HIGHER SCHOOL CERTIFICATE EXAMINATION TRIAL PAPER

2006

MATHEMATICS

Time Allowed – Three Hours
(Plus 5 minutes reading time)

Directions to Candidates

- **Attempt ALL questions.**
- **All questions are of equal value.**
- **All necessary working should be shown in every question.**
- **Board-approved calculators may be used.**

QUESTION 1 — MARKS

a) Find the value of $\log_4 93$ correct to three significant figures. 1

b) Factorise $9x^2 - 4y^2$ 1

c) Differentiate $\dfrac{2x}{x^2+4}$ 2

d) Rationalise the denominator of $\dfrac{2\sqrt{5}}{7+3\sqrt{5}}$ 2

e) Solve $(x-3)^2 = 13$ (give your answer in surd form) 2

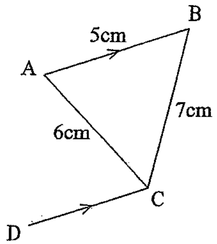
f) For the parabola $y = \dfrac{1}{4}x^2 + x$

i) Calculate the coordinates of the vertex. 2

ii) Find the equation of the directrix of this parabola. 2

QUESTION 2 MARKS

a) In ΔABC, $AB = 5$ cm, $AC = 6$ cm and $BC = 7$ cm.
AB is parallel to DC.



NOT TO SCALE

Calculate the size of angle ACD correct to the nearest minute. 3

b) $A(-3, -2)$, $B(0, -1)$ and $C(3, 5)$ are three points on the number plane

i) Calculate the gradient of CB 1

ii) Show that the equation of the line ℓ passing through the point A and parallel to CB, has the equation $2x - y + 4 = 0$ 2

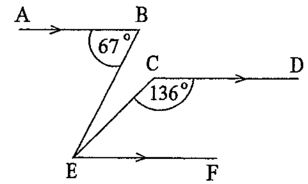
iii) The line ℓ intersect the y axis at the point D. Show that ABCD is a parallelogram. 2

iv) Calculate the perpendicular distance from B to the line ℓ . 2

v) Calculate the area of the parallelogram ABCD. 2

QUESTION 3 MARKS

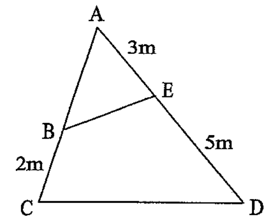
a) AB, CD and EF are parallel lines. $\angle ABE = 67°$ and $\angle DCE = 136°$



NOT TO SCALE

Calculate the size of $\angle BEC$, giving reasons. 2

b) i) In this diagram $\angle BCD + \angle BED = 180°$ 2
Prove that ΔABE is similar to ΔADC



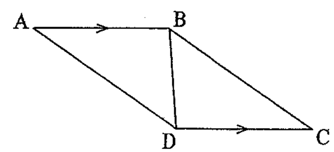
NOT TO SCALE

ii) Given that $AE = 3$m, $ED = 5$m and $BC = 2$m, calculate the length of AB. 2

c) Differentiate $2\tan 3x$ with respect to x. 1

d) Calculate the second derivative of $\log_e(\sin x)$ 2

e) AB is parallel to DC and $\angle ABC = \angle ADC$ 3



NOT TO SCALE

Prove that $\Delta ABD \equiv \Delta CDB$

QUESTION 4 MARKS

a) Evaluate $\int_0^9 (\sqrt{x}+1)^2\,dx$ 2

b) Find the equation of the normal to the curve $y = \ln x$ at the point where $x = e$. 3

c) i) Show that the derivative of $\frac{1}{3}\tan^3 x$ is $\tan^4 x + \tan^2 x$. 2

ii) Hence, evaluate $\int_0^{\frac{\pi}{3}} (\tan^4 x + \tan^2 x)\,dx$. 1

d) The second term of an arithmetic sequence is 9 and the fifth term is 30. Calculate the sum of the first 20 terms of this sequence 2

e) A class contains 25 students. 10 study French, 16 study Latin and 7 of these students study both languages. 2

If two students are selected at random from this class, what is the probability that both study neither of the two languages?

