HIGHER SCHOOL CERTIFICATE EXAMINATION TRIAL PAPER

2001

MATHEMATICS

EXTENSION 2

Time Allowed – Three Hours
(Plus 5 minutes reading time)

Examiner: Sami El Hosri

Directions to Candidates

- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Board-approved calculators may be used.

YEAR 12 – TRIAL 2001 – EXTENSION 2

QUESTION 1

MARKS

a) Find $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$ 2

b) i) Find a, b and c such that 2

$$\frac{16}{(x^2+4)(2-x)} = \frac{a\,x+b}{x^2+4} + \frac{c}{2-x}$$

ii) Find $\int \frac{16}{(x^2+4)\,(2-x)}\,dx$ 2

c) Find $\int \frac{\ln x}{x^2}\,dx$ 4

d) Use the substitution $t = \tan\frac{\theta}{2}$ to show that 5

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$$

QUESTION 2 **MARKS**

a) The complex number Z moves such that $\operatorname{Im}\left(\dfrac{1}{\overline{Z}-i}\right)=1$. 3

Show that the locus of Z is a circle and find its centre and radius.

b) i) Find the square root of the complex number $5 - 12i$ 2

ii) Given that $Z = \dfrac{1+\sqrt{5-12i}}{2+2i}$ and is purely imaginary, 2

find Z^{400}

c) i) Shade the region in the argand diagram containing all points representing the complex numbers Z such that 3

$|Z-1-i| \le 1$ and $-\dfrac{\pi}{4} \le \operatorname{Arg}(Z-i) \le \dfrac{\pi}{4}$

ii) Let ϕ be the complex number of minimum modulus satisfying the inequalities of i). 1

Express ϕ in the form $x + yi$

d) Express $\phi = \dfrac{-1+i}{\sqrt{3}+i}$ in modulus / argument form. 4

Hence, evaluate $\cos\dfrac{7\pi}{12}$ in surd form.

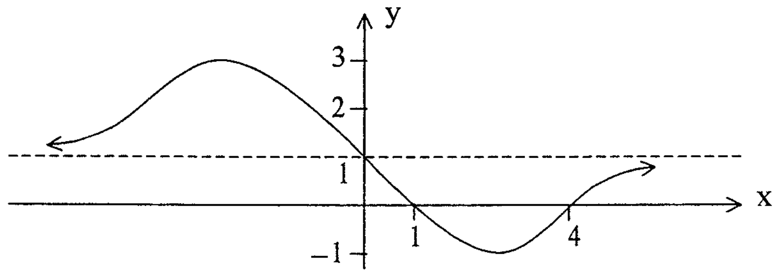
QUESTION 3 **MARKS**

a) Consider the equation $x^3 + 7x - 6i = 0$.

i) Given that this equation has no purely real root, show that none of the roots is a conjugate of any of the others. 1

ii) If 2i is one of the roots and the other two roots are purely imaginary, find the other two roots. 2

b)



The above diagram shows the graph of $y = f(x)$.
Sketch on separate diagrams the following curves, indicating clearly any turning points and asymptotes.

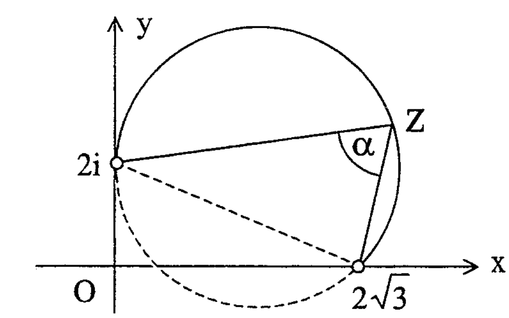
i) $y = \dfrac{1}{f(x)}$ 1

ii) $y = f(|x|)$ 2

iii) $y = \ln f(x)$ 2

iv) $y = \sin^{-1}(f(x))$ 2

c)



The locus of a point Z, moving in complex plane such that $\text{Arg}\,(Z - 2\sqrt{3}) - \text{Arg}\,(Z - 2i) = \dfrac{\pi}{3}$, is a part of a circle.

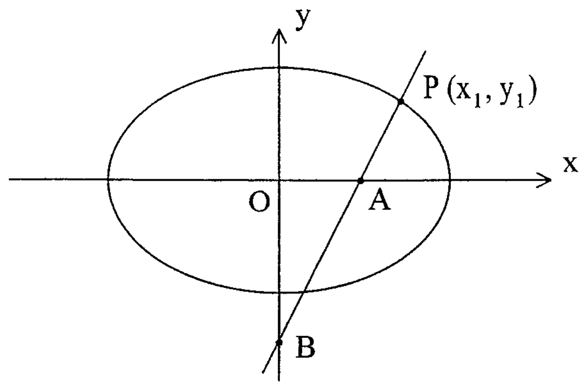
The angle between the lines from 2i to Z and from $2\sqrt{3}$ to Z is α as shown in the diagram.

i) Show that $\alpha = \dfrac{\pi}{3}$ 2

ii) Find the centre and the radius of the circle. 3

QUESTION 4 **MARKS**

a)



The point $P(x_1, y_1)$, where $x_1 > 0$ and $y_1 > 0$, lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The normal at P intersects the x axis at A and the y axis at B.

i) Show that the equation of the normal is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$ 3

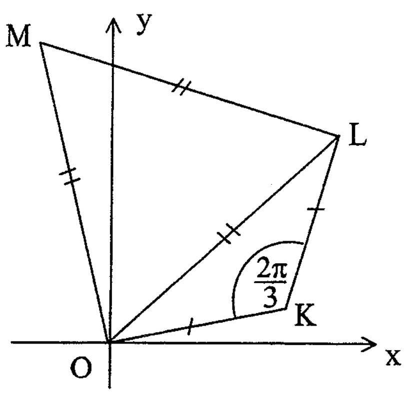
ii) Explain why the point A cannot be the focus of the ellipse. 2

iii) Find the ratio in which A divides the interval BP internally. 2

iv) Find the midpoint M of AB in terms of x_1 and y_1. 1

v) Given that H divides the interval OM in the ratio 4:1, show that the locus of H is an ellipse . 3

b)



The points K and M in a complex plane represent the complex numbers α and β respectively. The triangle OKL is isosceles and $\angle OKL = \frac{2\pi}{3}$. The triangle OLM is equilateral. 4

Show that $3\alpha^2+\beta^2=0$

QUESTION 5 **MARKS**

a) Solve $x^x a^{\ln x} = x$, where $a > 0$ and $x > 1$ 2

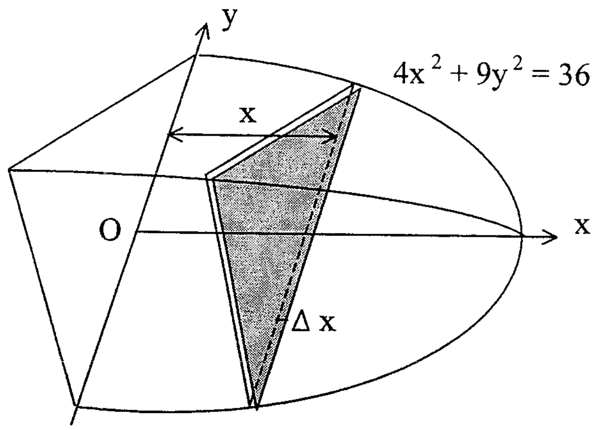
b) A farmer plants tomato seeds on his farm. After p days, he stops planting as he starts to collect his grown tomatoes. Assume that the equation for the number of seeds planted when $0 \le t \le p$ is $\frac{dS}{dt} = AS$ where A is a positive constant, and that when $t = 0$, $S = S_0$

i) Find an expression for $S(p)$ 2

ii) Assume that the number of tomatoes to be collected is given by $\frac{dT}{dt} = 2B\,S(p)t$ where B is a positive constant, and that when $t = p$, $T = 0$ show that $T = BS_0 e^{Ap}(t^2 - p^2)$ 2

iii) Given that $A = 0.032$, find the value of S for which the number of tomatoes T can be maximized on 75th day (i.e. $t = 75$). 3

c)



The base of the solid K shown in the diagram is the region in the xy plane enclosed between the semi-ellipse $4x^2 + 9y^2 = 36$ and the y axis. Each cross section perpendicular to the x axis is an equilateral triangle.

