1 & C_2 intersect at A & B.

Aim: To prove $CF^2 = CA \times CD$

construction: Join FA & FD.



i) Proof: In this part CF is a tangent to C_1

consider $\triangle CFA$ & $\triangle CDF$.

let $\angle CFA = \alpha$, $\angle CDF = \alpha$ ($\angle$ in the alternate segment between tangent CF and chord FA).

$\angle ACF$ is common angle.

$\angle FAC = \angle CFD$ (remaining $\angle$ of $\triangle CFA$ & $\triangle CDF$).

$\therefore \triangle CFA \,|||\, \triangle CDF$ (Equiangular)

Since the $\triangle$'s are similar their corresponding sides are proportional.

$\therefore \frac{CF}{CD} = \frac{CA}{CF}$ $\therefore CF^2 = CD\cdot CA$.

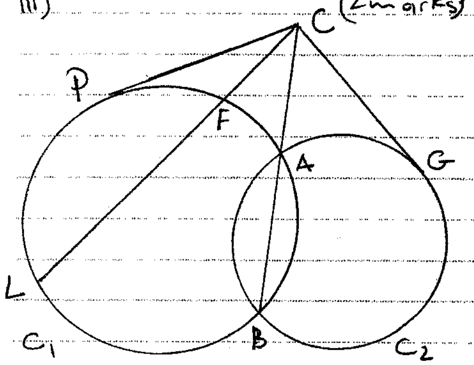
(2 marks)

ii) In this part CF is tangent to (C_1) $\therefore CF^2 = CA.CD$ (proven in part i). Similarly since CG is tangent to C_2, $\therefore CG^2 = CA.CE$

But $CF = CG$ (given) $\therefore CA\cdot CD = CA\cdot C$

$\therefore CD = CE$. & since the points C, E, & D are colliner points $\therefore E$ coincides with D. But since E belongs to C_2 & D belongs to C_1, & both are the second intersection of CA with both circles, they must lie on the point of intersection of C_1 & C_2 which is B. Hence the points B, E & D coincide under the provided conditions. (2 marks)

iii)



Assume on the contrary that CF is not a tangent to C_1 but CP is tangent to C_1.

construct: CP, tangent to C_1.

Extend CF to meet C_1 at L, under our assumption.

Proof: By similar argument to that in part (i) we have $CP^2 = CA \times CB$ and $CP^2 = CF \times CL$

We are given $CF^2 = CA \times CB$

$\therefore CF \times CL = CA \times CB = CF^2$

$\therefore CF \times CL = CF^2$, $\therefore CL = CF$

Now: C, F & L are colliner

$\therefore$ the points F & L coincide,

$\therefore CF$ must be a tangent at point F to C_1.

(3 marks)

7b i) $f(x) = x^n e^{-x}$ where $x > 0$

using the product rule

$u = x^n \qquad v = e^{-x}$

$u' = nx^{n-1} \qquad v' = -e^{-x}$

$$\frac{df}{dx} = nx^{n-1}e^{-x} - x^n e^{-x}$$

$$\frac{df}{dx} = x^{n-1}e^{-x}(n-x)$$

let $\frac{df}{dx} = 0$ to find possible turning points.

since e^{-x} is always positive and $x > 0$ $\frac{df}{dx}$ relies on the sign of $(n-x)$.

$\therefore n = x$

x	$n-1$	n	$n+1$
$\frac{df}{dx}$	$+$	0	$-$
$f(x)$	↗	Max	↘

By testing $x = n+1$ & $x = n-1$ we obtain the table.

$\therefore f(x) < f(n)$ (since n is maximum point, from table).

$\therefore x^n e^{-x} < n^n e^{-n}$ for $x \neq n$.