QUESTION 5 MARKS

a) i) Find the turning points of the curve $y = x^4 - 8x^2 + 7$ and determine their nature. 2

ii) Find the points of intersection of the curve with the x axis. 2

iii) Sketch the curve showing the turning points and the x and y intercepts. 2

b) Sketch the graph of $y = f(x)$ using the following: 2

- $f(0) = 0$
- $f'(x) > 0$ for all x
- $f''(0) = 0$
- $f''(x) > 0$ at $x < 0$
- $f''(x) < 0$ at $x > 0$

c) The local radio controlled car club is to hold two races. In each race eight cars, 3 green, 4 red and 1 blue are to race. Each car has an equal chance of winning each race.

What is the probability that:

i) both races are won by a green car? 1

ii) the blue car wins one race only? 1

iii) each race is won by a car of a different colour? 2

QUESTION 6 MARKS

a) Solve the equation $\tan x = \sqrt{3}$ for $0 \le x \le 2\pi$ 2

b) Sketch the curve $y = 4e^{-2x}$ 1

c) Consider the series $2e^{x} + 8e^{-x} + 32e^{-3x} + \ldots\ldots$

i) Show that this series is geometric. 1

ii) By using the graph in part (b), find the values of x for which this series has a limiting sum. 2

iii) Find the limiting sum of this series in terms of x. 1

d) The change in the population of a certain organism is proportional to the time t, measured in years, therefore $\frac{dP}{dt} = kt$.

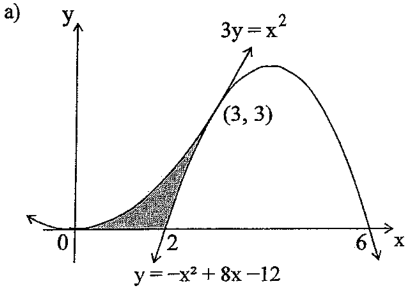
i) Given that the initial population is 400, write an expression for the size of the population P. 1

ii) Calculate the size of the population after 3 years given that it is 448 after 2 years. 2

iii) Find the time taken for this organism's population to reach 700. 2

QUESTION 7 MARKS

a)



NOT TO SCALE

The graphs of $3y = x^2$ and $y = -x^2 + 8x - 12$ are shown on the same set of axes. 3

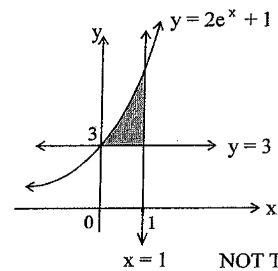
These curves meet at (3, 3) as shown.

The region enclosed by these curves in the first quadrant has been shaded.

Calculate the area of this region.

b) Use Simpson's Rule, with 2 equal sub intervals, to find the area between the curve $y = \sin^2 x$, the x axis, and the lines $x = 0$ and $x = 2$. Give your answer correct to 3 decimal places. 2

c)



NOT TO SCALE

The area enclosed by the curve $y = 2e^{x} + 1$ and the lines $x = 1$ and $y = 3$ is shaded as shown in the diagram.

i) Find the exact area of this shaded region. 2

ii) α) Show that the volume of the solid formed when this shaded region is rotated about the x axis can be expressed as 2

$$V = 4\pi \int_0^1 (e^{2x} + e^{x} - 2)\,dx.$$

β) Calculate the exact volume of the solid formed. 1

d) For what values of m does the equation $2x^2 + mx + 8 = 0$ have two positive unequal real roots. 2

QUESTION 8 | MARKS

a) A particle initially at a point 216m to the left of the origin starts to move. Its velocity at any time t seconds is given by $v = 3t^2 - 18t + 24$.

i) Find the displacement equation for this particle. 1

ii) Find the times at which the particle changes direction and its position at each time. 2

iii) Show that the particle only passes through the origin once, and calculate when this occurs. 2

iv) Describe the motion of this particle during the first 10 seconds. 2

b) Vanessa has just finished year 8 and wants to buy a car in 4 years, when she finishes year 12. She estimates that she will need \$8000 for the car and insurance. Her bank has an account which offers interest of 9% p.a, compounding monthly.

i) Calculate the single deposit she would need to put into this account so that she will receive \$8000 at the end of 4 years ? 1

ii) An alternative would be for Vanessa to deposit a regular amount at the beginning of each month into this account. 3

Calculate the size of this regular monthly deposit that will result in a total balance of \$8000 at the end of 4 years ?

iii) Which of these two alternatives will cost Vanessa the least amount of money, and how much cheaper would it be ? 1

