HIGHER SCHOOL CERTIFICATE EXAMINATION TRIAL PAPER

2005

MATHEMATICS

EXTENSION 2

Time Allowed – Three Hours
(Plus 5 minutes reading time)

Directions to Candidates
- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Board-approved calculators may be used.

YEAR 12 –TRIAL 2005 – EXTENSION 2

QUESTION 1 — MARKS

a) Use integration by parts to find $\int x \cos x \, dx$ — 2

b) Find $\int \frac{dx}{3x^2+24x+51}$ — 2

c) Using the substitution $u = \sqrt{1-x}$, or otherwise,

evaluate $\int_0^1 (x^2-1)\sqrt{1-x}\,dx$ — 3

d) Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\cos x}\,dx$ — 3

e) i) Find the real numbers a and b such that

$$\frac{x^2+3x}{(x^2+1)(x+1)} \equiv \frac{ax+1}{x^2+1} + \frac{b}{x+1}$$ — 2

ii) Find $\int_0^1 \frac{x^2+3x}{(x^2+1)(x+1)}\,dx$ — 3

QUESTION 2 — MARKS

a) Find the complex number z if $\operatorname{Re}(z+2\bar{z}-1)=0$ and $\operatorname{Re}(z) = 2\operatorname{Im}(z)$ — 2

b) If $z = 1 - i$ and $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, find in the form $x + iy$, $\omega^9(\bar{z})^2$ — 2

c) i) Find in modulus argument form the three roots of $z^3 + 8 = 0$ — 2

ii) Find a cubic equation of degree three whose roots are obtained by rotating the roots of $z^3+8=0$ through an angle $\frac{\pi}{6}$ anticlockwise. — 2

d) Let A be a point in the Argand diagram representing a complex number z where A is not the origin and does not lie on the real axis. Considering that z always satisfies the equation $\operatorname{Arg}\left(\frac{z+1}{\bar{z}+1}\right) = \operatorname{Arg} z$

i) Show that $|z| = 1$ — 3

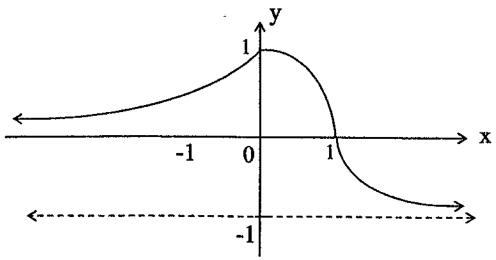
ii) Sketch the locus of z. — 1

iii) The points B and C represent the complex numbers $\bar{z}$ and w respectively. Given that OACB is a parallelogram, find the complex number w in terms of z and $\bar{z}$. Hence find the locus of C as A moves in the complex plane. — 3

QUESTION 3

MARKS

a)



The above diagram shows the graph of $y = f(x)$.
Sketch on separate diagrams the following curves, indicating clearly any turning points and asymptotes.

i) $y = \dfrac{1}{f(x)}$ 2

ii) $y = \sqrt{f(x)}$ 2

iii) $y = f(|x|)$ 1

iv) $y = \tan^{-1}(f(x))$ 2

b) The region bounded by the curve $y = \sin x$ where $0 \le x \le \pi$ and the line $y = c$, where c is any constant, is rotated about the line $y = c$ to form a solid.

i) Find the volume of the solid in terms of c. 2

ii) Show that for $c = \dfrac{2}{\pi}$ the volume of the solid is a minimum. 2

c) i) Show that $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. 2

ii) Use the result in (i), or otherwise, to show that 2

$$\int_0^1 \frac{(1-x)^n}{x^n + (1-x)^n}\,dx = \frac{1}{2}, \text{ for any integer n.}$$

QUESTION 4

MARKS

a) Consider the polynomial $Q(x) = \dfrac{x^4}{4} - \dfrac{x^3}{3} + x^2 - 2x + p$, where p is a real number.

i) Show that Q(x) has exactly one turning point for all values of p. 1

ii) For what values of p does the equation $Q(x) = 0$ have: 3

α) no roots

β) 2 roots.

b) A particle of mass m falls from rest under gravity and the resistance to its motion is mkv^3, where v is its speed and k is a positive constant.

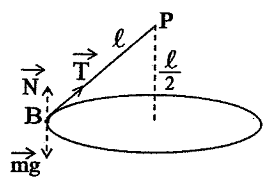
i) Show that the equation of motion is given by $\ddot{x} = g - kv^2$, where x is measured from the point of fall. 2

ii) Show that $v^2 = \dfrac{g}{k}(1 - e^{-2kx})$ 2

iii) Show that the particle reaches half its terminal velocity when $x = \dfrac{1}{2k}\ln\left(\dfrac{4}{3}\right)$ 2

c) A particle B of mass m is suspended by a string of length ℓ from a point P which is at height $\dfrac{\ell}{2}$ above a smooth horizontal table.
While the particle B is moving on the surface of the table in a horizontal circle at an angular velocity w radians per second, the string remains taut.

i) Find the tension T along the string and the normal reaction N exerted by the table on the particle B 3



ii) Show that the particle B remains in contact with the table with the string taut if $w^2 \le \dfrac{2g}{\ell}$ 2

QUESTION 5 MARKS

a) Consider the equation $x^{2n+1} - ax + b = 0$, where a, b are real numbers and n is a positive integer.

i) If $x = 1$ is a double root, show that $a = 2n + 1$ and $b = 2n$. 2

ii) If $n = 1$ and the roots of the equation are 1, 1 and α, find the monic polynomial $Q(x)$ of degree 3 whose roots are 3, 3, 3α. 1

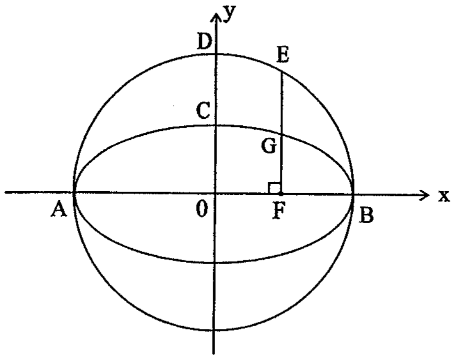
b) Consider the function $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

i) Find its domain and range 2

ii) Sketch the graph of the function. 2

c) The ellipse shown in the diagram touches the circle at A and B and intersects the y axis at C(0, a). C is the mid point of OD.
The perpendicular from a point E $(\sqrt{3}\,a, a)$ on the circle meets the x axis at F and intersects the ellipse at G.

i) Find the equation of the ellipse. 1



ii) Find the coordinates of F and show that F is a focus of the ellipse. 2

iii) Find the coordinates of G. 1

iv) Show that the tangent to the ellipse at G is $\sqrt{3}\,x + 2y - 4a = 0$. 1

v) Show that the tangent to the circle at E is $\sqrt{3}\,x + y - 4a = 0$. 1

vi) Let M be the point of intersection of the two tangents. Find the coordinates of M and determine its locus. 2