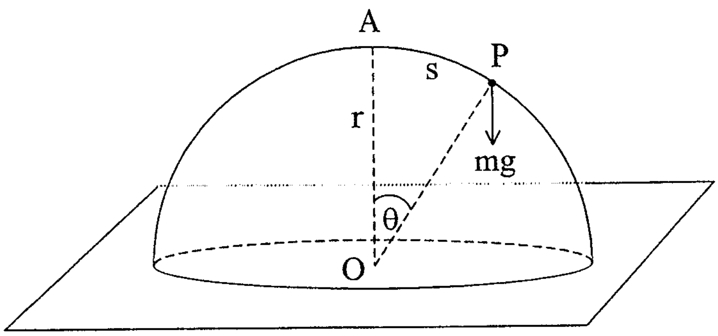
i) Consider a slice of the solid with thickness Δx and distant x from the y axis. Find the area of this slice in terms of x. 2

ii) Find the volume of the solid K. 2

iii) Solid J has the same base as solid K but its perpendicular cross sectional slice is an isosceles right angled triangle with its hypotenuse in the xy plane.

Find the ratio of volumes of solid K to solid J. 2

QUESTION 6 **MARKS**



a) A particle P of mass m is initially at rest on the highest point A of a hemisphere centred at O and with radius r.

The particle P is given a horizontal velocity $v = \frac{1}{4}\sqrt{gr}$ and begins to slide down the surface of the hemisphere forming an arc AP of a circle.

Let s be the length of the arc AP and θ be the angle subtended by this arc at the centre of the hemisphere.

Assuming that there are no frictional forces acting on the particle P, and g is the acceleration due to gravity,

i) Show that the tangential acceleration of P is given by **2**

$$\frac{d^2s}{dt^2} = \frac{1}{r}\frac{d}{d\theta}\left(\frac{1}{2}v^2\right)$$

ii) Show that $v^2 = \frac{gr}{16}(33 - 32\cos\theta)$ **3**

iii) Find the normal reaction N exerted by the hemisphere on the particle P. **3**

iv) Find the value of the angle θ at which the particle leaves the hemisphere, and its velocity at the moment of leaving. **2**

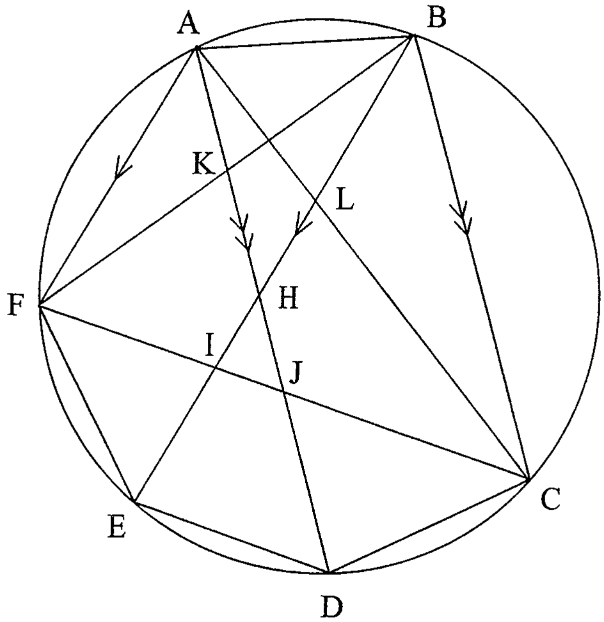
b) i) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \frac{4\cos x}{1 + 4\sin^2 x}\,dx$ **2**

ii) Show that for $n \geq 2$ **3**

$$\frac{\pi}{2} \leq \int_0^{\frac{\pi}{6}} \frac{4\cos x\,dx}{1 + 4\sin^n x} \leq 2$$

QUESTION 7 **MARKS**

a)



ABCDEF is a cyclic hexagon. Diagonal BE is parallel to AF and intersects diagonals AC and FC at L and I respectively. Diagonal AD is parallel to BC and intersects diagonals BF and FC at K and J respectively. AD and BE intersect at H.

i) Show that ABLK is a cyclic quadrilateral. 3

ii) Show that triangle BKL is similar to triangle BFI. 3

iii) Show that $\frac{BK}{BF} \times \frac{AJ}{AK} \times \frac{FI}{JC} = 1$ 3

b) Let $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^n x \, dx$, where n is a positive integer.

i) Using integration, show that 4

$$(n-1)\, I_n = 2^{n-2}\sqrt{3} + (n-2)\, I_{n-2}$$

ii) Evaluate $J = \int_0^{\frac{\pi}{3}} \sec^4 x \, dx$ 2

QUESTION 8 **MARKS**

a) Consider the polynomial $x^5 - i = 0$

i) Show that $1 - ix - x^2 + ix^3 + x^4 = 0$ for $x \neq i$ 2

ii) Show that

$$(x - i)\left(x^2 - 2i\sin\frac{\pi}{10}x - 1\right)\left(x^2 + 2i\sin\frac{3\pi}{10}x - 1\right) = 0$$ 4

iii) Show that $\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = \frac{1}{4}$ 2

b) Grace and David were at school together, but lost contact after finishing the HSC. Years later they each won two free lunches at a certain restaurant, to be used on any Sunday in a given period of n weeks, where $n > 3$.

Assume that they are equally likely to choose any of the Sundays in this period, what is the probability that:

i) They will meet at the restaurant on the first Sunday. 2

ii) They will meet at the restaurant only once. 2

iii) They will never both attend the restaurant on the same Sunday so never meet. 3

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq 1; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

YEAR 12 - TRIAL - 2001 - Extension 2 - Solutions + MAPPING GRID

© S & G Publishing and coaching Academy

Question 1

a) $\displaystyle\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$

let $u = \tan x$ $\quad\therefore du = \sec^2 x\,dx$

$\therefore \displaystyle\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c$

$= \sin^{-1}(\tan x) + c$

(2 marks)

b) $\dfrac{16}{(x^2+4)(2-x)} = \dfrac{ax+b}{x^2+4} + \dfrac{c}{2-x}$

i) $c = \lim\limits_{x\to 2} \dfrac{16}{x^2+4} = \dfrac{16}{8} = 2$

$\lim\limits_{x\to 2i} \dfrac{16}{2-x} = \lim\limits_{x\to 2i} (ax+b)$

$\therefore \dfrac{16}{2-2i} = 2ai + b$

$\therefore \dfrac{8}{1-i} = 2ai + b \quad \therefore 4(1+i) = 2ai + b$

$\therefore 4 + 4i = 2ai + b$

Equating reals and and Im's,

$\therefore 2a = 4,\ a = 2,\ b = 4$

$\therefore \dfrac{16}{(x^2+4)(2-x)} = \dfrac{2x+4}{x^2+4} + \dfrac{2}{2-x}$

(2 marks)

ii) $\displaystyle\int \frac{16\,dx}{(x^2+4)(2-x)} = \int \frac{2x+4}{x^2+4}\,dx + \int \frac{2\,dx}{2-x}$

$= \displaystyle\int \frac{2x}{x^2+4}\,dx + \int \frac{4}{x^2+4}\,dx + \int \frac{2}{2-x}\,dx$

$= \log_e(x^2+4) + 4 \times \frac{1}{2}\tan^{-1}\frac{x}{2} - 2\log_e|2-x| + c$

$= 2\tan^{-1}\dfrac{x}{2} + \log_e\left(\dfrac{x^2+4}{(2-x)^2}\right) + c$

(2 marks)

c) $I = \displaystyle\int \frac{\ln x}{x^2}\,dx$

let $u = \ln x$ $\quad dv = \dfrac{1}{x^2}\,dx$

$du = \dfrac{1}{x}\,dx$ $\quad\therefore v = -\dfrac{1}{x}$

$\therefore I = -\dfrac{\ln x}{x} + \displaystyle\int \frac{1}{x^2}\,dx$

$= -\frac{1}{x}\ln x - \frac{1}{x} + c$

$= -\frac{1}{x}(\ln x + 1) + c$ (4 marks)

d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6}$ $\quad$ let $t = \tan\dfrac{\theta}{2}$