QUESTION 9 | MARKS

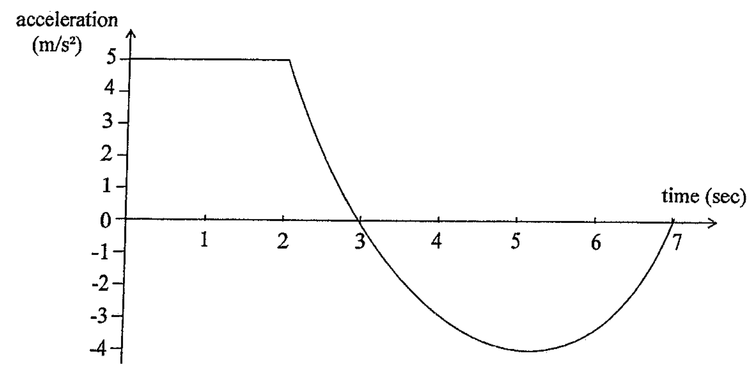
a) A group of friends decide to hire a ship to travel 150km from one port to another. When travelling at a speed s km/h, the cost per hour in dollars of hiring the ship and the fuel can be calculated by $\frac{360+s^2}{9}$.
In addition the ship's captain must be paid $50 + 24t$ dollars, where t is the time of the trip in hours.

i) Show that the cost of the trip can be expressed as

$$C = \frac{9600}{s} + \frac{50s}{3} + 50.$$ 2

ii) Calculate the speed that will result in the smallest cost. 3

b) This graph shows the acceleration of a particle during a 7 second interval.



Initially the particle is at rest at the origin.

i) Find the equation of the velocity for the time interval $0 \le t \le 2$ 1

ii) Given that the acceleration is $t^2 - 10t + 21$ for the time interval $2 \le t \le 7$, show that the equation of the velocity in this time interval is 3

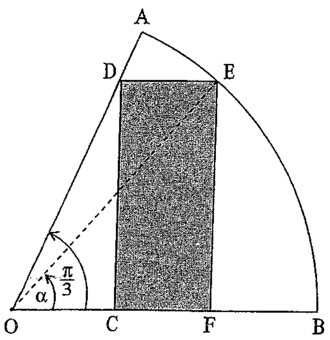
$$v = \frac{1}{3}t^3 - 5t^2 + 21t - 14\frac{2}{3}$$

iii) Sketch the velocity time graph of the particle for $0 \le t \le 7$. 2

iv) Calculate the distance traveled by the particle during this 7 seconds. 1

QUESTION 10 **MARKS**

a)



NOT TO SCALE

AOB is a sector of a circle with centre at O and radius r such that $\angle OAB = \frac{\pi}{3}$.
CDEF is a rectangle drawn in the sector and $\angle EOF = \alpha$ as shown in the diagram.

i) Show that $CF = r\cos\alpha - \frac{r\sin\alpha}{\sqrt{3}}$ **3**

ii) Given that $\frac{1}{2}\sin 2\alpha = \sin\alpha\cos\alpha$, **2**
show that the area of rectangle BDEF can be expressed as

$$A = r^2\left(\frac{1}{2}\sin 2\alpha - \frac{\sqrt{3}}{3}\sin^2\alpha\right)$$

iii) Find the value for α which will produce the rectangle of maximum area. **2**

b) Yvonne borrows $12 000 to buy a car and agrees to repay the loan, including the interest, in equal monthly instalments over a period of 15 months.
Interest of 12% p.a. is charged on the balance owing at the end of each month, immediately prior to Yvonne making her payment.
Her payments increase each month and form a sequence as follows:
$\ln p,\ 2\ln p,\ 4\ln p,\ 8\ln p, \ldots\ldots$

i) Show the amount owed immediately after Yvonne's third payment is **2**

$$A_3 = 12\,000 \times 1.01^3 - \ln p\,(1.01^2 + 2 \times 1.01 + 2^2)$$

ii) Calculate the value of p required for Yvonne to pay her loan off in 15 months. **2**

iii) Calculate the total amount paid by Yvonne over the 15 month period. **1**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq 1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Year 12 - Trial 2006 - Mathematics