(3 marks)

ii) let $x = n+1$ in the equation $x^n e^{-x} < n^n e^{-n}$

we get $(n+1)^n e^{-(n+1)} < n^n e^{-n}$

divide both sides by $e^{-n} > 0$

we get $(n+1)^n e^{-1} > n^n$

taking reciprocals we get

$$\frac{e}{(n+1)^n} > \frac{1}{n^n} \quad \therefore e > \left(\frac{n+1}{n}\right)^n$$

$\therefore e > \left(1+\frac{1}{n}\right)^n \therefore \left(1+\frac{1}{n}\right)^n < e$ (2 marks)

iii) $g(x) = x^{n+1} e^{-x}$, $x > 0$

using the product rule.

$u = x^{n+1} \qquad v = e^{-x}$

$u' = (n+1)x^n \qquad v' = -e^{-x}$

$$\frac{dg}{dx} = (n+1)x^n e^{-x} - x^{n+1}e^{-x}$$

$$\frac{dg}{dx} = x^n e^{-x}(n+1-x)$$

let $\frac{dg}{dx} = 0$ to find possible turning points.

$x^n e^{-x}(n+1-x) = 0$

now e^{-x} is always positive & $x > 1$ $\therefore \frac{dg}{dx}$ relise on the sign of $(n+1-x)$

$\therefore x = n+1$

x	n	$n+1$	$n+2$
$\frac{dg}{dx}$	$+$	0	$-$
$g(x)$	↗	max	↘

By testing $x = n$ & $x = n+2$.

since $g(x)$ is increasing when, +1 $\therefore g(n) < g(n+1)$

$n^{n+1} e^{-n} < (n+1)^{n+1} e^{-(n+1)}$

by dividing by e^{-n} on both sides.

$n^{n+1} < (n+1)^{n+1} e^{-1}$

$$\frac{n^{n+1}}{(n+1)^{n+1}} < \frac{1}{e}$$

By taking reciprocals on both sides we get:

$$\frac{(n+1)^{n+1}}{n^{n+1}} > e$$

$$\left(\frac{n+1}{n}\right)^{n+1} > e \quad \therefore e < \left(\frac{n+1}{n}\right)^{n+1}$$

$\therefore e < \left(1+\frac{1}{n}\right)^{n+1}$ from part (ii) we have $\left(1+\frac{1}{n}\right)^n < e$

$$\therefore \left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$$

(3 marks).

Question 8.

a i) $f(x) = x^{a+1} - 1 - (a+1)(x-1)$ where $x > 0$ & $a > 0$

$\frac{df}{dx} = (a+1)x^a - (a+1)$ gradient function

let $\frac{df}{dx} = 0$ to find possible turning points. $\therefore (a+1)x^a - (a+1) = 0$

Since $x > 0$ & $a > 0$ $x = 1$ is the only turning point.

now $\frac{d^2f}{dx^2} = a(a+1)x^{a-1}$

when $x = 1$ $\frac{d^2f}{dx^2} > 0$ $\therefore$ the curve is concave up. $\therefore x = 1$ is a minimum turning point.

(2 marks).

ii) From part (i) $x = 1$ is the minimum turning point $\therefore f(x) \geq f(1)$

now $f(1) = 0$ $\therefore f(x) \geq 0$

$\therefore x^{a+1} - 1 - (a+1)(x-1) \geq 0$

$\therefore x^{a+1} \geq 1 + (a+1)(x-1)$

(1 mark).

iii) let $x = \frac{p}{q}$ (where $q > 0$ & $p > 0$) in the inequation in part (ii).

$x^{a+1} > 1 + (a+1)(x-1)$

note: as $q > p \geq 0$, $x \neq 1$

$\therefore \left(\frac{p}{q}\right)^{a+1} > 1 + (a+1)\left(\frac{p}{q}-1\right)$

$\frac{p^{a+1}}{q^{a+1}} > 1 + (a+1)\left(\frac{p}{q}-1\right)$

$p^{a+1} > q^{a+1} + (a+1) q^{a+1}\left(\frac{p}{q}-1\right)$

$p^{a+1} - q^{a+1} > (a+1) q^a (p-q)$

note: $p - q < 0$ $\therefore$ the inequation would change.