QUESTION 6 MARKS

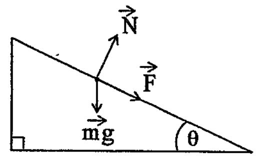
a) The region enclosed by the curve $y^2 = x(x-1)^2$ and the line $x = 1$ is rotated about the line $x = -1$ to form a solid. 3
Use the method of cylindrical shells to find the volume the solid.

b) Let $u_1 = 1$ and $u_{n+1} = \sqrt{2 + u_n}$, for all positive integers $n \geq 1$.

i) By using mathematical induction show that: 3
$u_n > 0$, $u_{n+1} > u_n$, and $u_n < 2$.

ii) Show that $\lim_{n \to \infty} u_n = 2$. 1

c) A car is travelling at a speed v ms^{-1} around a circular track of radius r meters, which is banked at an angle θ to the horizontal. Assume that the car is represented by a point mass m, and that the forces acting on the car are the gravitational force mg, a lateral friction force F, and a normal reaction N to the road.



i) By resolving the horizontal and vertical components of forces, show that 2

$$F \sin\theta = N\cos\theta - mg$$

$$F\cos\theta = \frac{mv^2}{r} - N\sin\theta$$

ii) Show that $F = \frac{m(v^2 - gr\tan\theta)}{r}\cos\theta$ 2

iii) The car has no tendency to slip when travelling at a speed of $30ms^{-1}$ round the track with $r = 200m$. 2
Find the angle θ. Take $g = 9.8\ ms^{-2}$.

iv) Explain what happens when the car is travelling around the track at a speed $v > 30\ ms^{-1}$ 2

QUESTION 7 — MARKS

a) Let $I_n = \int_1^e (\ln x)^n \, dx$ for $n \geq 0$.

i) Using integration by parts, show that — 2

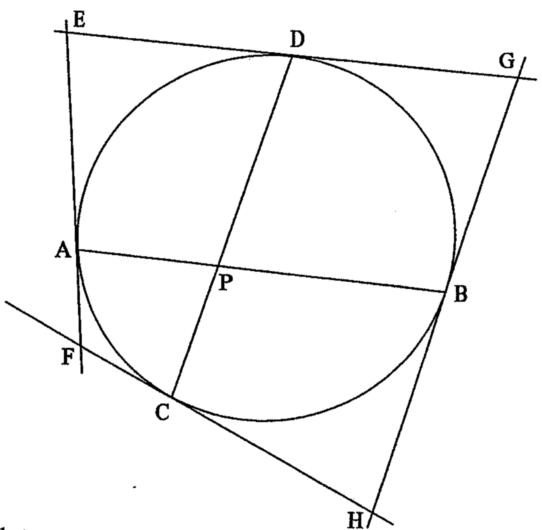
$I_n + n\, I_{n-1} = e$, for $n \geq 1$.

ii) Evaluate I_2 — 1

iii) Explain why $I_n < I_{n-1}$, hence show — 3

$\frac{e}{n+2} < I_n < \frac{e}{n+1}$

b) The diagram shows a circle in which two chords AB and CD intersect at P. EF, FH, HG and GE are tangents to the circle at A, C, B and D respectively.



i) Show that — 3
$\angle PAE + \angle PBH = 180°$

ii) By considering the quadrilaterals PAED and PBHC, or otherwise, show that — 3
$2\angle APC + \angle DEA + \angle CHB = 360°$

iii) Show that E, F, H and G are concyclic if and only if the chords AB and CD are perpendicular. — 3

QUESTION 8 — MARKS

a) Let ω be a root of the equation $P(x) = 1 + x + x^2 + ... + x^7 = 0$

i) Show that $\omega^8 = 1$ — 1

ii) Show that $\omega = -1$ is the only real root of the equation. — 2

iii) Show that for any integer $n \geq 0$, ω is a root of — 2

$Q(x) = 1 + x^{8n+1} + x^{8n+2} + ... + x^{8n+7} = 0$

Deduce that $P(x)$ is a factor of $Q(x)$. Give reasons.

b) Consider the functions $f(x) = \ln(1+x) - \frac{x}{1+x}$

and $g(x) = \ln(1+x) - x$ where $x > 0$.

i) Show that $x > \ln(1+x) > \frac{x}{1+x}$. — 2

ii) Let $u_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln n$, where n is a positive integer. — 3

Show that $u_n > 0$, $u_{n+1} - u_n < 0$ and explain why $\lim_{n \to \infty} u_n$ exists.

c) Three identical dice each numbered from 1 to 6 are rolled.

i) What is the probability that the sum of the numbers shown on the dice is no more than 6? — 2

ii) Is it profitable to bet even money on the occurrence of at least two triple sixes in 150 throws? — 3

QUESTION 1

(a) $I = \int x\cos x\,dx$

Let $u = x$ $\quad dv = \cos x\,dx$

$\frac{du}{dx} = 1$ $\quad v = \sin x$

$\therefore I = x\sin x - \int \sin x\,dx$

$= x\sin x + \cos x + C$

(2 marks)

b) $\int \frac{dx}{3(x^2+8x+17)} = \frac{1}{3}\int \frac{dx}{(x+4)^2+1}$

$= \frac{1}{3}\tan^{-1}(x+4) + C$

(2 marks)

c) $u = \sqrt{1-x} = (1-x)^{\frac{1}{2}}$

$\frac{du}{dx} = -\frac{1}{2}(1-x)^{-\frac{1}{2}} = \frac{-1}{2u}$

$dx = -2u\,du$

Changing domain:

$u = \sqrt{1-1} = 0$ for $x = 1$

$u = \sqrt{1-0} = 1$ for $x = 0$

and $u^2 = 1 - x$

$\therefore x = 1 - u^2$,

$x^2 = 1 - 2u^2 + u^4$

$\therefore \int_0^1 -2u \times (1 - 2u^2 + u^4 - 1)u\,du$

$= -2\int_0^1 u^2(u^4 - 2u^2)\,du$

$= 2\int_1^0 (u^6 - 2u^4)\,du$

$= 2\left[\frac{u^7}{7} - \frac{2u^5}{5}\right]_1^0$

$= 2[0] - 2\left[\frac{1}{7} - \frac{2}{5}\right] = \frac{18}{35}$

(3 marks)

d) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\cos x}dx = \int_0^{\frac{\pi}{2}} \frac{\cos x + 1 - 1}{1+\cos x}dx$

$= \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{1+\cos x}\right)dx.$

But $\cos x = \cos^2(x/2) - \sin^2(x/2)$

$= 2\cos^2(x/2) - 1$

$\therefore \frac{1}{1+\cos x} = \frac{1}{2\cos^2(x/2)}$

$= \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\cos x}dx = \int_0^{\frac{\pi}{2}}\left(1 - \frac{1}{2}\sec^2\left(\frac{x}{2}\right)\right)dx$