$\therefore 4\sin\theta - 2\cos\theta + 6$

$= \dfrac{8t}{1+t^2} - \dfrac{2-2t^2}{1+t^2} + 6 = \dfrac{8t - 2 + 2t^2 + 6 + 6t^2}{1+t^2}$

$= \dfrac{8t^2 + 8t + 4}{1+t^2}$

$t = \tan\dfrac{\theta}{2}$ $\quad\therefore \dfrac{\theta}{2} = \tan^{-1} t$ $\quad\therefore \theta = 2\tan^{-1} t$

$\therefore d\theta = \dfrac{2}{1+t^2}\,dt.$ when $\theta = 0,\ t = 0$

$\theta = \frac{\pi}{2},\ t = 1$

$\therefore \displaystyle\int_0^1 \frac{1+t^2}{8t^2+8t+4} \cdot \frac{2\,dt}{1+t^2}$

$= \displaystyle\int_0^1 \frac{dt}{4t^2+4t+2} = \int_0^1 \frac{dt}{1+(1+2t)^2}$

Question 1 - continued

let $u = 2t+1 \therefore du = 2dt$

$\therefore dt = \frac{1}{2}du$

$= \int_1^3 \frac{\frac{1}{2}du}{1+u^2} = \frac{1}{2}[\tan^{-1}u]_1^3$

$= \frac{1}{2}\left(\tan^{-1}3 - \tan^{-1}1\right)$

let $\alpha = \tan^{-1}3 \therefore \tan\alpha = 3 \quad 0<\alpha<\frac{\pi}{2}$

$\beta = \tan^{-1}1 \therefore \tan\beta = 1, \quad 0<\beta<\frac{\pi}{2}$

$\therefore \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$= \frac{3-1}{1+3\times1} = \frac{2}{4} = \frac{1}{2}$

$\therefore \tan(\alpha-\beta) = \frac{1}{2} \quad \therefore \alpha-\beta = \tan^{-1}\frac{1}{2}$

$\therefore \tan^{-1}3 - \tan^{-1}1 = \tan^{-1}\frac{1}{2}$

$\therefore \int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6} = \frac{1}{2}\tan^{-1}\frac{1}{2}$

(5 marks)

Question 2

a) let $z = x+iy \quad \therefore \bar{z} = x-iy$

$\bar{z} - i = x - iy - i = x - i(y+1)$

$\therefore \frac{1}{\bar{z}-i} = \frac{1}{x-i(y+1)} \times \frac{x+i(y+1)}{x+i(y+1)}$

$= \frac{x+i(y+1)}{x^2+(y+1)^2} = \frac{x}{x^2+(y+1)^2} + \frac{i(y+1)}{x^2+(y+1)^2}$

$\therefore \frac{y+1}{x^2+(y+1)^2} = 1 \quad \therefore x^2+(y+1)^2 = y+1$

$\therefore x^2+y^2+2y+1 = y+1$

$\therefore x^2+y^2+y = 0 \quad \therefore x^2+y^2+y+\frac{1}{4} = \frac{1}{4}$

$\therefore x^2 + \left(y+\frac{1}{2}\right)^2 = \frac{1}{4}$

$\therefore$ The locus of z is a circle centred at $(0, -\frac{1}{2})$ with radius $\frac{1}{2}$ units. (3 marks)

b) i) $\sqrt{5-12i} = x+iy$

$\therefore x^2 - y^2 = 5$ ① (Real = Real)

$x^2+y^2 = 13$ ② (mod = mod)

$2xy = -12$ (Im = Im)

$\therefore xy = -6$ ③

① + ② gives $2x^2 = 18 \quad \therefore x^2 = 9 \quad \therefore x = \pm 3$

for $x = 3, y = -2$ (from ③)

$x = -3, y = 2$

$\therefore \sqrt{5-12i} = 3-2i$ or $-3+2i$ (2 marks)

ii) $z = \frac{1+\sqrt{5-12i}}{2+2i} \times \frac{2-2i}{2-2i}$

$= \frac{(2-2i)(1+\sqrt{5-12i})}{8}$

$= \frac{(1-i)(1+3-2i)}{4}$ or $\frac{(1-i)(1-3+2i)}{4}$

$= \frac{(1-i)(2-i)}{2}$ or $\frac{(1-i)(-1+i)}{2}$

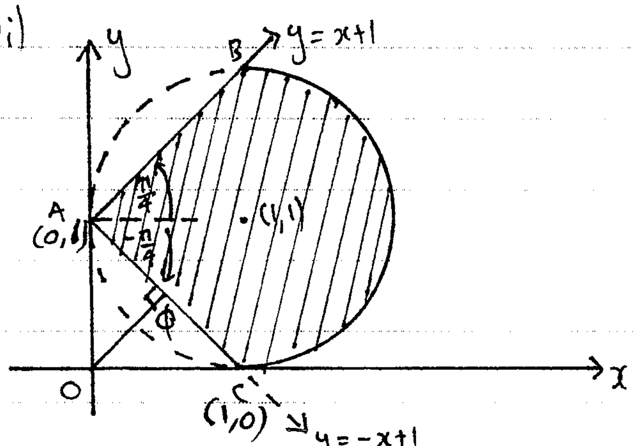
$= \frac{1}{2} - \frac{3}{2}i$ or i

$\therefore z = i$ (as it is purely imaginary)

$\therefore z^{400} = i^{400} = (i^4)^{100} = 1^{100} = 1$

(2 marks)

c) i)



$|z-1-i| \leq 1$ is the region inside the circle centred at (1,1) and with radius 1 unit.

$-\frac{\pi}{4} \leq \text{Arg}(z-i) \leq \frac{\pi}{4}$ is the region of the Argand diagram between the rays AB and AC. ∴ The region shaded in the diagram satisfies the above conditions. (3 marks)

ii) From the diagram ϕ shoud be the foot of the perpendicular from O to AC. Since OA = OC

∴ ΔOAC is isosceles ∴ C is the midpoint of AC ∴ $C\left(\frac{1}{2}, \frac{1}{2}\right)$

$\therefore \phi = \frac{1}{2}(1+i)$ (1 mark)

d) $\phi = \frac{-1+i}{\sqrt{3}+i}$ let $z_1 = -1+i$, $z_2 = \sqrt{3}+i$

$\phi = \frac{z_1}{z_2}$ $|z_1| = \sqrt{(-1)^2+1} = \sqrt{2}$

$\arg z_1 = \frac{3\pi}{4}$ $\therefore z_1 = \sqrt{2}\,\text{cis}\frac{3\pi}{4}$

$|z_2| = \sqrt{3+1} = 2$ $\text{Arg}\, z_2 = \frac{\pi}{6}$

$\therefore z_2 = 2\,\text{cis}\frac{\pi}{6}$

$\therefore \phi = \frac{\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)}{2\,\text{cis}\frac{\pi}{6}} = \frac{\sqrt{2}}{2}\,\text{cis}\left(\frac{7\pi}{12}\right)$

$\therefore \phi = \frac{\sqrt{2}}{2}\,\text{cis}\frac{7\pi}{12}$ (mod/arg form)

consider $\phi = \frac{-1+i}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i}$

$\phi = \frac{(-1+i)(\sqrt{3}-i)}{4} = \frac{1-\sqrt{3}+i(1+\sqrt{3})}{4}$

$\therefore \frac{1-\sqrt{3}}{4} = \frac{\sqrt{2}}{2}\cos\frac{7\pi}{12}$ (equating real's)

$\therefore \cos\frac{7\pi}{12} = \frac{2}{\sqrt{2}} \times \frac{1-\sqrt{3}}{4} = \frac{1-\sqrt{3}}{2\sqrt{2}}$

$\therefore \cos\frac{7\pi}{12} = \frac{\sqrt{2}-\sqrt{6}}{4}$ (4 marks)

Question 3

a i) Assume that the roots of the equation $x^3+7x-6i=0$ are in the form $a+ib$, $a-ib$ and $x+iy$. ie: 2 of them are conjugates. The sum $= 2a+x+iy$. However, sum $=0$ (from eqⁿ). Hence $y=0$. However, none of the roots is purely real ∴ sum cannot be zero if roots are conjugates.