$\frac{p^{a+1}-q^{a+1}}{p-q} < (a+1) q^a$

$\frac{q^{a+1}-p^{a+1}}{q-p} < (a+1)q^a$ —(1)

let $x = \frac{q}{p}$ (where $q > p > 0$)

in the inequation from part (ii)

$x^{a+1} > 1 + (a+1)(x-1)$

$\therefore \left(\frac{q}{p}\right)^{a+1} > 1 + (a+1)\left(\frac{q}{p}-1\right)$

$\frac{q^{a+1}}{p^{a+1}} > 1 + (a+1)\left(\frac{q}{p}-1\right)$

$q^{a+1} > p^{a+1} + (a+1)p^{a+1}\left(\frac{q}{p}-1\right)$

$q^{a+1} - p^{a+1} > (a+1)p^a(q-p)$

$\frac{q^{a+1}-p^{a+1}}{q-p} > (a+1)p^a$ —(2)

The desired conclusion follows from the inequations (1) & (2).

$(a+1)p^a < \frac{q^{a+1}-p^{a+1}}{q-p} < (a+1)q^a$

(3 marks).

iv) In the inequation in part (iii) when $p=0$, & $q=1$

$(a+1)0^a < 1^{a+1} < (a+1)2^a$

when $p=1$ & $q=2$

$(a+1)1^a < 2^{a+1} - 1^{a+1} < (a+1)2^a$

when $p=2$ & $q=3$

$(a+1)2^a < 3^{a+1} - 2^{a+1} < (a+1)3^a$

when $p=n-1$ & $q=n$

$(a+1)(n-1)^a < n^{a+1} - (n-1)^{a+1} < (a+1)n^a$

By adding all the above inequalities we get

$(a+1)[1^a+2^a+\dots(n-1)^a] < n^{a+1} < (a+1)[1^a+2^a+\dots n^a]$

$1^a+2^a+\dots(n-1)^a < \frac{n^{a+1}}{a+1} < 1^a+2^a+\dots n^a$

(3 marks).

b) i) The numbers divisible by 3 in the barrel are 3, 6, 9, 12 & 15.

In a single game, the probability to obtain the 3 balls divisible by 3 is $\frac{^5C_3}{^{15}C_3} = \frac{2}{91}$

$\therefore$ In a single game

The probability not to get the three balls divisible by 3 is $\frac{81}{91}$. Hence to obtain 3 numbers divisible by 3 in atleast 4 games out of five, we use binomal probabilit

$^5C_4\left(\frac{2}{91}\right)^4\left(\frac{81}{91}\right) + {}^5C_5\left(\frac{2}{91}\right)^5 \doteq 1.04 \times 10^{-6}$

(3 marks).

ii) Dividing the balls into 3 groups,

group 1: 3, 6, 9, 12, 15 (all divisible by 3).

group 2: 1, 4, 7, 10, 13 (each gives a remainder of 1 when divided by 3).

group 3: 2, 5, 8, 11, 14 (each gives a remainder of 2 when divided by 3)

To get 3 numbers, whose sum is divisible by 3, we have 2 cases.

Case 1: The three numbers have to be all

from group 1 or group 2 or group 3. The total possibilities for this case is:

$3 \times {}^5C_3 = 10 \times 3 = 30$

Case 2: Each of the numbers must be from a different group. The total possibilities are

${}^5C_1 \times {}^5C_1 \times {}^5C_1 = 5^3 = 125$

$\therefore$ There are $125 + 30 = 155$ possible ways to obtain a sum divisible by 3, when 3 balls are drawn.

The total possible ways to draw the 3 balls regardless of their sum is ${}^{15}C_3 = 455$

$\therefore$ The Probability required is

$\frac{155}{455} = \frac{31}{91}$. (3 marks).