$= \left[x - \frac{1}{2}\frac{\tan(x/2)}{(1/2)}\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} - [1 - 0] = \frac{\pi}{2} - 1$

(3 marks)

e)(i) $\frac{x^2+3x}{(x+1)(x^2+1)} = \frac{ax+1}{x^2+1} + \frac{c}{x+1}$

$\therefore x^2 + 3x = (ax+1)(x+1) + C(x^2+1)$

$= (a+c)x^2 + (a+1)x + C + 1$

Equating like coefficients:

$\begin{cases} a + C = 1 \\ a + 1 = 3 \\ C + 1 = 0 \end{cases}$

$\therefore C = -1,\ a = 1 - C = 2.$

$\therefore \frac{x^2+3x}{(x+1)(x^2+1)} = \frac{2x+1}{x^2+1} - \frac{1}{x+1}$

(2 mark)

ii) From part (i) we have

$\int_0^1 \frac{x^2+3x}{(x+1)(x^2+1)}dx = \int_0^1 \frac{2x+1}{x^2+1}dx - \int_0^1 \frac{1}{x+1}dx$

$= \int_0^1 \frac{2x}{x^2+1}dx + \int_0^1 \frac{1}{x^2+1}dx - \int_0^1 \frac{1}{x+1}dx$

$= \left[\ln(x^2+1)\right]_0^1 + \left[\tan^{-1}x\right]_0^1 - \left[\ln|x+1|\right]_0^1$

$= (\ln 2 - \ln 1) + [\tan^{-1}1 - \tan^{-1}0] - [\ln 2 - \ln 1]$

$= \pi/4$

(3 marks)

QUESTION 2.

a) Let $z = x + iy$. Then $\bar{z} = x - iy$.

$Re(z + 2\bar{z} - 1) = Re(x + iy + 2x - 2iy - 1)$

$= Re(3x - 1 - iy) = 3x - 1 = 0$

$\therefore x = \frac{1}{3}.$

Since $Re(\bar{z}) = 2\,Im(z)$,

$x = 2y.$

$\therefore y = \frac{1}{2}x = \frac{1}{2}\times\frac{1}{3} = \frac{1}{6}$

$\therefore z = \frac{1}{3} + \frac{1}{6}i.$ **(2 marks)**

b) $|\bar{z}| = |1 + i| = \sqrt{1^2 + (+1)^2} = \sqrt{2}$

$\text{Arg}\,\bar{z} = \tan^{-1}\left(\frac{+1}{1}\right) = +\frac{\pi}{4}$

$\therefore \bar{z} = \sqrt{2}\,\text{Cis}(+\pi/4)$

$|w| = \left|\frac{1}{2} + i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$

$\text{Arg}\,w = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \frac{\pi}{3}$

$w = \text{Cis}\left(\frac{\pi}{3}\right)$

$(\bar{z})^2 = \left[\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)\right]^2$

$= (\sqrt{2})^2\left[\cos\frac{\pi}{2} + i\sin\left(\frac{\pi}{2}\right)\right]$

$= 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$

(using de Moivre's theorem)

$\therefore (\bar{z})^2 = 2i.$

$(w)^9 = \left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]^9$

$= \cos\left(\frac{9\pi}{3}\right) + i\sin\left(\frac{9\pi}{3}\right)$

(using de Moivre's theorem)

$\therefore w^9 = \cos(3\pi) + i\sin(3\pi)$

$= -1$

$\therefore w^9(\bar{z})^2 = (-1)(2i) = -2i$

(2 marks)

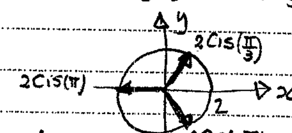
c)(i) $z^3 = -8 = 8[\cos(\pi) + i\sin(\pi)]$

$= 8\,\text{Cis}(\pi)$

$z = 8^{1/3}\,\text{Cis}\left(\frac{\pi}{3} + \frac{2\pi n}{3}\right),\ n = 0, 1, -1.$

(using de Moivre's theorem)

$z = 2\,\text{Cis}\left(\frac{\pi}{3}\right), 2\,\text{Cis}\left(-\frac{\pi}{3}\right), 2\,\text{Cis}(\pi)$



(2 marks)

(ii) Let w be a root of $z^3 + 8 = 0$.

The complex number z obtained by rotating w an angle of $\frac{\pi}{6}$ anticlockwise is: $z = w\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$

$\therefore z^3 = w^3\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]^3$

$= w^3\left[\cos\left(\frac{3\pi}{6}\right) + i\sin\left(\frac{3\pi}{6}\right)\right]$

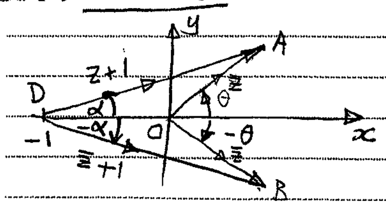
(using de Moivre's theorem),

$\therefore z^3 = w^3\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right] = w^3 i$

(2 marks)

$\therefore z^3 = w^3 i = -8i \quad \therefore z^3 + 8i = 0$

d) (i) METHOD 1



Let D and B be the points on the Argand diagram representing $z = -1$, and $\bar{z}$.
Let $\text{Arg}(z+1) = \alpha$ and $\text{Arg}(z) = \theta$.
As A and B are symmetrical about the x-axis, $\text{Arg}(\bar{z}+1) = -\alpha$ and $\text{Arg}\,\bar{z} = -\theta$. By substitution in the given equation:

$$\arg\left(\frac{z+1}{\bar{z}+1}\right) = \arg(z+1) - \arg(\bar{z}+1) = \arg(z)$$, we obtain: $\alpha - (-\alpha) = \theta$

$\therefore \quad 2\alpha = \theta$

$\therefore \angle A\hat{O}B = 2\angle A\hat{D}B$

$\therefore$ A, B, D are points on the circle with centre O and radius 1 i.e. $|z| = 1$.

Note: Angle at center = twice the angle subtended by the same chord at circumference).

Method 2.

Let $z = x + iy$.

$\therefore \theta = \text{Arg}\, z = \tan^{-1} y/x$.

$z + 1 = x + 1 + iy$

$\therefore \alpha = \text{Arg}(z+1) = \tan^{-1}\left(\frac{y}{x+1}\right)$

(3 marks)

$\bar{z} + 1 = x + 1 - iy$.

$$\text{Arg}(\bar{z}+1) = \tan^{-1}\left(\frac{-y}{x+1}\right) = -\alpha.$$

Since $\theta = \arg z = \arg\left(\frac{z+1}{\bar{z}+1}\right)$

$= \arg(z+1) - \arg(\bar{z}+1) = \alpha - (-\alpha)$

$= 2\alpha$

$$\tan\theta = \tan 2\alpha = \frac{2\tan\alpha}{1-(\tan\alpha)^2}$$

$$\frac{2y/(x+1)}{1 - y^2/(1+x)^2} = y/x$$

Simplifying we obtain

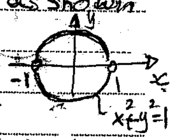
$y(x^2+y^2) = y$

$\therefore x^2 + y^2 = 1 \Longrightarrow |z| = \sqrt{x^2+y^2} = 1$.