Question 1

a) $\log_4 93 = \dfrac{\log_{10} 93}{\log_{10} 4} = 3.27$ (1 mark)

b) $9x^2 - 4y^2 = (3x)^2 - (2y)^2$

$= (3x+2y)(3x-2y)$

(1 mark)

c) $\dfrac{dy}{dx} = \dfrac{2(x^2+4) - 2x(2x)}{(x^2+4)^2}$

$= \dfrac{2x^2+8-4x^2}{(x^2+4)^2}$

$= \dfrac{8-2x^2}{(x^2+4)^2}$ (2 marks)

d) $\dfrac{2\sqrt5}{7+3\sqrt5} \times \dfrac{7-3\sqrt5}{7-3\sqrt5}$

$= \dfrac{14\sqrt5 - (6\times5)}{49 - 21\sqrt5 + 21\sqrt5 - 45}$

$= \dfrac{14\sqrt5 - 30}{4} = \dfrac{7\sqrt5 - 15}{2}$ (2 marks)

e) $(x-3)^2 = 13$

$x - 3 = \pm\sqrt{13}$

$\therefore x = 3 \pm \sqrt{13}$ (2 marks)

f) i) $4y = x^2 + 4x$

$4y + 4 = x^2 + 4x + 4$

$4(y+1) = (x+2)^2$

$\therefore$ Vertex at $(-2,-1)$ (2 marks)

ii) Focal length = 1 unit

$\therefore$ Directrix is 1 unit below $(-2,-1)$

so directrix is $y = -2$ (2 marks)

Question 2

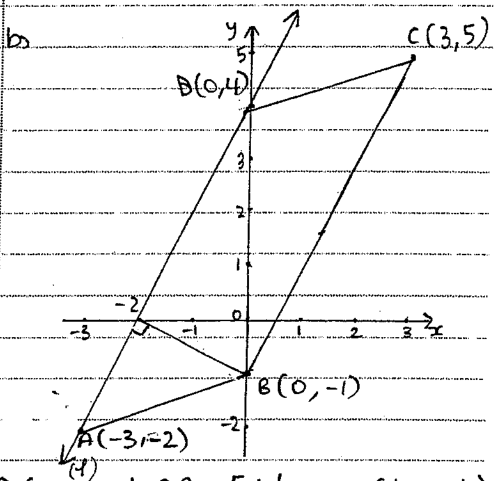
a) $\angle ACD = \angle CAB$ (alternate $\angle$'s, $AB \parallel DC$)

$\cos \angle CAB = \dfrac{5^2+6^2-7^2}{2\times5\times6}$

$\therefore \angle CAB = 78°28'$

$\therefore \angle ACD = 78°28'$ (3 marks)

b)



i) Gradient $CB = \dfrac{5+1}{3-0} = 2$ (1 mark)

ii) $m(l) = m\,BC = 2$ $(l) \parallel BC$

Using point-gradient formula

$y + 2 = 2(x+3)$

$y + 2 = 2x + 6$

$\therefore 2x - y + 4 = 0$ (2 marks)

iii) At D, $x = 0$

$\therefore D = 2\times0 - y + 4$

i.e. $y = 4$ so $D(0,4)$

$CB \parallel AD$ (from above)

$CB = \sqrt{(3-0)^2 + (5+1)^2}$

$= \sqrt{9+36} = \sqrt{45} = 3\sqrt5$ units

$AD = \sqrt{(0+3)^2 + (4+2)^2}$

$= \sqrt{9+36} = \sqrt{45} = 3\sqrt5$ units

$\therefore ABCD$ is a parallelogram as $CB \parallel AD$ and $CB = AD = 3\sqrt5$ units (one pair of opposite sides parallel and equal).

Alternatively: We can prove gradient CD = gradient BC, hence ABCD will be a parralelogram as the two pairs of opposite sides are parallel. (2 marks)

iv) Perpendicular distance $= \dfrac{|2\times0 - 1\times -1 + 4|}{\sqrt{2^2+(-1)^2}} = \dfrac{5}{\sqrt5} = \sqrt5$ (2 marks)

v) A = base × perpendicular height

$= 3\sqrt5 \times \sqrt5 = 15$ units^2

(2 marks)

Question 3

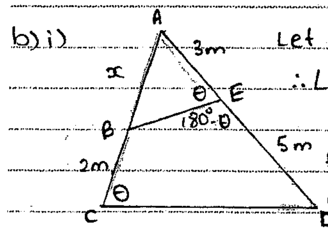
a) $\angle CEF = 44°$ (co-interior to $\angle ECD$, $CD \parallel EF$)

$\angle BEF = 67°$ (alternate to $\angle ABE$, $AB \parallel EF$)