Hence, none of the roots is a conjugate of any of the others (1 mark)

aii) let the roots be

$x = 2i$, $x = ib$, $x = iv$

sum of roots:

$i(2+b+v) = 0$

$\therefore b+v = -2 \quad \cdots ①$

product of roots

$2i(iv)(ib) = 6i$

$\therefore bv = -3$. Forming a new quadratic equation with roots b, v:

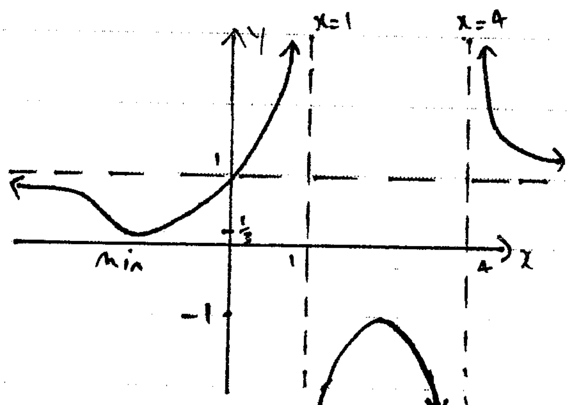
$x^2 + 2x - 3 = 0 \quad \therefore x = 1$ or $x = -3$

$\therefore b = 1, \; v = -3$

$\therefore$ the roots of $x^3 + 7x - 6i = 0$ are

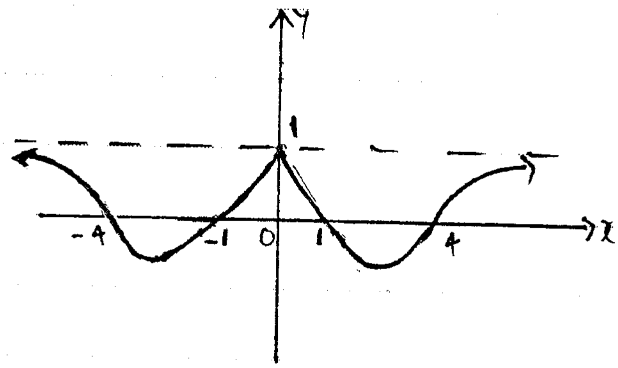
$x = 2i$, $x = -3i$, $x = i$ (2 marks)

bi)



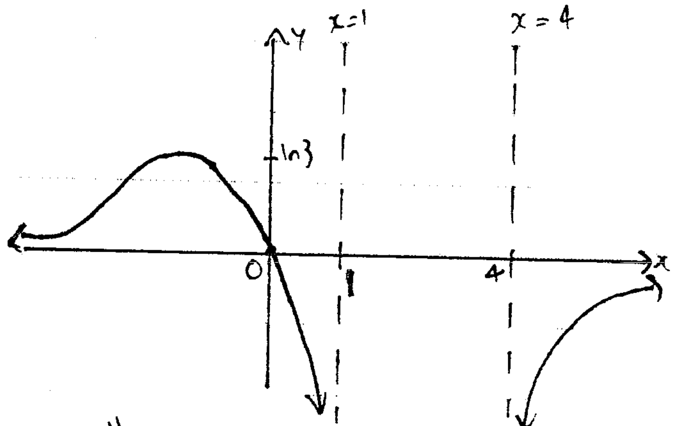
$y = \frac{1}{f(x)}$. Zero's become asymptotes, minimums become maximums & vice versa.

ii)



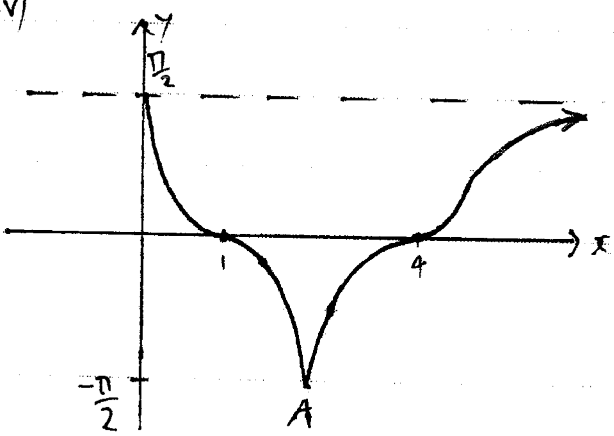
$y = f(|x|)$. The positive x-values are reflected along the y-axis. There is no turning point at $(0,1)$ - it is an angular critical point.

iii)



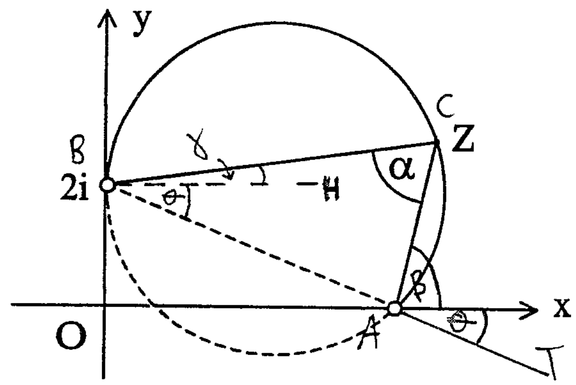
$y = \ln(f(x))$ is defined only when $f(x) > 0$. Zero's become asymptotes. Values between 0 and 1 become negative.

iv)



$y=\sin^{-1}(f(x))$. Only $-1\le y\le 1$ are defined. There is an asymptote at $y=\frac{\pi}{2}$. Point A is not a turning point.

c) i)



let $Arg(z-2\sqrt{3})=\beta$ and

$Arg(z-2i)=\gamma$ and also

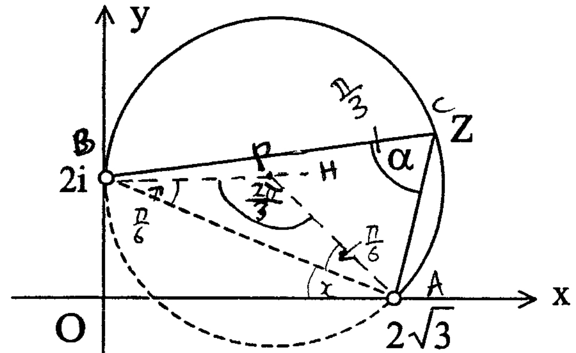
let $\angle HBA=\theta$ $\therefore \angle TAX=\theta$

(corresponding $\angle$'s, $\parallel$ lines)

$\alpha+\gamma+\theta=\beta+\theta$ (exterior $\angle$ of ΔABC = sum of opposite interior $\angle$'s).

$\therefore \alpha=\beta-\gamma$. But as

$\beta-\gamma=\frac{\pi}{3}$ $\therefore \alpha=\frac{\pi}{3}$ (2 marks)



Let P be the centre of the circle. $\therefore \angle BPA=\frac{2\pi}{3}$ ($\angle$ at centre is twice $\angle$ at circumference when subtended by the same arc)

As $BP=PA$ (equal radii of same circle)

$\therefore \Delta BPA$ isosceles. $\therefore \angle PAB=\angle PBA$

(base $\angle$'s of isosceles Δ are equal).

$\therefore \angle PAB=\frac{\pi}{6}$ ($\angle$ sum of ΔPAB)

Now, in ΔOAB, $\tan x=\frac{OB}{OA}=\frac{2}{2\sqrt{3}}=\frac{1}{\sqrt{3}}$

$\therefore x=\frac{\pi}{6}$. Since $\angle BPA+\angle PAO$

$=\frac{2\pi}{3}+\frac{\pi}{6}+\frac{\pi}{6}=\pi$ and they are co-interior angles. $\therefore BP\parallel OA$ ie: the centre P lies on the line BP with equation $y=2$. Equation of line PA:

$m=\tan\frac{2\pi}{3}=-\sqrt{3}$. Using pt-gradient formulae, $y-0=-\sqrt{3}(x-2\sqrt{3})$

$\therefore y=-\sqrt{3}x+6$ sub $y=2$

$2=-\sqrt{3}x+6$ $\therefore x=\frac{4\sqrt{3}}{3}$.