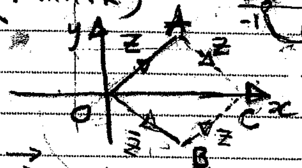
(Notice $y \neq 0$ since $\text{Im}\, z \neq 0$ is given)

(ii) As $y \neq 0$, the locus of z is as shown

(1 mark)



(iii)



$\overrightarrow{AC} = \overrightarrow{OB}$ (OACB is a parallelogram)

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$, from ΔOAC.

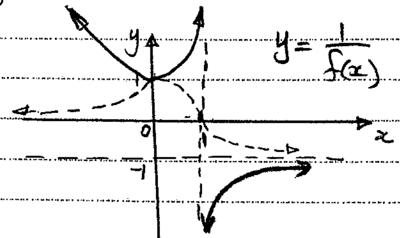
$\therefore \overrightarrow{OC}$ is represented by $w = z + \bar{z} = 2x$.

$\therefore$ w moves on the x-axis as z moves on the circle $|z| = 1$. Since $-1 < x < 1$, $w = 2x$ moves on the x-axis between -2 and 2.

(3 marks)

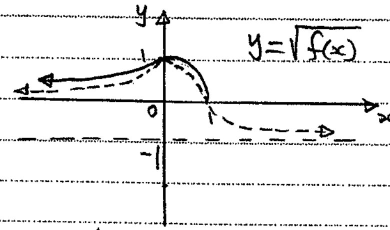
QUESTION 3

a) (i)



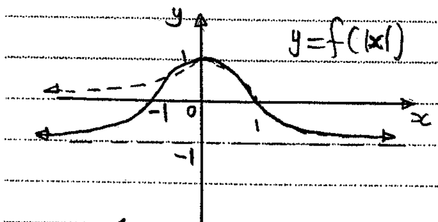
(2 marks)

(ii)



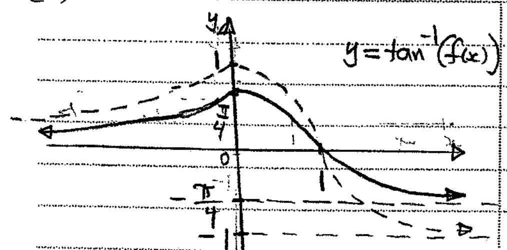
(2 marks)

(iii)



(1 mark)

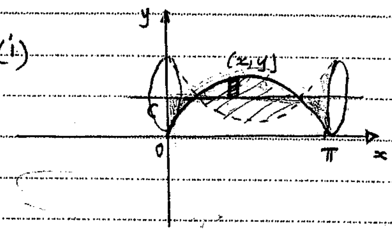
(iv)



(2 marks)

b)

(i)



A vertical section is a disc with radius $(y-c)$ and volume

$\delta V = \pi(y-c)^2\delta x$

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{x=\pi} \pi(y-c)^2 \delta x$$

$$= \pi\int_0^{\pi} (y-c)^2\,dx$$

$$= \pi\int_0^{\pi} (\sin(x) - c)^2\,dx$$

$$= \pi\int_0^{\pi} (\sin^2(x) - 2c\sin(x) + c^2)\,dx$$

$$= \pi\int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos(2x) - 2c\sin(x) + c^2\right)dx$$

$$= \pi\left[\frac{1}{2}x - \frac{1}{4}\sin(2x) + 2c\cos(x) + c^2x\right]_0^{\pi}$$

$$= \pi\left[\frac{\pi}{2} - 0 - 4c + \pi c^2\right]$$

$$= \frac{\pi^2}{2} - 4c\pi + c^2\pi^2$$

(2 marks)

(ii) V is minimum when $\frac{dV}{dc} = 0$ and $\frac{d^2V}{dc^2} > 0$.

$$\frac{dV}{dc} = -4\pi + 2c\pi^2 = 0$$

$\therefore c = \frac{2}{\pi}$

$$\frac{d^2V}{dc^2} = 2\pi^2 > 0$$

$\therefore c = \frac{2}{\pi}$ gives minimum volume, as required.

(2 marks)

c)(i) Substitute $u=a-x$, $du=-dx$

$x=0$, $u=a$

$x=a$, $u=0$

$$\int_0^a f(x)\,dx=\int_a^0 f(a-u)(-du)$$

$$=-\int_a^0 f(a-u)\,du$$

$$=\int_0^a f(a-x)\,dx$$

(2 marks)

(ii) Let $f(x)=\dfrac{(1-x)^n}{x^n+(1-x)^n}$ and $a=1$. From (i) we have

$$\int_0^1\frac{(1-x)^n}{x^n+(1-x)^n}dx=\int_0^1\frac{[1-(1-x)]^n}{(1-x)^n+[1-(1-x)]^n}dx$$

$$=\int_0^1\frac{x^n}{(1-x)^n+x^n}dx$$

$$\therefore \int_0^1\frac{(1-x)^n}{x^n+(1-x)^n}dx=\frac{1}{2}\int_0^1\frac{(1-x)^n}{x^n+(1-x)^n}dx+\frac{1}{2}\int_0^1\frac{(1-x)^n}{x^n+(1-x)^n}dx$$

$$=\frac{1}{2}\int_0^1\frac{(1-x)^n}{x^n+(1-x)^n}dx+\frac{1}{2}\int_0^1\frac{x^n}{x^n+(1-x)^n}dx$$

$$=\frac{1}{2}\left[\int_0^1\frac{(1-x)^n+x^n}{x^n+(1-x)^n}dx\right]=\frac{1}{2}\int_0^1 1\,dx$$

$$=\frac{1}{2},$$

as required

(2 marks)

QUESTION 4

a)(i) $\dfrac{d}{dx}Q(x)=x^3-x^2+2x-2$

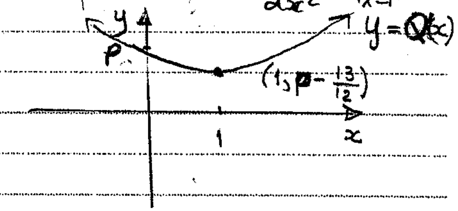
$$=x^2(x-1)+2(x-1)$$

$$=(x^2+2)(x-1)$$

Since $\dfrac{d}{dx}Q(x)=0$ only if $x=1$,

and $\dfrac{d^2}{dx^2}Q(x)=3x^2-2x+2$

$=3-2+2=1>0$, at $x=1$, therefore $x=1$ gives the only turning point of $Q(x)$ for all values of p. This point is a minimum since $\dfrac{d^2Q(x)}{dx^2}\Big|_{x=1}>0$.

ii) 

$Q(1)=\frac{1}{4}-\frac{1}{3}+1-2+p$

$=p-\frac{13}{12}$

If $p-\frac{13}{12}>0$, $Q(x)\geq Q(1)=p-\frac{13}{12}>0$,

since at $x=1$, $Q(x)$ has a minimum

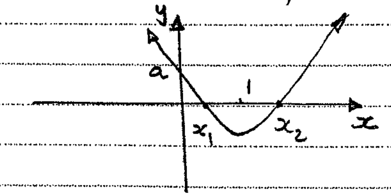
$\therefore Q(x)=0$ has no real roots if $p-\frac{13}{12}>0$ $\therefore$ $p>\frac{13}{12}$.