$\therefore \angle BEC = \angle BEF - \angle CEF$

$= 67° - 44° = 23°$ (2 marks)

b) i)



Let $\angle BCD = \theta$

$\therefore \angle BED = 180° - \theta$

(Data)

Hence $\angle AEB = \theta$

(AED straight line)

So in ΔAEB and ΔACD

$\angle EAB = \angle CAD$ (common angle)

$\angle AEB = \angle ACD$ (proven above)

$\angle ABE = \angle ADC$ (remaining angles)

$\therefore \Delta AEB \sim \Delta ACD$ (equiangular)

(2 marks)

ii) Since $\Delta ABE \sim \Delta ADC$, their corresponding sides are in the same ratio. i.e. $\dfrac{AB}{AD} = \dfrac{AE}{AC}$

Let $AB = x$,

$\dfrac{x}{8} = \dfrac{3}{x+2}$

so $x^2 + 2x = 24$

$x^2 + 2x - 24 = 0$

$(x+6)(x-4) = 0$

$\therefore x = -6,\ x = 4$

$x > 0$ (as x represents the length of AB)

so $AB = 4$m (2 marks)

c) $\dfrac{d}{dx}(2\tan 3x) = 2 \times 3 \times \sec^2 3x$

$= 6\sec^2 3x$

(1 mark)

d) $y = \ln(\sin x)$

$\therefore \dfrac{dy}{dx} = \dfrac{\cos x}{\sin x}$ $u = \cos x$ $v = \sin x$

$u' = -\sin x$ $v' = \cos x$

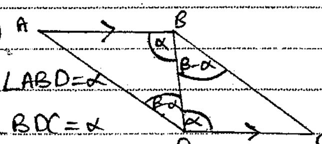
$\dfrac{d^2y}{dx^2} = \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x}$

$= \dfrac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= \dfrac{-1}{\sin^2 x}$ or $-\operatorname{cosec}^2 x$

(2 marks)

(e)



Let $\angle ABD = \alpha$

$\therefore \angle BDC = \alpha$

(alternate $\angle$'s, $AB \parallel DC$)

Let $\angle ABC = \beta$

$\therefore \angle ADC = \beta$ (Data)

$\therefore \angle BDA = \angle DBC = \beta - \alpha$

BD is common side

$\therefore \Delta ABD \equiv \Delta BDC$ (ASA)

(3 marks)

Question 4

a) $\int_0^9 (\sqrt{x}+1)^2 dx$

$= \int_0^9 (x + 2x^{\frac{1}{2}} + 1) dx$

$= \left[\frac{x^2}{2} + \frac{4x^{\frac{3}{2}}}{3} + x \right]_0^9$

$= \left(\frac{81}{2} + 36 + 9\right) - (0) = 85\frac{1}{2}$

(2 marks)

b) $y = \ln x$

$\frac{dy}{dx} = \frac{1}{x}$ (gradient function)

when $x = e$

$\frac{dy}{dx} = \frac{1}{e}$

$\therefore m_{tangent} = \frac{1}{e}$

$\therefore m_{normal} = -e$

Also, when $x = e$, $y = 1$.

Using point-gradient formula:

$y - 1 = -e(x - e)$

so $y = -ex + e^2 + 1$

(3 marks)

c) i) $y = \frac{1}{3}\tan^3 x$

$\therefore \frac{dy}{dx} = \frac{1}{3} \times 3\tan^2 x \times \sec^2 x$

$= \sec^2 x . \tan^2 x = (\tan^2 x + 1)\tan^2 x$

$= \tan^4 x + \tan^2 x$ (2 marks)

ii) $\int_0^{\frac{\pi}{3}} (\tan^4 x + \tan^2 x) dx$

$= \left[\frac{1}{3}\tan^3 x \right]_0^{\frac{\pi}{3}} = \frac{1}{3}(\sqrt{3})^3 - \frac{1}{3} \times 0$

$= \frac{1}{3} \times 3\sqrt{3}$

$= \sqrt{3}$

(1 mark)

d) $T_2 = a + d = 9$ ①

$T_5 = a + 4d = 30$ ②

② − ① gives $3d = 21$

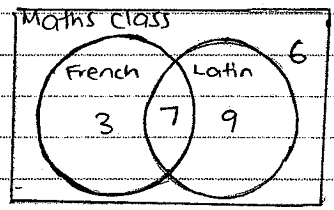
$\therefore d = 7$

$\therefore a + 7 = 9$

$a = 2$

$S_{20} = \frac{20}{2}(2 \times 2 + 19 \times 7) = 1370$

(2 marks)

e) 

Probability of neither: $\frac{6}{25} \times \frac{5}{24} = \frac{1}{20}$

(2 marks)

Question 5

a) i) $y = x^4 - 8x^2 + 7$

$\frac{dy}{dx} = 4x^3 - 16x$ (gradient function)

let $\frac{dy}{dx} = 0$ to find possible stationary turning points.

