HIGHER SCHOOL CERTIFICATE EXAMINATION TRIAL PAPER

2005

MATHEMATICS

EXTENSION 1

Time Allowed – Two Hours
(Plus 5 minutes reading time)

Directions to Candidates

- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Board-approved calculators may be used.

YEAR 12 – TRIAL 2005 – EXTENSION 1

QUESTION 1

MARKS

a) Given that $\sin 2\alpha = \frac{7}{10}$ and $\sin\alpha = \frac{2}{5}$, find the value of $\cos\alpha$ — 2

b) Evaluate $\lim_{x\to 0} \frac{\sin 2x}{3x}$ — 2

c) The variable point $P\,(t + \frac{1}{t}, t^2 + \frac{1}{t^2})$ lies on a parabola. — 3
Find the equation of this parabola.

d) Find the coordinates of the point A that divides the interval joining $(-3, -4)$ and $(-2, 2)$ externally in the ratio 3 : 5 — 2

e) Use the substitution $u = 1 + \sin^2 x$ to evaluate — 3

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+\sin^2 x}\,dx$$

QUESTION 2

MARKS

a) Consider the curve $y = 4\sin^{-1}\frac{x}{3}$

i) State the domain and the range. — 2

ii) Sketch the graph of the curve. — 1

b) Find $\frac{d}{dx}\tan^{-1}(\sin x)$ — 2

c) Solve the inequation $\frac{1}{x+3} > \frac{1}{x+4}$ — 3

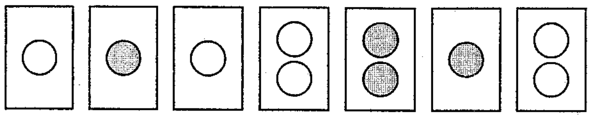
d) Evaluate $\int_{\frac{\pi}{5}}^{\frac{\pi}{2}} 4\cos^2 5x\,dx$ — 2

e) Find the coefficient of x^3 in the binomial expansion of — 2

$$\left(x^2 + \frac{2}{x^3}\right)^{14}$$

QUESTION 3 **MARKS**

a) How many arrangements along a straight line can be made using all the cards shown below? 1



b) How many different ways can a group of 5 letters be chosen from the word Kingdom if the letter k is to be included in every group? 2

c) Given that the roots of the equation 3

$2x^4 - 5x^3 + kx^2 + 45x - 18 = 0$ are $\alpha, \frac{1}{\alpha}, \beta, -\beta$.

Find the value of k.

d) The displacement of a particle moving along the x axis is given by

$x = 4 - 3\sin 2t$, where t is the time in seconds.

i) Show that the motion of the particle is simple harmonic. 2

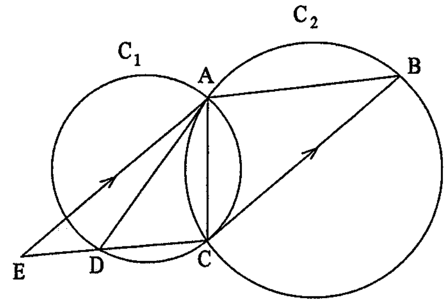
ii) How long does it take the particle to first reach its minimum displacement? 1

e) The equation $4e^{-x} + \sin x - 1 = 0$ has a root near $x = 2$. Use one application of Newton's method to find a better approximation for this root. 3

Write your answer correct to four decimal places.

QUESTION 4 **MARKS**

a) CB is a tangent to circle C_1 and AD is a tangent to circle C_2.
AE is parallel to BC



NOT TO SCALE

Show that ABCE is a parallelogram. 4

b) Jasmine puts a pot of boiled water on the bench in her kitchen where the temperature is 25°C.
The temperature T of the water, measured in degrees Celsius, decreases according to the equation
$\frac{dT}{dt} = -k(T - 25)$ where k is a positive constant and t is the time in minutes.

i) If the initial temperature of the boiled water is 90°C, and it cools to 70°C after 5 minutes, find the value of k. 2

ii) How long will it take for the boiled water to cool from its initial temperature of 90°C to 50°C? 3

c) A basket contains tennis balls 30% of which are never used.
John selects 8 balls from the basket. What is the probability that