If $p-\frac{13}{12}=0$, $Q(1)=0$ and $\frac{dQ}{dx}(1)=0$, so $x=1$ is a double root of $Q(x)=0$

If $p-\frac{13}{12}<0$, $Q(1)=p-\frac{13}{12}<0$

(2 marks)

and the graph of $Q(x)$ intersects the x axis at two points x_1, x_2.



$Q(x_1)=0$ and $Q(x_2)=0$.

$\therefore$ x_1, x_2 are two roots of $Q(x)=0$.

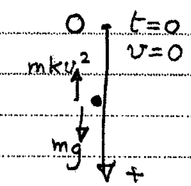
$\therefore$ $Q(x)=0$ has two real roots if $p-\frac{13}{12}\leq 0$, as required.

b)(i) Choose initial position as the origin, and direction of motion as positive.

Initial conditions:

$t=0$, $x=0$, $v=0$

Force acting on the particle are mg downward, and mkv^2 upward.



Using Newton's second law, the equation of motion is:

$m\ddot{x}=mg-mkv^2$

$\ddot{x}=g-kv^2$ (1)

(2 marks)

(ii) $\ddot{x}=\dfrac{dv}{dt}=\dfrac{dv}{dx}\dfrac{dx}{dt}=v\dfrac{dv}{dx}$

$=\dfrac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\therefore$ $\dfrac{d}{dx}\frac{1}{2}v^2=g-kv^2$ by (1)

$\therefore$ $\dfrac{-k}{g-kv^2}dv^2=-2k\,dx$

Integrating gives:

$\ln|g-kv^2|=-2kx+C$,

where C is a constant.

$x=0$, $v=0 \Rightarrow C=\ln g$

$\therefore$ $\ln\left|\dfrac{g-kv^2}{g}\right|=-2kx$

$\dfrac{g-kv^2}{g}=e^{-2kx}$

$v^2=\dfrac{g}{k}(1-e^{-2kx})$. (2)

(2 marks)

(iii) The terminal velocity v_T is obtained by setting $\ddot{x}=0$ in (1).

$\therefore$ $v_T=\sqrt{\dfrac{g}{k}}$

Using (2), we have

$\left(\frac{1}{2}\right)^2=\left(\dfrac{v}{v_T}\right)^2=\dfrac{v^2}{(g/k)}=(1-e^{-2kx})$

$\therefore$ $e^{-2kx}=1-\frac{1}{4}=\frac{3}{4}$

$2kx=\ln(4/3)$

$\therefore$ $x=\dfrac{1}{2k}\ln(4/3)$,

as required.

(2 maks)

c)(i) Resolving radially,

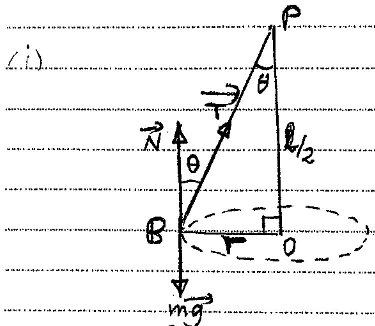
$T\sin\theta=m\omega^2 r$ (1)

Resolving vertically,

$T\cos\theta+N-mg=0$ (2)

since there is no vertical motion

(i)



$\cos\theta = \frac{\ell/2}{\ell} = \frac{1}{2}$

$\therefore\ \theta = \frac{\pi}{3}$ and $r = \ell\sin\theta = \frac{\sqrt{3}}{2}\ell$

From (1) and (2), we have

$\therefore\ T = \frac{m\omega^2 r}{\sin\theta} = m\omega^2\ell,$

and

$N = mg - T\cos\theta = mg - \frac{1}{2}m\omega^2\ell$

(3 marks)

(ii) For the particle to remain on table, we must have $N \geq 0$.

$\therefore\ N = mg - \frac{1}{2}m\omega^2 L \geq 0.$

$\therefore\ \omega^2 \leq \frac{2g}{L},$

as required. (2 marks)

Note: The string is always taut as $T > 0$.

QUESTION 5

a) (i) Let $p(x) = x^{2n+1} - ax + b$.

Since $x = 1$ is a double root of $p(x)$, $p(1) = 0 = p'(1)$.

ie. $1 - a + b = p(1) = 0$

$(2n+1)\times 1 - a = p'(1) = 0$

$\therefore\ a = 2n+1 \quad b = a - 1 = 2n.$

(2 marks)

(ii) $1, 1, \alpha$ are the roots of $x^3 - 3x + 2 = 0$.

$\therefore\ 3, 3, 3\alpha$ are the roots of

$\left(\frac{x}{3}\right)^3 - 3\left(\frac{x}{3}\right) + 2 = 0$

ie. $x^3 - 27x + 54 = 0$

So $Q(x) = x^3 - 27x + 54$.

(1 marks)

b) (i) $x^{2/3} \geq 0$ for all x, also $y^{2/3} \geq 0$ for all y. Now, if (x, y) is a point on the graph, $x^{2/3} + y^{2/3} = 1$.

$\therefore\ |x^{2/3}| \leq 1$ and $|y^{2/3}| \leq 1$.

i.e. $|x| \leq 1$ and $|y| \leq 1$.

But $(\pm 1, 0)$, $(0, \pm 1)$ belong to the graph. Therefore,

domain: $-1 \leq x \leq 1$

range: $-1 \leq y \leq 1$

(2 marks)

(ii) For any point (x, y) on the graph,

$(-x)^{2/3} + y^{2/3} = x^{2/3} + y^{2/3} = 1.$

$\therefore\ (-x, y)$ is also on the graph.

$\therefore$ The graph is symmetrical about the y axis.