$4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$4x(x-2)(x+2) = 0$

$\therefore x = 0, 2, -2$

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	-60	0	12	0	-12	0	60
y	↘	min	↗	max	↘	min	↗

By substituting values as shown in the table, we could find the sign of $\frac{dy}{dx}$.

From the table we can see that when $x < -2$, $\frac{dy}{dx} < 0$, y is decreasing. When $-2 < x < 0$, $\frac{dy}{dx} > 0$ y is increasing.

$\therefore$ There is a minimum turning point at $(-2, -9)$.

When $0 < x < 2$, $\frac{dy}{dx} < 0$, y is decreasing.

$\therefore$ There is a maximum turning point at $(0, 7)$

When $x > 2$, $\frac{dy}{dx} > 0$, y is increasing.

$\therefore$ There is a minimum turning point at $(2, -9)$. (2 marks)

ii) $y = x^4 - 8x^2 + 7$

To find the intersections the curve makes with the x-axis, we let $y = 0$

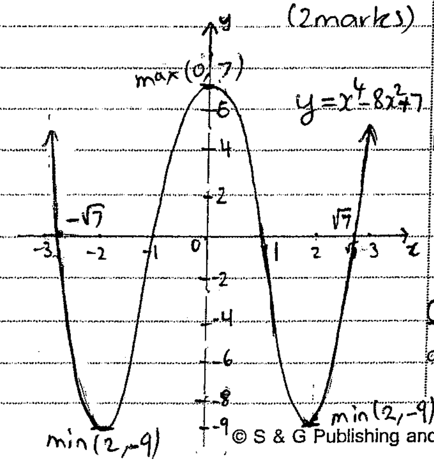
$x^4 - 8x^2 + 7 = 0$

$\therefore (x^2 - 7)(x^2 - 1) = 0$

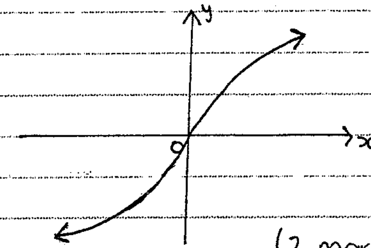
$\therefore x^2 = 7$, $x^2 = 1$

$\therefore x = \pm\sqrt{7}$, ± 1 (2 marks)

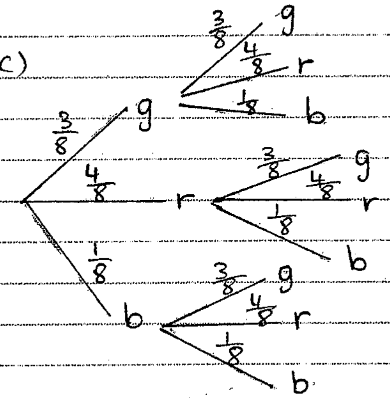
iii) (2 marks)



b) The curve is always increasing. It concaves up for $x < 0$. Then it concaves down for $x > 0$. As it passes through the origin it makes a point of inflexion.



(2 marks)

c) 

i) Probability (g, g): $\frac{3}{8} \times \frac{3}{8} = \frac{9}{64}$ (1 mark)

ii) Probability: $\frac{3}{8} \times \frac{1}{8} + \frac{1}{2} \times \frac{1}{8} + \frac{1}{8} \times \frac{1}{2} + \frac{1}{8} \times \frac{3}{8} = \frac{7}{32}$ (1 mark)

iii) Probability (different colour):

Probability (1 − same colour)

$= 1 - \left(\frac{3}{8} \times \frac{3}{8} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{8} \times \frac{1}{8}\right) = \frac{19}{32}$

(2 marks)