$P(\frac{4\sqrt{3}}{3},2)$. Radius is equal to x value of $P=\frac{4\sqrt{3}}{3}$ units.

Question 4

ai) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Using implicit differentiation

$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

$\therefore \frac{dy}{dx} = \frac{-b^2x}{a^2y}$

At $P(x_1, y_1)$, $m_{tan} = \frac{-b^2x_1}{a^2y_1}$

$\therefore m_{norm} = \frac{a^2y_1}{b^2x_1}$

Using the gradient point formula we get:

$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1)$

$\therefore b^2x_1y - b^2x_1y_1 = a^2y_1x - a^2y_1x_1$

$\therefore a^2y_1x - b^2x_1y = x_1y_1(a^2-b^2)$

$\therefore \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$ (equation of normal)

(3marks)

ii) Let $y=0$ to find A.

$\therefore \frac{a^2x}{x_1} = a^2 - b^2 = a^2e^2$

$\therefore x = e^2x_1$ $\therefore A(e^2x_1, 0)$

If A were to be the focus

$\therefore x = ae$ $\therefore ae = e^2x_1$

$\therefore x_1 = \frac{a}{e}$

However, $x_1 = \frac{a}{e}$ is the equation of the directrix and does not lie on the ellipse

$\therefore$ A cannot be the focus.

(2marks)

iii) Let $x=0$ to find B.

$\therefore -\frac{b^2y}{y_1} = a^2 - b^2$

$\therefore y = \frac{(a^2-b^2)y_1}{-b^2} = \frac{a^2e^2y_1}{-b^2}$

$\therefore B\left(0, \frac{-a^2e^2y_1}{b^2}\right)$

Let $AP = m$, $BA = n$

$\therefore \frac{nx_1 + m\times 0}{m+n} = e^2x_1$

as $x_1 \neq 0$

$\therefore n = e^2(m+n)$ $\therefore n = e^2m + e^2n$

$\therefore n(1-e^2) = e^2m$ $\therefore \frac{n}{m} = \frac{e^2}{1-e^2}$.

$\therefore$ The ratio is $e^2 : 1-e^2$

(2marks)

Alternatively,

$$\frac{\frac{-a^2e^2y_1m}{b^2} + ny_1}{m+n} = 0$$

$\therefore \frac{-a^2e^2y_1m}{b^2} + ny_1 = 0$

$\therefore -a^2e^2m + b^2n = 0$

$\therefore ma^2e^2 = b^2n$

$\therefore \frac{n}{m} = \frac{a^2e^2}{a^2(1-e^2)} = \frac{e^2}{1-e^2}$.

iv) $A(e^2x_1, 0)$ $B\left(0, -\frac{a^2e^2y_1}{b^2}\right)$

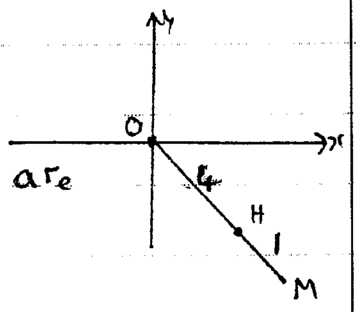
$\therefore M_x = \frac{e^2x_1 + 0}{2}$, $M_y = \frac{-\frac{a^2e^2y_1}{b^2}}{2}$.

$\therefore M\left(\frac{e^2x_1}{2}, \frac{-a^2e^2y_1}{2b^2}\right)$

$\therefore M\left(\frac{a^2-b^2}{2a^2}x_1, -\frac{(a^2-b^2)y_1}{2b^2}\right)$

(as $b^2 = a^2(1-e^2)$).

v)



co-ordinates of H are

$$X = \frac{2(a^2-b^2)}{5a^2}x_1$$

$$Y = -\frac{2(a^2-b^2)}{5b^2}y_1$$

To find the locus of H we must find a relationship between X and Y independent of x_1 and y_1.

Let $A = \frac{2(a^2-b^2)}{5a^2}$, $B = -\frac{2(a^2-b^2)}{5b^2}$.

$$\frac{X}{A} = x_1, \quad \frac{Y}{B} = y_1$$

$$\frac{X^2}{a^2A^2} = \frac{x_1^2}{a^2} \text{ (1)}, \quad \frac{Y^2}{b^2B^2} = \frac{y_1^2}{b^2} \text{ (2)}$$

(1) + (2) gives

N.B since $P(x_1, y_1)$ lies on the ellipse $\therefore \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$.

$$\therefore \frac{X^2}{a^2A^2} + \frac{Y^2}{b^2B^2} = 1$$

$$\therefore \frac{X^2}{a^2 \cdot \frac{4}{25a^4}(a^2-b^2)^2} + \frac{Y^2}{b^2 \cdot \frac{4}{25b^4}(a^2-b^2)^2} = 1$$

$$\therefore \frac{X^2}{\frac{4(a^2-b^2)^2}{25a^2}} + \frac{Y^2}{\frac{4(a^2-b^2)^2}{25b^2}} = 1$$

which is the locus of an ellipse since a, b are constant. NB the semi-major axis

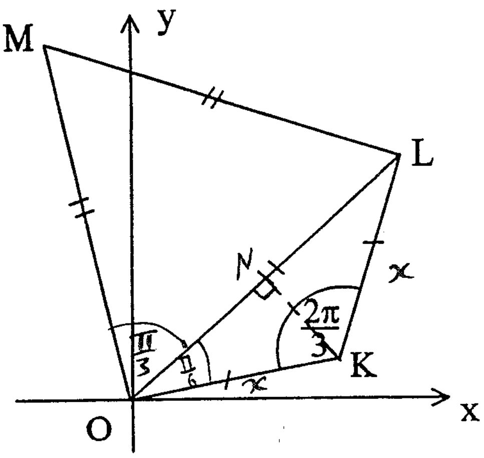
$$\frac{2(a^2-b^2)}{5a} = \frac{2a^2e^2}{5a} = \frac{2ae^2}{5}$$

and semi-minor axis

$$\frac{2a^2e^2}{5b} = \frac{2e^2a}{5\sqrt{1-e^2}}$$

(3 marks)

b)



Let $OK = x$ $\therefore KL = x$ (equal sides in isosc Δ)

Construct a perpendicular KN to base OL. $\therefore \angle OKN = \frac{\pi}{3}$. (KN bisects $\angle OKL$)

$\therefore \angle NOK = \frac{\pi}{6}$ ($\angle$ sum of Δ KON)

$\cos\frac{\pi}{6} = \frac{ON}{x}$ (in Δ OKN)

$\therefore ON = \frac{x\sqrt{3}}{2}$ $\quad \therefore OL = \frac{x\sqrt{3}}{2} \times 2 = x\sqrt{3}$

$\therefore OM = x\sqrt{3}$ (equal sides in equilateral Δ MOL)

$\therefore |\beta| = x\sqrt{3} = \sqrt{3}|\alpha|$ (as $OM = |\beta|$, $OK = |\alpha|$).

now, $\angle KOM = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$.

$\therefore \beta = \alpha\sqrt{3}\,\text{cis}\,\frac{\pi}{2}$ (as we are rotating α by $\frac{\pi}{2}$ about the origin, with enlargement of $\sqrt{3}$ to get β).

$\therefore \beta = \alpha\sqrt{3}\,i$ (By squaring both sides)

$\therefore \beta^2 = -3\alpha^2$

$\therefore 3\alpha^2 + \beta^2 = 0$ (4 marks).