Similarly,

$x^{2/3} + (-y)^{2/3} = x^{2/3} + y^{2/3} = 1$

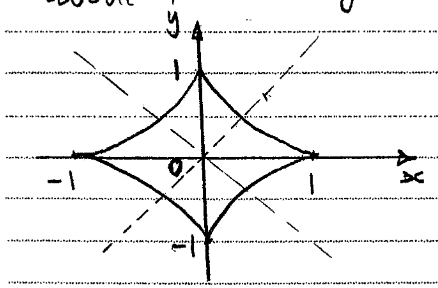
$\therefore\ (x, -y)$ is also on graph.

$\therefore$ The graph is symmetrical about x axis.

also $y^{2/3} + x^{2/3} = x^{2/3} + y^{2/3} = 1$.

$\therefore\ (y, x)$ is on the graph.

$\therefore$ graph is symmetrical about the lines $y = \pm x$.



Implicit differentiation gives

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0.$

$\therefore\ \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} = \pm\frac{\sqrt{1 - x^{2/3}}}{x^{1/3}}$

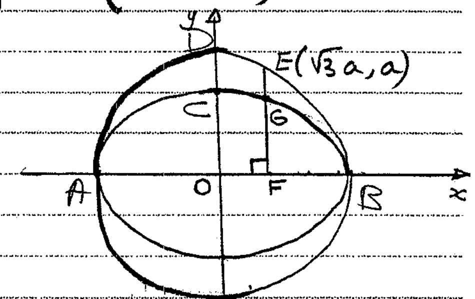
So $\frac{dy}{dx} \to \pm\infty$ as $x \to 0$,

and $\frac{dy}{dx} \to 0$ as $x \to \pm 1$.

$\therefore$ the graph has vertical tangents at $(0, \pm 1)$ and horizontal tangents at $(\pm 1, 0)$.

(2 marks)

c)



(i) $\frac{x^2}{(2a)^2} + \frac{y^2}{a^2} = 1$ (1 mark)

or $x^2 + 4y^2 = 4a^2$ (1)

(ii) $a^2 = (2a)^2(1 - e^2)$.

$\therefore\ e = \sqrt{3}/2$

$\therefore$ Foci are $(\pm 2ae, 0) = (\pm\sqrt{3}a, 0)$

$E = (\sqrt{3}a, a)$ given

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$\therefore$ $F = (\sqrt{3}a, 0)$ is one of the foci.

(2 marks)

iii) To find G, substitute $x = \sqrt{3}a$ in the equation of the ellipse:

$$3a^2 + 4y^2 = 4a^2$$

So $y = \pm a/2$

$\therefore$ $G = (\sqrt{3}a, a/2)$

(1 mark)

(iv) Differentiating (1), we obtain:

$$2x + 8y\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = -\frac{x}{4y} = -\frac{\sqrt{3}}{2} \text{ at } G.$$

$\therefore$ the equation of tangent to ellipse at G is:

$$\frac{y - a/2}{x - \sqrt{3}a} = -\frac{\sqrt{3}}{2}$$

or $\sqrt{3}x + 2y - 4a = 0$ (2)

(1 mark)

(v) Differentiating $x^2 + y^2 = 4a^2$, we have:

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -x/y = \frac{-\sqrt{3}a}{a} \text{ at } E.$$
$$= -\sqrt{3}$$

The tangent equation at E is:

$$\frac{y - a}{x - \sqrt{3}a} = -\sqrt{3}$$ (1 mark)

or $\sqrt{3}x + y - 4a = 0$ (3)

(vi) To find M, we solve (2), (3):

(2) $-$ (3) $\Longrightarrow$ $y = 0$

$\therefore$ $x = \dfrac{4a}{\sqrt{3}}$ by (1)

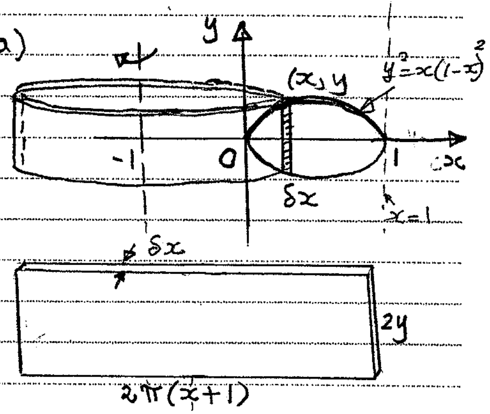
$\therefore$ $M = \left(\dfrac{4a}{\sqrt{3}}, 0\right)$.

$\therefore$ M moves on the x axis as a varies.

(2 marks)

QUESTION 6

a)



Volume of shell $= \delta V = 2\pi(x+1) \times 2y\,\delta x$.

$$= 4\pi(x+1)x^{1/2}(1-x)\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{x=1} 4\pi(x+1)(1-x)x^{1/2}\delta x$$

$$= 4\pi\int_0^1 (x+1)(1-x)x^{1/2}\,dx$$

$$= 4\pi\int_0^1 (x^{1/2} - x^{5/2})\,dx$$

$$= 4\pi\left[\frac{2}{3}x^{3/2} - \frac{2}{7}x^{7/2}\right]_0^1$$

$$= 4\pi\left(\frac{2}{3} - \frac{2}{7}\right) = \frac{32}{21}\pi$$

(3 marks)

b) (i) Define the statement

$S(n)$: $u_n > 0$, $u_{n+1} > u_n$ and $u_n < 2$.

Step 1: $S(1)$ is true:

$u_1 = 1 > 0$, $u_2 = \sqrt{2 + u_1} > u_1$ and $u_1 = 1 < 2$

Step 2: Assume $S(k)$ is true. That is:

$u_k > 0$, $u_{k+1} > u_k$ and $u_k < 2$.

We show that $S(k+1)$ is true:

$u_{k+1} = \sqrt{2 + u_k} > 0$, by the assumption $u_k > 0$.

$u_{k+2} = \sqrt{2 + u_{k+1}} > \sqrt{2 + u_k}$, by the assumption $u_{k+1} > u_k$.

$\therefore$ $u_{k+2} > \sqrt{2 + u_k} = u_{k+1}$.

$u_{k+1} = \sqrt{2 + u_k} < \sqrt{2+2} = 2$ by the assumption $u_k < 2$.

$\therefore$ $S(k+1)$ is true if $S(k)$ is true.

By step 1, $S(1)$ is true. Hence $S(2)$ is true by step 2, and $S(3)$, $S(4)$, ... are true.

$\therefore$ $S(n)$ is true for $n = 1, 2, 3, \ldots$

(3 marks)

(ii) By (i) the sequence $\{u_n\}_{n=1}^{\infty}$ is increasing and bounded above by 2. Hence $\lim_{n \to \infty} u_n = L$ exists.