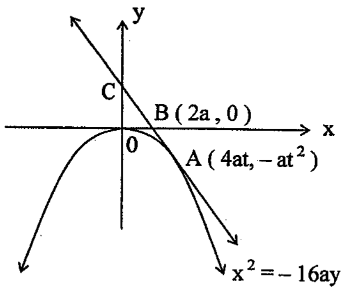
i) only two of them are never used? 1

ii) no more than two are never used? 1

iii) at least one of the balls never used? 1

QUESTION 5 MARKS

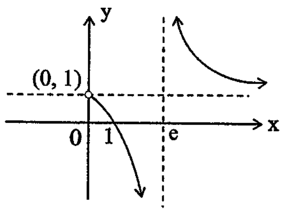
a) The point A $(4at, -at^2)$, is a variable point on the parabola $x^2 = -16ay$.
The tangent at A meets the x-axis at the point B $(2a, 0)$ and the y-axis at C.



i) Find the equation of the tangent to the parabola at A. 2

ii) Find the coordinates of the point C. 2

b) The graph of the function $f(x) = \dfrac{\ln x}{\ln x - 1}$ is as shown below.



i) Explain why f (x) has an inverse function. 1

ii) Find the equation of the inverse function $y = f^{-1}(x)$. 2

c) Sand is being poured from the back of a truck onto flat ground at the rate of 0.2m^3 per minute. As it falls, it forms a pile in the shape of a cone where the semi-vertical angle is 30°.

i) Find the rate at which its height is increasing when the height of the pile is 8m. 3

ii) Find the rate at which its base area is increasing when the radius is 10m. 2

QUESTION 6 MARKS

a) The constant terms in each of the following binomials 3

$\left(x + \dfrac{4}{x}\right)^{2n}$ and $\left(\dfrac{x}{a} + \dfrac{b}{x}\right)^{2n}$ are equal.

If $a + b = -5$, find the value of a and b.

b) The velocity of a particle P moving along the x axis is given at time t by

$V = ax + b$ where a and b are constants and x is the displacement of P from the origin O after t seconds.

The particle starts to move from the origin with a velocity of 600m/s. After travelling 70m, its velocity decreases to 180m/s.

i) What is the velocity of the particle when it is 80m from the origin? 2

ii) Find an expression for the displacement of the particle in terms of t. 1

iii) How long does it take the particle to reach a velocity of 120m/s? Give your answer correct to 3 decimal places. 2

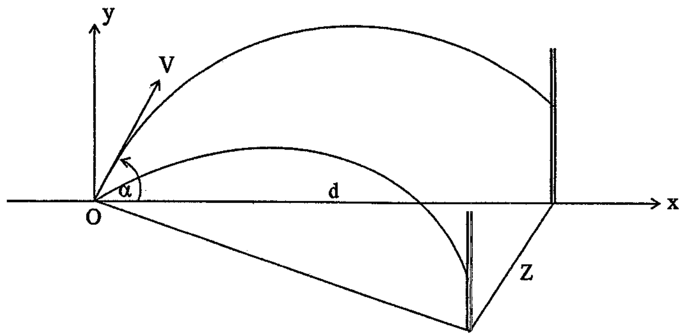
iv) What is the maximum displacement of the particle? 1

c) The displacement in metres of a particle P moving along the x axis is given by $x = 8(\sin^6 t + \cos^6 t)$, where t is the time in seconds. 3

Show that the motion of the particle P is simple harmonic .

QUESTION 7 MARKS

a)



Jane projected a golf ball from a point O with a velocity V at an angle α to the horizontal so that it would hit a pole at a distance d due east of her.

Use axes as shown in the diagram, and assume that there is no air resistance and that g is the acceleration due to gravity.