Question 5

a) $x^x a^{\ln x} = x \qquad a>0,\ x>1$

$\ln(x^x a^{\ln x}) = \ln x$

$\ln x^x + \ln a^{\ln x} = \ln x$

$x\ln x + \ln x . \ln a = \ln x$

$\therefore x + \ln a = 1$ (Divide by $\ln x \neq 0$)

$\therefore x = 1 - \ln a$ as $x > 1$

(2 marks)

b) i $\frac{ds}{dt} = As$

$\therefore \int_{S_0}^{S} \frac{ds}{s} = \int_0^t A\,dt$

$\therefore [\ln S]_{S_0}^{S} = [At]_0^t$

$\therefore \ln\left(\frac{S}{S_0}\right) = At$

$\therefore S(t) = S_0 e^{At}$

$\therefore S(p) = S_0 e^{Ap}$ (2 marks)

ii) $\frac{dT}{dt} = 2B\,S(p)t$

$\frac{dT}{dt} = 2B\,S_0\,e^{Ap}t$

$\therefore \int_0^T dT = 2BS_0e^{Ap}\int_p^t t\,dt$

$\therefore [T]_0^T = BS_0e^{Ap}[t^2]_p^t$

$\therefore T = BS_0e^{Ap}(t^2 - p^2)$, when $t \geq p$

(2 marks).

iii) $T = BS_0e^{Ap}(t^2 - p^2)$

given $A = 0.032$ & $t = 75$

$\therefore T = BS_0e^{0.032p}(5625 - p^2)$

$\therefore \frac{dT}{dp} = 0.032BS_0e^{0.032p}(5625 - p^2) - 2pBS_0e^{0.032p}$

$\therefore \frac{dT}{dp} = BS_0e^{0.032p}(180 - 0.032p^2 - 2p)$

Let $\frac{dT}{dp} = 0$ to find possible stationary points.

$\therefore p^2 + 62.5p - 5625 = 0$

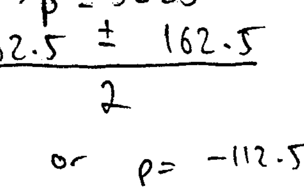
$\therefore p = \frac{-62.5 \pm 162.5}{2}$

p		50	
$\frac{dT}{dp}$	$+$	0	$-$
T	$\nearrow$	max	$\searrow$

$\therefore p = 50$ or $p = -112.5$ (not valid)

$\therefore p = 50$ days (3 marks)

c) i



Consider the triangle above, taken as a slice of the solid with thickness Δx.

$\tan 30 = \frac{y}{h} \quad \therefore h = y\sqrt{3}$.

$\therefore$ Area of slice $= y\sqrt{3} \times 2y \times \frac{1}{2} = y^2\sqrt{3}$

As, $4x^2 + 9y^2 = 36 \therefore y^2 = \frac{36 - 4x^2}{9}$

$\therefore$ Area of the slice $= \sqrt{3}\left(\frac{36 - 4x^2}{9}\right)$ (2 marks)

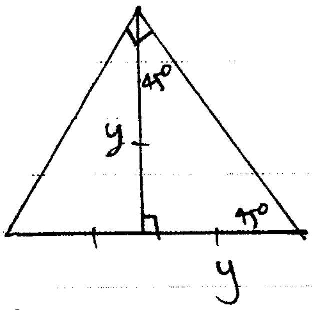
$\therefore$ Volume of $K = \frac{\sqrt{3}}{9}\int_0^3 (36 - 4x^2)\,dx$

$= \frac{\sqrt{3}}{9}\left[36x - \frac{4x^3}{3}\right]_0^3 = \frac{\sqrt{3}}{9}\left[108 - 36\right]$

$= 8\sqrt{3}$ units^3.

(2 marks)

iii)



Area of slice $= 2y \times y \times \frac{1}{2} = y^2$.

$\therefore$ Volume of J

$$= \int_0^3 y^2\,dx$$

$$= \frac{1}{9}\int_0^3 (36 - 4x^2)\,dx.$$

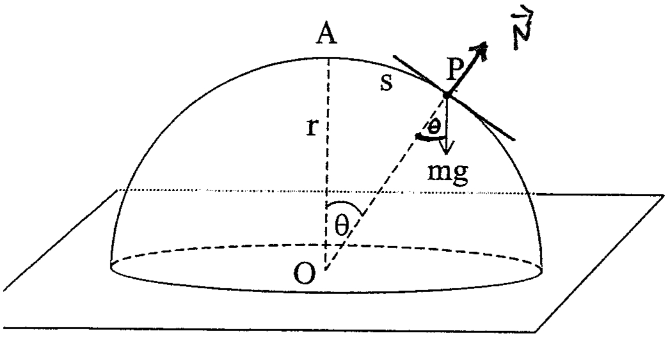
$$= \frac{1}{9}\left[36x - \frac{4x^3}{3}\right]_0^3$$

$$= \frac{1}{9}[108 - 36] = 8 \text{ units}^3.$$

$\therefore$ Ratio of volumes of solid K to solid J is $8\sqrt{3} : 8$

$= \sqrt{3} : 1$ (2 marks)

Question 6



The system is the mass m at P.

Forces applicable are $\vec{mg}$ and $\vec{N}$.

$\vec{mg}$ the weight of the particle vertically downwards

$\vec{N}$ the Normal reaction of the hemisphere.

i) $s = r\theta$

$$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

$\therefore v = r\dfrac{d\theta}{dt}$ $\left(\dfrac{ds}{dt} = v\right)$.

now, $\dfrac{d^2s}{dt^2} = \dfrac{dv}{dt} = \dfrac{dv}{d\theta}\cdot\dfrac{d\theta}{dt} = \dfrac{dv}{d\theta}\cdot\dfrac{v}{r}$

$= \dfrac{1}{r}\dfrac{d\left(\frac{1}{2}v^2\right)}{d\theta}$ (2 marks)

ii) Projecting tangentially,

$mg\sin\theta = m\dfrac{d^2s}{dt^2} = \dfrac{m}{r}\dfrac{d\left(\frac{1}{2}v^2\right)}{d\theta}$

$\therefore \frac{1}{2}v^2 = \int gr\sin\theta\,d\theta$

$\frac{1}{2}v^2 = -gr\cos\theta + c$

when $\theta = 0$, $v = \frac{1}{4}\sqrt{gr}$ $\therefore c = \frac{33}{32}gr$

$\therefore \frac{1}{2}v^2 = -gr\cos\theta + \frac{33}{32}gr$

$\therefore v^2 = \frac{33gr}{16} - 2gr\cos\theta$

$\therefore v^2 = \frac{gr}{16}(33 - 32\cos\theta)$. (3 marks)

iii) Projecting towards the normal,

$mg\cos\theta - N = \dfrac{mv^2}{r}$

$N = mg\cos\theta - \dfrac{mv^2}{r}$

$= mg\cos\theta - \dfrac{m}{r}\cdot\dfrac{gr}{16}(33 - 32\cos\theta)$

$= mg\left(3\cos\theta - \frac{33}{16}\right)$

$\therefore N = \frac{mg}{16}(48\cos\theta - 33)$ (3 marks)

iv) Particle leaves the hemisphere when $N = 0$.

ie: $48\cos\theta = 33$

$\therefore \cos\theta = \frac{11}{16}$

at this value of θ,

$v^2 = \frac{gr}{16}\left(33 - 32 \times \frac{11}{16}\right)$

$v^2 = \frac{11gr}{16}$ $\therefore v = \frac{\sqrt{11gr}}{4}$ m/s (2 marks)

b) i) $\int_0^{\frac{\pi}{6}} \frac{4\cos x}{1+4\sin^2 x}\,dx$

let $u = \sin x$ for $x = \frac{\pi}{6}, u = \frac{1}{2}$

$du = \cos x\,dx$ $x = 0, u = 0.$

$= \int_0^{\frac{1}{2}} \frac{4du}{1+4u^2} = 2\int_0^{\frac{1}{2}} \frac{\frac{1}{2}du}{\frac{1}{4}+u^2} = \left[2\tan^{-1}2u\right]_0^{\frac{1}{2}}$

$= \frac{\pi}{2}$. (2 marks)

ii) For $n \geq 2$ and $0 \leq x \leq \frac{\pi}{6}$,

$0 \leq \sin^n x \leq \sin^2 x$ (since $0 \leq \sin x \leq 1$) NB: The higher the power, the lower the term is.