But $\lim_{n\to\infty} u_n = \sqrt{2 + \lim_{n\to\infty} u_{n-1}}$

$\therefore$ $L = \sqrt{2 + L}$

$L^2 = 2 + L$

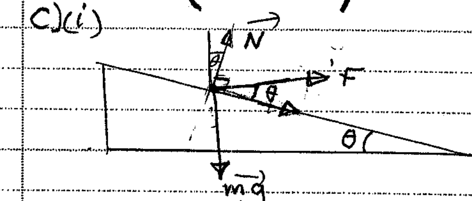
$L^2 - L - 2 = 0$

$(L-2)(L+1) = 0$

$\therefore$ $L = 2$, as $L \geq 0$.

(1 mark)

C) (i)



Using $\Sigma F = ma$ (Newton's law), we have:

Horizontal components:

$$F\cos\theta + N\sin\theta = \frac{mv^2}{r}$$

Vertical components:

$$N\cos\theta - F\sin\theta - mg = 0$$

(no vertical motion).

Rearranging:

$$F\sin\theta = N\cos\theta - mg \quad (1)$$

$$F\cos\theta = \frac{mv^2}{r} - N\sin\theta \quad (2)$$

(2 marks)

(ii) (1) $\times \sin\theta$

$$F\sin^2\theta = N\sin\theta\cos\theta - mg\sin\theta$$

(2) $\times \cos\theta$

$$F\cos^2\theta = \frac{mv^2}{r}\cos\theta - N\sin\theta\cos\theta$$

Adding gives:

$$F = m(v^2 - gr\tan\theta)\frac{\cos\theta}{r}$$

(2 marks)

(iii) $v = 30\ ms^{-1}$, $r = 200m$, $g = 9.8\ ms^{-2}$.

Since there is no tendency to slip, $F = 0$.

$\therefore\ v^2 - gr\tan\theta = 0$ by (ii). (notice that $0 < \theta < \pi/2$ so $\cos\theta \neq 0$).

$\therefore\ \tan\theta = \frac{v^2}{gr} = 0.459$

$\theta = 24.7^\circ$

(2 marks)

iv) When $v > 30\ ms^{-1}$, $F > 0$. Therefore, there is a reaction force down the track. Hence the car tends to slip up the track.

(2 marks)

QUESTION 7

(i) $I_n = \int_1^e 1.(\ln x)^n\,dx$

Let $u = (\ln x)^n$, $dv = dx$

Then $du = \frac{n}{x}(\ln x)^{n-1}$, $v = x$

Therefore

$I_n = [x(\ln x)^n]_1^e - \int_1^e x.\frac{n}{x}(\ln x)^{n-1}\,dx$

$= [e(\ln e)^n - 1(\ln 1)^n] - n\int_1^e (\ln x)^{n-1}\,dx$

$= e - nI_{n-1}$.

$\therefore\ I_n + nI_{n-1} = e$, $n \geq 0$.

[2 marks]

(ii) $I_2 + 2I_1 = e$ by (i).

$\therefore\ I_2 = e - 2I_1$

$= e - 2[e - I_0]$ by (i).

But $I_0 = \int_1^e (\ln x)^0\,dx$

$= \int_1^e 1\,dx = e - 1$.

$\therefore\ I_2 = e - 2[e - e + 1]$

$= e - 2$.

[1 mark]

(iii) For $1 < x < e$, $0 < \ln x < 1$.

Therefore

$(\ln x)^n < (\ln x)^{n-1}$,

$\int_1^e (\ln x)^n\,dx < \int_1^e (\ln x)^{n-1}\,dx$.

$\therefore\ I_n < I_{n-1}$ (1)

$\therefore\ e = I_n + nI_{n-1} > I_n + nI_n$

$= (n+1)I_n$ by (1)

We also have, by (i) and (1),

$e = I_{n+1} + (n+1)I_n$

$< I_n + (n+1)I_n$

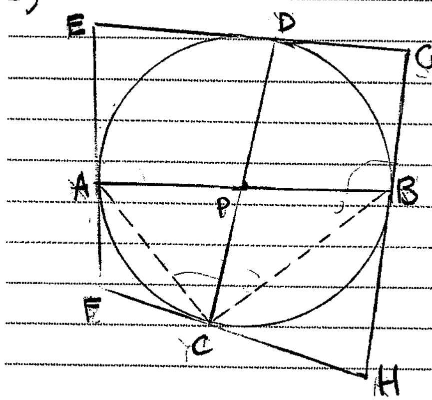
$= (n+2)I_n$

Therefore

$\frac{e}{n+2} < I_n < \frac{e}{n+1}$.

[3 marks]

b)



(i) Join AC and BC.

$\angle PAE = \angle ACB$ and

$\angle PBG = \angle ACB$

since $\angle$ between chord and tangent equals $\angle$ in alternate segment.

$\therefore\ \angle PAE = \angle PBG$,

$\angle PAE + \angle PBH = \angle PBG + \angle PBH = 180^\circ$ (1)

[3 marks]

ii) by joining AC and BC and using a similar reasoning, we obtain

$\angle PDE + \angle PCH = 180^\circ$ (2)

Consider now the quadrilaterals PDEA and PCHB. Since the sum of the interior angles of a quadrilateral is 360°, we have:

$\angle APD + \angle PDE + \angle DEA + \angle PAE = 360^\circ$,

$\angle BPC + \angle PCH + \angle CHB + \angle PBH = 360^\circ$

Adding the above equations gives

$(\angle APD + \angle BPC) + (\angle DEA + \angle CHB) + (\angle PAE + \angle PBH) + (\angle PDE + \angle PCH) = 720^\circ$

Since $\angle APD = \angle BPC$ (opposite angles), it follows from (1) and (2) that

$2\angle APD + \angle DEA + \angle CHB = 360^\circ$

[3 marks]

(iii) Assume first that E, F, H, G are concyclic, so

$$\angle DEA + \angle GHB = 180^\circ$$

From (ii) we then have

$$2\angle APD = 180^\circ, \therefore \angle APD = 90^\circ$$

so AB and CD are perpendicular.

Conversely, if $AB \perp CD$,

$\angle APD = 90^\circ$, and

From (ii) we obtain

$$\angle DEA + \angle CHB = 180^\circ.$$

Hence E, F, H, G are concyclic.

This shows that E, F, H, G are concyclic if and only if the chords AB and CD are perpendicular

[3 marks]

QUESTION 8

a)(i) Since w is a root,

$$0 = p(w) = 1 + w + w^2 + \cdots + w^7$$

But $(1-x^8) = (1-x)(1+x+x^2+\cdots+x^7)$

$$\therefore (1-w^8) = (1-w)(1+w+w^2+\cdots+w^7) = 0$$

$$\therefore w^8 = 1 \quad \textbf{(1 mark)}$$

Notice that $w \neq 1$ since $p(1) \neq 0$.

(ii) From

$$(1-x^8) = (1-x)(1+x+x^2+\cdots+x^7),$$

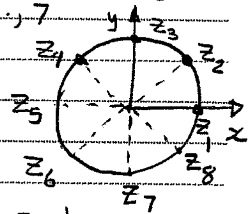
the roots of $p(x)=0$ are the roots of $x^8 = 1$ (1)

except $x = 1$.