You may consider that the equations of motion of this ball are

$x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$

i) Show that the equation of the motion of the ball is **2**

$$y = x\tan\alpha - \frac{gx^2}{2V^2}(1+\tan^2\alpha)$$

ii) Show that the maximum height the ball at which can hit the pole is **3**

$$\frac{V^4 - g^2d^2}{2V^2g}$$

iii) Jane hits a second ball with a velocity of $V = 3\sqrt{gd}$ towards another pole Z metres south of the first pole. Find the maximum height at which this ball can hit the second pole. **2**

iv) How far south should the second pole be from first pole in order that the second ball hits the pole just at its base? **2**

b) Use mathematical induction to prove that, for all integers n with $n \geq 1$, **3**

$$\frac{1}{\operatorname{cosec} x} + \frac{1}{\operatorname{cosec} 3x} + \frac{1}{\operatorname{cosec} 5x} + \dots + \frac{1}{\operatorname{cosec}(2n-1)x} = \frac{\operatorname{cosec} x}{\operatorname{cosec}^2 nx}$$

Given that $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq 1; \quad x \neq 0, \quad \text{if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

a) $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\therefore \frac{7}{10} = 2\times\frac{2}{3}\times\cos\alpha$

$\frac{7}{10} = \frac{4}{5}\times\cos\alpha$

so $\cos\alpha = \frac{7}{8}$ (2 marks)

b) $\lim_{x\to 0}\frac{\sin 2x}{3x} = \lim_{x\to 0}\frac{2}{3}\cdot\frac{\sin 2x}{2x}$

$= \frac{2}{3}$ (2 marks)

c) $x = t+\frac{1}{t}$, $y = t^2+\frac{1}{t^2}$

so $x^2 = \left(t+\frac{1}{t}\right)^2$

$x^2 = t^2+2+\frac{1}{t^2}$

$x^2-2 = t^2+\frac{1}{t^2}$

so $x^2-2 = y$ (3 marks)

d) $(-3,-4)$, $(-2,2)$ $3:5$ $(m=-3)$ [$x_1\ y_1$, $x_2\ y_2$, $m:n$]

$x_A = \frac{-3\times-2+5\times-3}{-3+5}$, $y_A = \frac{-3\times 2+5\times-4}{-3+5}$

$x_A = -\frac{9}{2}$ $y_A = \frac{-26}{2}$

$\therefore$ A has co-ordinates $(-4\frac{1}{2},-13)$ (2 marks)

e) As $u = 1+\sin^2 x$

$\therefore \frac{du}{dx} = 2\cos x.\sin x$

so $du = \sin 2x.\,dx$

when $x=\frac{\pi}{2}$, $u=2$

$x=0$, $u=1$

$\therefore \int_0^{\frac{\pi}{2}}\frac{\sin 2x\,dx}{1+\sin^2 x}$

$= \int_1^2\frac{du}{u} = \left[\ln u\right]_1^2$

$= \ln 2-\ln 1 = \ln 2$ (3 marks)

Question 2

a) For $y = 4\sin^{-1}\frac{x}{3}$

i) Domain: $-1\le\frac{x}{3}\le 1$

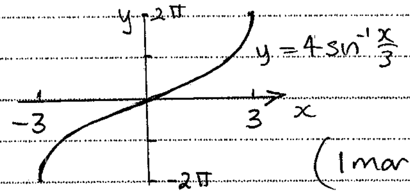
$\therefore -3\le x\le 3$.

Range: $-\frac{\pi}{2}\le\sin^{-1}\frac{x}{3}\le\frac{\pi}{2}$

$\therefore -2\pi\le 4\sin^{-1}\frac{x}{3}\le 2\pi$

so $-2\pi\le y\le 2\pi$ (2 marks)

ii)



(1 mark)

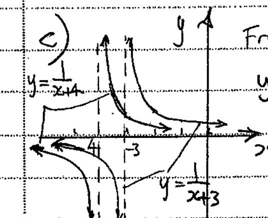
b) let $u=\sin x$ $\therefore y = \tan^{-1}u$

$\frac{du}{dx} = \cos x$, $\frac{dy}{du} = \frac{1}{1+u^2}$

$\therefore \frac{dy}{dx} = \cos x\times\frac{1}{1+\sin^2 x}$

$= \frac{\cos x}{1+\sin^2 x}$ (2 marks)

c)



From the graphs of $y=\frac{1}{x+3}$ and $y=\frac{1}{x+4}$ it can be seen that the sol^{n} is $x>-3$ or $x<-4$ (3 marks)