$\therefore 0 \leq 4\sin^n x \leq 4\sin^2 x$

$\therefore 1 \leq 1+4\sin^n x \leq 1+4\sin^2 x$

$\therefore 1 \geq \frac{1}{1+4\sin^n x} \geq \frac{1}{1+4\sin^2 x}$

Now $4\cos x > 0$, as $0 \leq x \leq \frac{\pi}{6}$

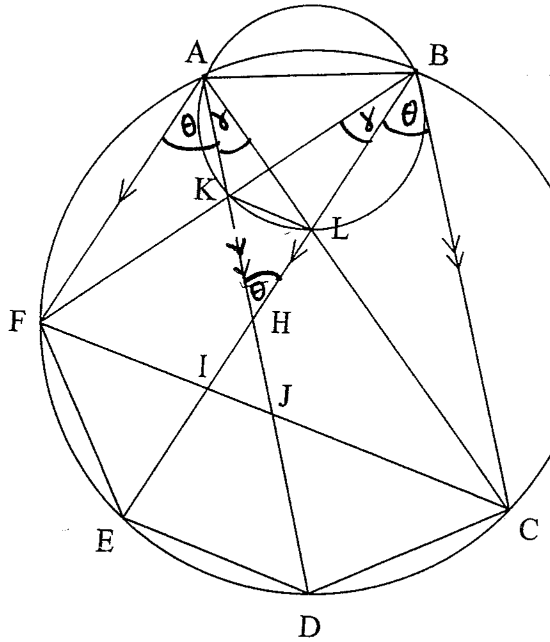
$4\cos x \geq \frac{4\cos x}{1+4\sin^n x} \geq \frac{4\cos x}{1+4\sin^2 x}$

$\therefore \int_0^{\frac{\pi}{6}} \frac{4\cos x\,dx}{1+4\sin^2 x} \leq \int_0^{\frac{\pi}{6}} \frac{4\cos x\,dx}{1+4\sin^n x} \leq \int_0^{\frac{\pi}{6}} 4\cos x\,dx$

$\therefore \frac{\pi}{2} \leq \int_0^{\frac{\pi}{6}} \frac{4\cos x}{1+\sin^n x}dx \leq 2$. (3 marks)

Question 7:

a)



Data: $BE \parallel AF$, $AD \parallel BC$

Aim: show ABLK cyclic quadrilateral

Proof: Let $\angle FAD = \theta$

$\therefore \angle AHL = \theta$ (alt ∠'s, $AF \parallel BE$)

$\therefore \angle EBC = \theta$ (alt ∠'s, $AD \parallel BC$)

$\therefore \angle FAD = \angle EBC = \theta$. Now, since these two angles are angles at the circumference of the circle and equal ∴ They must be subtended by equal arcs. ∴ Arc FD = Arc EC. But, as Arc ED is a common arc.

∴ Arc EF = Arc DC. Let $\angle EBF = \gamma$

$\therefore \angle DAC = \gamma$ (angle at circumference of a circle subtended by equal Arcs.

$\therefore$ ABLK is cyclic (angles at the circumference of a circle subtended by chord LK are equal)

(3 marks)

ii) Aim: $\Delta BKL \,|||\, \Delta BFI$.

construction: construct circle ABKL. construct chord KL.

Proof: $\angle LAB = \angle BKL$

(angles on a circumference subtended by the same arc LB are equal)

Also, $\angle LAB = \angle BFC$.

(angles on a circumference subtended by the same arc BC are equal).

$\therefore \angle BKL = \angle BFC$.

$\therefore KL \parallel FC$

(Corresp. $\angle$'s are equal, converse theorem)

Hence, since $\angle BKL = \angle BFC$, and $\angle IBF$ common, $\Delta BKL \,|||\, \Delta BFI$ (equiangular).

(3 marks)

iii) Aim: $\dfrac{BK}{BF} \times \dfrac{AJ}{AK} \times \dfrac{FI}{JC} = 1$.

since $KL \parallel FC \therefore \angle AKL = \angle AJC$

(corresponding $\angle$'s, $KL \parallel FC$).

$\angle LAK$ common

$\therefore \Delta AKL \,|||\, \Delta AJC$ (equiangular).

Ratio of sides

$\therefore \dfrac{AK}{AJ} = \dfrac{KL}{JC}$ $\quad \therefore AK.JC = AJ.KL$

$\therefore KL = \dfrac{AK.JC}{AJ}$ ①

also, $\dfrac{BK}{BF} = \dfrac{KL}{FI}$ (Ratio of sides of $\Delta BKL \,|||\, \Delta BFI$)

$\therefore BK.FI = BF.KL$ $\quad \therefore KL = \dfrac{BK.FI}{BF}$ ②

From ① and ②

$\therefore \dfrac{BK.FI}{BF} = \dfrac{AK.JC}{AJ}$.

$\therefore \dfrac{AJ\,BK\,FI}{AK\,BF\,JC} = 1$

$\therefore \dfrac{BK}{BF} \times \dfrac{AJ}{AK} \times \dfrac{FI}{JC} = 1$ (3 marks)

b) $I_n = \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^n x \, dx$

$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx$

N.B: $y = \cot x$

$\dfrac{dy}{dx} = \dfrac{\cos x}{\sin x}$

$\therefore \dfrac{dy}{dx} = \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x}$

$= \dfrac{-1}{\sin^2 x}$

$= -\operatorname{cosec}^2 x$

Using by parts:

let $u = \operatorname{cosec}^{n-2} x$

$du = -(n-2)\operatorname{cosec}^{n-3} x \cot x \operatorname{cosec} x \, dx$

$dv = \operatorname{cosec}^2 x \, dx \quad \therefore v = -\cot x$.

$$\therefore I_n = \left[\cot x \operatorname{cosec}^{n-2} x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$- (n-2) \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^{n-2} x \cot^2 x \, dx.$$

$$= 2^{n-2}\sqrt{3} - (n-2)\left[\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^n x \, dx - \int \operatorname{cosec}^{n-2} x \, dx\right]$$ (since $\cot^2 x = \operatorname{cosec}^2 x - 1$)

$$\therefore I_n = 2^{n-3}\sqrt{3} - (n-2)(I_n - I_{n-2})$$

$$= 2^{n-2}\sqrt{3} - (n-2)I_n + (n-2)I_{n-2}$$

$$\therefore (n-1)I_n = 2^{n-2}\sqrt{3} + (n-2)I_{n-2}.$$

(4 marks)

ii) $\displaystyle\int_0^{\frac{\pi}{3}} \sec^4 x \, dx$

let $u = \frac{\pi}{2} - x$, $\sec(\frac{\pi}{2} - u) = \operatorname{cosec} u.$

$-du = dx$

$$\therefore J = \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} -\operatorname{cosec}^4 u \, du$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^4 u \, du = I_4$$

From i) $(n-1)I_n = 2^{n-2}\sqrt{3} + (n-2)I_{n-2}$

For $n=4$, $3I_4 = 4\sqrt{3} + 2I_2.$

For $n=2$, $I_2 = \sqrt{3}$

$$\therefore 3I_4 = 4\sqrt{3} + 2\sqrt{3} = 6\sqrt{3}$$

$$\therefore I_4 = 2\sqrt{3}$$

$$\therefore \int_0^{\frac{\pi}{3}} \sec^4 x \, dx = 2\sqrt{3}$$ (2 marks)

Question 8

a i) $x^5 - i = x^5 - i^5$ (as $i^5 = i$)

$= (x-i)(x^4 + ix^3 + i^2x^2 + i^3x + i^4)$

$= (x-i)(x^4 + ix^3 - x^2 - ix + 1)$

but since $x \neq i$

$1 - ix - x^2 + ix^3 + x^4 = 0.$ (2 marks)

ii) $x^5 - i = 0$

$x^5 = i$ let $x = r\operatorname{cis}\theta$. using De moivre's Th. $x^5 = r^5 \operatorname{cis} 5\theta$