By de Moivre's theorem, the roots of (1) are

$$z = \cos\frac{2\pi k}{8} + i\sin\frac{2\pi k}{8},$$

$k = 0, 1, 2, 3, \ldots, 7$



$z_1 = 1$ and $z_5 = -1$ are the real roots. Since $z_1 = 1$ is not a root of $p(x) = 0$, $z_5 = -1$ is the only real root of $p(x) = 0$.

(2 marks)

(iii) Let w be any root of $p(x)=0$.

By (i), $w^8 = 1$.

$$\therefore w^{8n+k} = (w^8)^n w^k = 1 w^k = w^k,$$

$k = 0, 1, \ldots, 7$.

$$\therefore q(w) = 1 + w^{8n+1} + \cdots + w^{8n+7}$$

$$= 1 + w + w^2 + \cdots + w^7 = 0.$$

Let $w_1, w_2, \ldots, w_7$ be the roots of $p(x) = 0$. Hence $w_1, \ldots, w_7$ are roots of $q(x) = 0$.

Therefore

$(x-w_1)(x-w_2)\cdots(x-w_7)$ is a factor of $q(x)$.

But $p(x)$ is the product of its factors.

$\therefore$ $p(x)$ is a factor of $q(x)$.

(2 marks)

b)(i) METHOD 1

$$f'(x) = \frac{1}{1+x} - \frac{(1+x)\cdot 1 - x}{(1+x)^2}$$

$$= \frac{1}{1+x} - \frac{1}{(1+x)^2} > 0$$

for $x > 0$, since $(1+x)^2 > (1+x)$.

Therefore, f is strictly increasing

Hence

$$f(x) > f(0) = \ln(1) - 0 = 0$$

$$\therefore \ln(1+x) - \frac{x}{1+x} > 0, x > 0.$$

Similarly, $g'(x) = \dfrac{1}{1+x} - 1 < 0$

for $x > 0$. Therefore, g is strictly decreasing and

$$g(x) < g(0) = \ln 1 - 0 = 0$$

$$\therefore \ln(1+x) - x < 0$$

Set $x = \frac{1}{n}$ in the above inequalities to obtain

$$\frac{1}{n} > \ln\left(1+\frac{1}{n}\right) > \frac{\frac{1}{n}}{1+\frac{1}{n}} = \frac{1}{1+n}$$

METHOD 2

Since

$$1 < 1+t < (1+t)^2$$

for $t > 0$,

$$1 > \frac{1}{1+t} > \frac{1}{(1+t)^2}, \quad t > 0.$$

Integration over the interval $[0, x]$ gives

$$\int_0^x dt > \int_0^x \frac{1}{1+t}\,dt > \int_0^x \frac{1}{(1+t)^2}\,dt$$

$$[t]_0^x > \Big[\ln|1+t|\Big]_0^x > \left[\frac{-1}{1+t}\right]_0^x$$

$$x > [\ln(1+x) - 0] > \left[\frac{-1}{1+x} + 1\right]$$

$$x > \ln(1+x) > \frac{x}{1+x}, \quad x > 0.$$

$$\frac{1}{n} > \ln\left(1+\frac{1}{n}\right) > \frac{\frac{1}{n}}{1+\frac{1}{n}}$$

$$\frac{1}{n} > \ln\left(1+\frac{1}{n}\right) > \frac{1}{1+n}.$$

(2 marks)

(ii) We first show that $u_n > 0$, $n = 1, 2, \ldots$:

Using the inequality

$$\frac{1}{n} > \ln\left(1+\frac{1}{n}\right),$$

we obtain

$$u_n > \ln(1+1) + \ln\left(1+\frac{1}{2}\right) + \cdots + \ln\left(1+\frac{1}{n}\right) - \ln(n)$$

$$= \ln\left(2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n}{n-1}\cdot\frac{n+1}{n}\right) - \ln n$$

$= \ln(n+1) - \ln n = \ln\left(\frac{n+1}{n}\right) > 0.$

Next we show that

$u_{n+1} < u_n,\ n = 1, 2, \ldots$

It follows from the definition of u_n that

$$u_{n+1} - u_n = \frac{1}{n+1} - \ln(n+1) + \ln n$$

$$= \frac{1}{n+1} - \ln\left(\frac{n+1}{n}\right)$$

$$= \frac{1}{n+1} - \ln\left(1+\frac{1}{n}\right) < 0$$

by (i).

Since $\{u_n\}_{n=1}^{\infty}$ is a decreasing sequence of positive numbers, it has a limit.

(3 marks)

c)(i) We denote each outcome of throwing the three dice by a triple of integers (a, b, c), where a, b, and c may take any value from 1 to 6. Therefore the set of all possible outcomes has $6^3 = 216$ elements.

Therefore the sample space consists of 216 elements and we assign the probability $\frac{1}{216}$ to each outcome. The event A in question is the set of all triples satisfying $3 \leq a+b+c \leq 6$. If $A(n)$ denotes the set of (a,b,c) for which $a+b+c=n$, then A is the union of the sets $A(3), A(4), A(5), A(6)$. Direct enumeration shows that:

$A(3) = \{(1,1,1)\}$.

$A(4) = \{(1,2,1), (1,1,2), (2,1,1)\}$.

$A(5) = \{(1,1,3), (1,3,1), (1,2,2), (2,1,2), (2,2,1), (3,1,1)\}$

$A(6) = \{(1,2,3), (1,3,2), (1,1,4), (1,4,1), (2,1,3), (2,3,1), (2,2,2), (3,1,2), (3,2,1), (4,1,1)\}$

Therefore A has 20 elements and $P(A) = \frac{20}{216} = \frac{5}{54}$

(2 marks)

(ii) We have 150 independent trials, each is the throw of a 3 dice. Success means throwing 3 sixes, failure otherwise. To answer the question we need to calculate the probability of the event "at least two triple sixes in 150 throws". There are $6^3 = 216$ possible outcomes in any throw of the three dice.

∴ probability of one triple six in each throw is $\frac{1}{216}$. Let $p = \frac{1}{216}$ and $q = 1 - p = \frac{215}{216}$.

If X denotes the number of triple sixes occuring in 150 throws of three dice, then X is a binomial variable which can assume the values $0, 1, 2, \ldots, 150$.

$P(X \geq 2)$ = probability that $X \geq 2$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[q^{150} + {}^{150}C_1 q^{149} p\right]$$

$$= 1 - \left[\left(\frac{215}{216}\right)^{150} + 150\left(\frac{215}{216}\right)^{149}\frac{1}{216}\right]$$

$$= 1 - [0.498547 + 0.3478236]$$

$$= 0.1536$$

Since $p(X \geq 2) < 0.5$, it is not profitable to bet even money on the occurance of at least two triple sixes