Question 2 (continued)

d) $2\cos^2 x = \cos 2x+1$

$\therefore 4\cos^2 5x = 2(\cos 10x+1)$

so $\int_{\frac{\pi}{5}}^{\frac{\pi}{2}}4\cos^2 5x\,dx = 2\int_{\frac{\pi}{5}}^{\frac{\pi}{2}}(\cos 10x+1)\,dx$

$= 2\left[\frac{1}{10}\sin 10x+x\right]_{\frac{\pi}{5}}^{\frac{\pi}{2}}$

$= 2\left(\frac{1}{10}\sin 5\pi+\frac{\pi}{2}-\left(\frac{1}{10}\sin 2\pi+\frac{\pi}{5}\right)\right)$

$= 2\left(0+\frac{\pi}{2}-\left(0+\frac{\pi}{5}\right)\right) = \frac{3\pi}{5}$ (2 marks)

e) General term $= {}^{14}C_k(x^2)^{14-k}(2x^{-3})^k$

$= {}^{14}C_k\,2^k x^{28-2k-3k}$

For x^3, $28-5k=3$

so $k=5$

$\therefore$ Term is ${}^{14}C_5\,2^5x^3$ and so co-efficient is ${}^{14}C_5\cdot 2^5 = 64064$ (2 marks)

Question 3

a) "Arrangement" means order is important

$\therefore$ number of arrangements $= \frac{7!}{2!\times 2!\times 2!}$

$= \frac{5040}{8} = 630$ (1 mark)

b) "Group" means order is not important. 1 letter must be K $\therefore$ 4 more letters to choose from remaining 6 letters

$\therefore$ number of different ways $= {}^6C_4$

$= 15$

(2 marks)

c) Σ roots 4 at a time $= \frac{e}{a}$

so $\alpha\times\frac{1}{\alpha}\times\beta\times-\beta = \frac{-18}{2}$

ie $-\beta^2 = -9$ ——— ①

Σ roots two at a time $= \frac{c}{a}$

so $1+\alpha\beta-\alpha\beta+\frac{\beta}{\alpha}-\frac{\beta}{\alpha}-\beta^2 = \frac{k}{2}$

ie $1-\beta^2 = \frac{k}{2}$ ——— ②

$\therefore 1-9 = \frac{k}{2}$, so $k=-16$ (3 marks)

d) i) The motion of a particle is SHM oscillating about the position $x=b$ if $\ddot{x} = -n^2(x-b)$.

given $x = 4-3\sin 2t$

$\dot{x} = -6\cos 2t$

$\ddot{x} = -12\sin 2t$

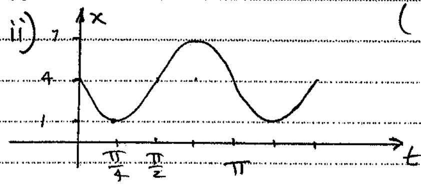
so $\ddot{x} = -4(-3\sin 2t)$

$= -4(4-3\sin 2t-4)$

$= -4(x-4)$

$\therefore$ motion is SHM about $x=4$. (2 marks)

ii)



From the graph of $x=4-3\sin 2t$ min x first reached when $t=\frac{\pi}{4}$ sec. (1 mark)

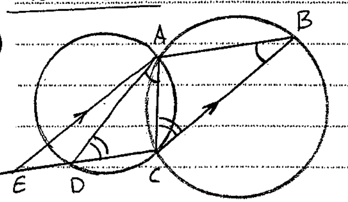
e) $f'(x) = -4e^{-x}+\cos x$

$a_1 = 2-\frac{4e^{-2}+\sin 2-1}{-4e^{-2}+\cos 2}$

$\doteqdot 2\cdot 4706$ (4 d.p) (3 marks)

Question 4

a)



$D\hat{A}C = A\hat{B}C$ ($\angle$ between chord and tangent = $\angle$ in alt seg)

$B\hat{C}A = A\hat{D}C$ (as above)

$\therefore \triangle DAC \parallel\!\mid \triangle CBA$ (equiangular)

so $B\hat{A}C = A\hat{C}D$ ($\angle$ sum $\triangle$)

hence $AB \parallel DC$ (alternate $\angle$'s equal $B\hat{A}C = A\hat{C}D$)

also $EA \parallel CB$ - (Data.)