$\therefore r^5 \operatorname{cis} 5\theta = \operatorname{cis}\frac{\pi}{2}$

$\therefore r^5 = 1 \quad \therefore r = 1$

$\operatorname{cis} 5\theta = \operatorname{cis}\frac{\pi}{2}$

$\therefore \cos 5\theta = \cos\frac{\pi}{2}$

$\therefore 5\theta = \frac{\pi}{2} + 2k\pi$

$\theta = \frac{\pi}{10} + \frac{2k\pi}{5}$

$k=0, \theta = \frac{\pi}{10}$

$k=1, \theta = \frac{\pi}{2}$

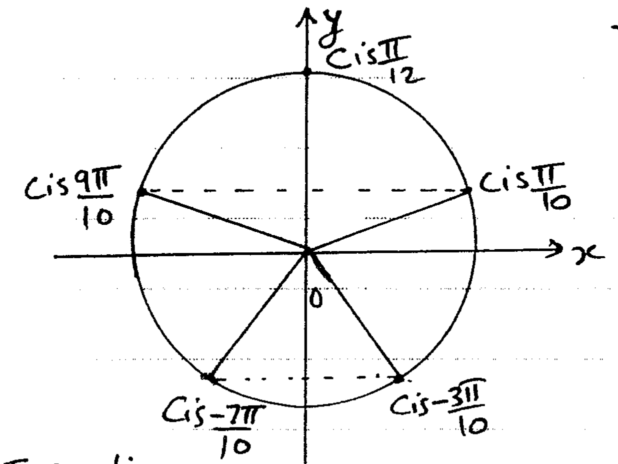
$k=2, \theta = \frac{9\pi}{10}$

$k=-1, \theta = -\frac{3\pi}{10}$

$k=-2, \theta = -\frac{7\pi}{10}.$

Hence $x^5 - i = 0$ could be expressed as

$$\left(x - \operatorname{cis}\frac{\pi}{10}\right)\left(x - \operatorname{cis}\frac{9\pi}{10}\right)\left(x - \operatorname{cis}\left(-\frac{7\pi}{10}\right)\right)\left(x - \operatorname{cis}\left(-\frac{3\pi}{10}\right)\right)\left(x - \operatorname{cis}\frac{\pi}{2}\right) = 0$$



From diagram,

let $\alpha = cis\frac{\pi}{10}$ $\therefore$ $cis\frac{9\pi}{10} = -\bar{\alpha}$

let $\beta = cis\left(\frac{-3\pi}{10}\right)$ $\therefore$ $cis\left(\frac{-7\pi}{10}\right) = -\bar{\beta}$

$\therefore (x-\alpha)(x-\bar{\alpha})(x-\beta)(x+\bar{\beta})(x-i)=0$

$\therefore \left(x^2+(\bar{\alpha}-\alpha)x-\alpha\bar{\alpha}\right)\left(x^2+(\bar{\beta}-\beta)x-\beta\bar{\beta}\right)(x-i)=0$

N.B $\alpha\bar{\alpha} = |\alpha|^2 = 1$ and $\beta\bar{\beta} = |\beta|^2 = 1$

Also, $\bar{\alpha}-\alpha = \cos\frac{\pi}{10} - i\sin\frac{\pi}{10} - \cos\frac{\pi}{10} - i\sin\frac{\pi}{10}$

$\therefore \bar{\alpha} = \alpha - -2i\sin\frac{\pi}{10}$

Similarly, $\bar{\beta}-\beta = -2i\sin\left(-\frac{3\pi}{10}\right)$

$= 2i\sin\left(\frac{3\pi}{10}\right)$ since $(-\sin\alpha = \sin(-\alpha))$

$\therefore \left(x^2-2i\sin\frac{\pi}{10}x-1\right)\left(x^2+2i\sin\frac{3\pi}{10}x-1\right)(x-i)=0$

(4 marks)

iii) From i) and ii) we get:

$1-ix-x^2+ix^3+x^4 = \left(x^2-2i\sin\frac{\pi}{10}x-1\right)\left(x^2+2i\sin\frac{3\pi}{10}x-1\right)$

By equating the coefficients of x^2, we get: $(-1)+\left(-2i\sin\frac{\pi}{10}\right)\left(2i\sin\frac{3\pi}{10}\right)+(-1)=-1$

$\therefore 4\left(\sin\frac{\pi}{10}\right)\left(\sin\frac{3\pi}{10}\right)=1$

$\therefore \sin\frac{\pi}{10}\sin\frac{3\pi}{10} = \frac{1}{4}$ (2marks)

b)

1st lunch (For either Grace or David)

2nd lunch (For either Grace or David)

	1	2	3	4	5	...	n
1							
2	X						
3	X	X					
4	X	X	X				
5	X	X	X	X			
6	X	X	X	X	X		
...							
n							

The above table shows the number of choices for the lunches according to the available weeks for either Grace or David. From the table we can see that the number of choices for each of them to attend the restaurant on any 2 sundays is the bottom part of this table.

Now, the total number of boxes in the table is n^2.

$\therefore$ The top and bottom parts without the diagonal is n^2-n.

$\therefore$ The bottom part is $\frac{1}{2}(n^2-n) = \frac{1}{2}n(n-1)$

$\therefore$ Each one of them has $\frac{1}{2}n(n-1)$ choices.

$\therefore$ The total number of choices for both of them is $\left[\frac{1}{2}n(n-1)\right]^2$.

i) To meet on the first sunday both of them must choose any one box

From the first column.

$\therefore$ There are $(n-1)$ possible choices for each of them.

$\therefore$ The total possible choices for both is $(n-1)^2$.

$$\therefore P = \frac{(n-1)^2}{\frac{n^2}{4}(n-1)^2} = \frac{4}{n^2}$$

(2 marks)

ii) In order for them to meet only once:

Weeks	Grace's total choices	David's possible choices (For every choice of Grace)	Total possible choices for both.
For $n=4$	$\frac{4}{2}(4-1)$	$4=2(4)-4$	$\frac{4}{2}(4-1).[2(4)-4]$
For $n=5$	$\frac{5}{2}(5-1)$	$6=2(5)-4$	$\frac{5}{2}(5-1)[2(5)-4]$
For $n=6$	$\frac{6}{2}(6-1)$	$8=2(6)-4$	$\frac{6}{2}(6-1)[2(6)-4]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
For $n=n$	$\frac{n}{2}(n-1)$	$2(n)-4$	$\frac{n}{2}(n-1)(2n-4)$

$\therefore$ The probability that they will meet only once in a period of n weeks is:

$$P = \frac{\frac{n}{2}(n-1)(2n-4)}{\left[\frac{n}{2}(n-1)\right]^2}$$

$$= \frac{2n-4}{\frac{n}{2}(n-1)} = \frac{4n-8}{n(n-1)}$$

$$\therefore P = \frac{4(n-2)}{n(n-1)}$$

(2 marks)

ii) In order for them not to meet:

Weeks	Grace's total choices	David's possible choices (for every choice of Grace	Total possible choices for both.
For $n=4$	$\frac{4}{2}(4-1)$	1	$\frac{4}{2}(4-1).(1)$
For $n=5$	$\frac{5}{2}(5-1)$	$3=1+2$	$\frac{5}{2}(5-1)(1+2)$
For $n=6$	$\frac{6}{2}(6-1)$	$6=1+2+3$	$\frac{6}{2}(6-1)(1+2+3)$
For $n=7$	$\frac{7}{2}(7-1)$	$10=1+2+3+4$	$\frac{7}{2}(7-1)(1+2+3+4)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
For $n=n$	$\frac{n}{2}(n-1)$	$1+2+3+\dots+(n-3)$ using the sum of arithmetic series, we get $\frac{(n-3)}{2}(n-2)$	$\frac{n}{2}(n-1)\frac{(n-3)(n-2)}{2}$

$\therefore$ The probability that they will never meet in a period of n weeks is:

$$P = \frac{\frac{n}{2}(n-1)\frac{(n-3)(n-2)}{2}}{\left[\frac{n}{2}(n-1)\right]^2}$$

$$\therefore P = \frac{(n-3)(n-2)}{n(n-1)}$$

(3 marks)