$\therefore$ ABCE is a parallelogram.

(4 marks)

b) $T = R + Ae^{-kt}$, $R = 25$

i) when $t=0$, $T = 90°$

$\therefore 90 = 25 + A$, so $A = 65$.

when $t=5$, $T = 70°$

$\therefore 70 = 25 + 65e^{-5k}$

$45 = 65e^{-5k}$

$\ln\left(\frac{45}{65}\right) = -5k$

$\cdot 073544\ldots = k$ (2 marks)

ii) find t when $T = 50$

$50 = 25 + 65e^{-\cdot 0735 t}$

$\frac{25}{65} = e^{-\cdot 0735 t}$

$\ln\left(\frac{25}{65}\right) = -\cdot 0735\ldots t$

$12\cdot 9922\ldots = t$

$13\text{min} \doteqdot t$ (3 marks)

c) $P(\text{used}) = 0\cdot 7$

$P(\text{not used}) = 0\cdot 3$

(i) $P(\text{two never used}) = {}^8C_2 \times \cdot 7^6 \times \cdot 3^2$

$= \frac{29647548}{1000000000} = 0\cdot 29647548$ (1 mark)

ii) P(no more than 2 never used)

Prob (2 never used + 1 never used + none never used)

$= {}^8C_2 \times \cdot 7^6 \times \cdot 3^2 + {}^8C_1 \times \cdot 7^7 \times \cdot 3^1 + {}^8C_0 \times \cdot 7^8 \times \cdot 3^0$

$= \cdot 29647548 + \cdot 19765032 + \cdot 05764801$

$= \cdot 55177381$ (1 mark)

(iii) P(at least one never used)

$= 1 - \text{Prob}(\text{all are used})$

$= 1 - {}^8C_0 \cdot 7^8 \times \cdot 3^0 = 1 - \cdot 05764801$

$= 0\cdot 94235199$ (1 mark)

Question 5

a) $y = \frac{-1}{16a}x^2$, $y' = \frac{-1}{8a}x$

i) when $x = 4at$, $y = -at^2$, $m = \frac{-4at}{8a} = \frac{-t}{2}$

$\therefore$ tang eqn $y + at^2 = -\frac{t}{2}(x - 4at)$

$y + at^2 = -\frac{tx}{2} + 2at^2$

so $y = -\frac{t}{2}x + at^2$ but as this tangent passes through $(2a, 0)$

$0 = -at + at^2 \quad \therefore 0 = at(t-1)$

so $t = 0, 1$, must be $t = 1$

so eqn is $y = \frac{-x}{2} + a$ (2 marks)

ii) when $x = 0$, $y = a$

$\therefore C$ is $(0, a)$ (2 marks)

Question 5 (continued)

b) i) $f(x)$ has an inverse function since for each y value there is only one x value (any horizontal line will only cut the graph once) (1 mark)

ii) For inverse function, make y the subject of $x = \frac{\ln y}{\ln y - 1}$

$x(\ln y - 1) = \ln y$

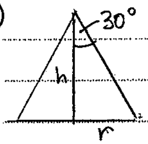
$x\ln y - \ln y = x$

$\ln y(x - 1) = x$

$\ln y = \frac{x}{x-1}$

$\therefore y = e^{\frac{x}{x-1}}$ (2 marks)

c)



$\tan 30 = \frac{r}{h}$

$\therefore r = \frac{h}{\sqrt{3}}$, $h = \sqrt{3}r$

i) $V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{\pi h^3}{9}$

$\frac{dV}{dh} = \frac{\pi h^2}{3}$; $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$\therefore \cdot 2 = \frac{\pi h^2}{3} \times \frac{dh}{dt}$; $\frac{dh}{dt} = \frac{\cdot 2 \times 3}{\pi h^2}$

so $\frac{dh}{dt} = \cdot 002984\text{ m/min}$ or $2\cdot 984\text{ mm/min}$. (3 marks)

(ii) $V = \frac{1}{3}\pi r^2 \cdot \sqrt{3}r = \frac{\sqrt{3}\pi r^3}{3}$

$\frac{dV}{dr} = \sqrt{3}\pi r^2$ Area $= \pi r^2$

$\frac{dA}{dr} = 2\pi r$

find $\frac{dA}{dt}$ when $r = 10$

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$ and $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$

so $\frac{dr}{dt} = \frac{dV}{dt} \div \frac{dV}{dr} \quad \therefore \frac{dA}{dt} = \frac{dA}{dr} \times \left(\frac{dV}{dt} \div \frac{dV}{dr}\right)$

ie $\frac{dA}{dt} = 2\pi r \times \cdot 2 \div \sqrt{3}\pi r^2$

$= 2 \times \pi \times 10 \times \cdot 2 \div (\sqrt{3} \times \pi \times 100)$

$= \cdot 023094\ldots \text{ m}^2/\text{min}$

(2 marks)

Question 6

a) Constant term in each is

${}^{2n}C_n\, x^n \left(\frac{4}{x}\right)^n$; ${}^{2n}C_n \left(\frac{x}{a}\right)^n \left(\frac{b}{x}\right)^n$

$= {}^{2n}C_n\, 4^n \qquad = {}^{2n}C_n \left(\frac{b}{a}\right)^n$

so ${}^{2n}C_n\, 4^n = {}^{2n}C_n \left(\frac{b}{a}\right)^n$

ie $4^n = \left(\frac{b}{a}\right)^n$

so $4 = \frac{b}{a}$ or $4a = b$

also as $a + b = -5$,

$a + 4a = -5$; $a = -1$

$\therefore b = -4$ (3 marks)

b) $\frac{dx}{dt} = ax + b$

(i) when $t = 0$, $x = 0$, $v = 600$

$\therefore 600 = a \times 0 + b \quad \therefore b = 600$

when $x = 70$, $v = 180$

$\therefore 180 = 70a + 600$

$-420 = 70a$

$-6 = a$ so $V = -6x + 600$

Question 6 (continued)

b(i) $V = -6 \times 80 + 600 = 120$ m/s (2 marks)

ii) $\frac{dt}{dx} = \frac{1}{ax+b}$ $\therefore t = \frac{1}{a}\ln(ax+b) + c$

from above, $b = 600$, $a = -6$

$\therefore t = -\frac{1}{6}\ln(-6x+600) + c$

when $t = 0$, $x = 0$

$\therefore 0 = -\frac{1}{6}\ln 600 + c$

$\therefore c = \frac{1}{6}\ln 600$

and $t = -\frac{1}{6}\ln(-6x+600) + \frac{1}{6}\ln 600$

$-6t = \ln\left(\frac{600-6x}{600}\right)$

$\therefore 600e^{-6t} = 600 - 6x$

$6x = 600 - 600e^{-6t}$

$x = 100 - 100e^{-6t}$ (1 mark)

iii) $\frac{dx}{dt} = 600e^{-6t}$

if $V = 120$ then $120 = 600e^{-6t}$

$0.2 = e^{-6t}$

$\ln(.2) = -6t$

so $t = 0.268$ seconds (2 marks)

iv) as $x = 100(1 - e^{-6t})$

as $t \to \infty$, $e^{-6t} \to 0$

$\therefore x \to 100(1-0)$

ie $x \to 100$

so maximum displacement is 100 m (1 mark)

c) The motion of the particle is SHM, oscillating about the position $x = b$ if $\ddot{x} = -n^2(x-b)$

$x = 8(\sin^6 t + \cos^6 t)$

$= 8\left((\sin^2 t)^3 + (\cos^2 t)^3\right)$

$= 8(\sin^2 t + \cos^2 t)(\sin^4 t - \sin^2 t\cos^2 t + \cos^4 t)$

$= 8.1.\left(\sin^2 t(1-\cos^2 t) - \sin^2 t\cos^2 t + \cos^2 t(1-\sin^2 t)\right)$

$= 8(\sin^2 t + \cos^2 t - 3\sin^2 t\cos^2 t)$

$= 8\left(1 - 3 \times \frac{1}{4}\sin^2 2t\right)$

$= 8\left(1 - \frac{3}{4}\left(\frac{1-\cos 4t}{2}\right)\right)$

$= 8 - 3 + 3\cos 4t$

$= 5 + 3\cos 4t$

$\therefore v = -12\sin 4t$

$a = -48\cos 4t$

$= -16(3\cos 4t)$

$= -16(5 + 3\cos 4t - 5)$

$= -16(x - 5)$

which is of the form

$\ddot{x} = -n^2(x-b)$ $\therefore$ motion of the particle is simple Harmonic Motion (3 marks)

Question 7

a) (i) $t = \frac{x}{v\cos\alpha}$

so $y = \frac{v \cdot x \cdot \sin\alpha}{v\cos\alpha} - \frac{1}{2} \times g \times \frac{x^2}{v^2\cos^2\alpha}$

$y = x\tan\alpha - \frac{gx^2}{2v^2} \times \sec^2\alpha$

$= x\tan\alpha - \frac{gx^2}{2v^2}(1 - \tan^2\alpha)$ (2 marks)

Question 7 a) continued

ii) Height at which the ball hits the pole is "y" when $x = d$

ie) $h = d\tan\alpha - \frac{gd^2}{2v^2}(1 + \tan\alpha)$

to find max height solve $\frac{dh}{d\alpha} = 0$

$\frac{dh}{d\alpha} = d\sec^2\alpha - \frac{gd^2}{2v^2}2\sec^2\alpha\tan\alpha$

ie $0 = d\sec^2\alpha\left(1 - \frac{gd}{v^2}\tan\alpha\right)$

as $\sec\alpha \neq 0$, $\tan\alpha = \frac{v^2}{gd}$

so $h = \frac{d \cdot v^2}{gd} - \frac{gd^2}{2v^2} \times \left(1 + \frac{v^4}{g^2d^2}\right)$

$= \frac{v^2}{g} - \frac{gd^2}{2v^2} - \frac{v^2}{2g}$

$= \frac{2v^4 - g^2d^2 - v^4}{2v^2g} = \frac{v^4 - g^2d^2}{2v^2g}$ (3 marks)

(iii) distance to new pole $= \sqrt{d^2 + z^2}$

velocity $= 3\sqrt{gd}$

$\therefore h_{max} = \frac{81g^2d^2 - g^2(d^2+z^2)}{2g \times 9gd}$

$= \frac{80g^2d^2 - g^2z^2}{18g^2d}$

$= \frac{80d^2 - z^2}{18d}$ (2 marks)

iv) If $h = 0$, $80d^2 - z^2 = 0$

$\therefore z^2 = 80d^2$

so $z = \sqrt{80}\,d$

$= 4\sqrt{5}\,d$ (2 marks)

b) $\frac{1}{\operatorname{cosec} x} + \frac{1}{\operatorname{cosec} 3x} + \dots + \frac{1}{\operatorname{cosec}(2n-1)x} = \frac{\operatorname{cosec} x}{\operatorname{cosec}^2 nx}$

is equivalent to

$\sin x + \sin 3x + \dots + \sin(2n-1)x = \frac{\sin^2 nx}{\sin x}$

• Prove true for $n = 1$

$LHS = \sin x$; $RHS = \frac{\sin^2 x}{\sin x} = \sin x = LHS$

• Assume true for $n = k$

ie $\sin x + \sin 3x + \dots + \sin(2k-1)x = \frac{\sin^2 kx}{\sin x}$

Prove true for $n = k+1$

ie $\sin x + \sin 3x + \dots + \sin(2k-1)x + \sin(2k+1)x = \frac{\sin^2(k+1)x}{\sin x}$

$LHS = \frac{\sin^2 kx}{\sin x} + \sin(2k+1)x$

$= \frac{\sin^2 kx + \sin x\sin(2k+1)x}{\sin x}$

$= \frac{\frac{1}{2} - \frac{1}{2}\cos 2kx + \frac{1}{2}[\cos((2k+1)x - x) - \cos((2k+1)x + x)]}{\sin x}$

$= \frac{\frac{1}{2} - \frac{1}{2}\cos(2k+2)x}{\sin x}$

$= \frac{\sin^2(k+1)x}{\sin x}$

$= RHS$

so if statement is true for $n=k$, also true for $n=k+1$.

From first step, statement is true for $n=1$, so from second step is true for $n=2$, and then $n=3$ and so on for all positive integers $n \geq 1$

(3 marks)