Question 6

a) $\tan x = \sqrt{3}$ $\quad 0 \le x \le 2\pi$

$\tan x = \tan \frac{\pi}{3}$

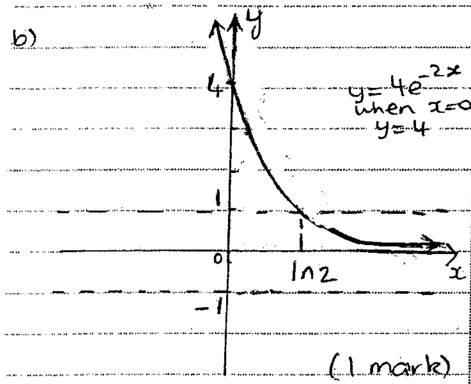
$\therefore x = \frac{\pi}{3} + \pi k$

when $k=0$, $x = \frac{\pi}{3}$

when $k=1$, $x = \frac{4\pi}{3}$

$\therefore x = \frac{\pi}{3}, \frac{4\pi}{3}$ in the domain $0 \le x \le 2\pi$. (2 marks)

b)



(1 mark)

c) $2e^{x} + 8e^{-x} + 32e^{-3x} + \ldots$

i) $\frac{T_2}{T_1} = \frac{8e^{-x}}{2e^{x}} = 4e^{-2x}$

$\frac{T_3}{T_2} = \frac{32e^{-3x}}{8e^{-x}} = 4e^{-2x}$

$\therefore \frac{T_3}{T_2} = \frac{T_2}{T_1}$

$\therefore$ The series is geometric where $a = 2e^{x}$ and $r = 4e^{-2x}$. (1 mark)

ii) A geometric series has a limiting sum when $-1<r<1$ $\therefore -1 < 4e^{-2x} < 1$

The solution of this inequality can be found by using part (b), and taking the part of the curve $y=4e^{-2x}$ between $y=-1$ and $y=1$.

But, $4e^{-2x} = 1$

$e^{-2x} = \frac{1}{4}$

$\ln e^{-2x} = \ln\left(\frac{1}{4}\right)$

$-2x = -\ln 4$

$-2x = -2\ln 2$

$x = \ln 2$

$\therefore$ when $x > \ln 2$, the infinite sum exists. (2 marks)

iii) $S_\infty = \frac{a}{1-r} = \frac{2e^{x}}{1-4e^{-2x}}$ (1 mark)

d) i) $\frac{dP}{dt} = kt$

$\int dP = \int kt \, dt$

$P = \frac{kt^2}{2} + C$

When $t=0$, $P=400$

$\therefore C = 400$

$\therefore P = \frac{kt^2}{2} + 400$ (1 mark)

ii) when $t=2$, $P=448$

$\frac{4k}{2} + 400 = 448$

$2k = 448 - 400$

$k = \frac{48}{2}$

$k = 24$

$\therefore P = 12t^2 + 400$

when $t=3$

$P = 108 + 400 = 508$ (2 marks)

iii) $P = 12t^2 + 400$

Let $P = 700$

$12t^2 + 400 = 700$

$12t^2 = 300$

$t^2 = \frac{300}{12}$

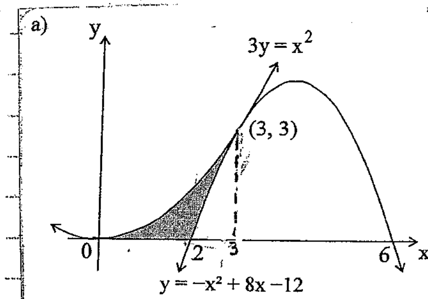
$t^2 = 25$

$t = \pm 5$

Since t represents time, then $t \ge 0$.

$\therefore t = 5$ years (2 marks)

Question 7



The area under the curve $y = \frac{x^2}{3}$ between $x=0$ and $x=3$ is A_1.

$A_1 = \int_0^3 \frac{x^2}{3}\,dx = \left[\frac{x^3}{9}\right]_0^3 = 3$ units^2

The area under the curve $y = -x^2 + 8x - 12$ between $x=2$ and $x=3$ is A_2.

$A_2 = \int_2^3 (-x^2 + 8x - 12)\,dx$

$= \left[-\frac{x^3}{3} + 4x^2 - 12x\right]_2^3$

$= (-9 + 36 - 36) - \left(-\frac{8}{3} + 16 - 24\right)$

$= 1\frac{2}{3}$ units^2.

The shaded area can be found by subtracting A_2 from A_1.

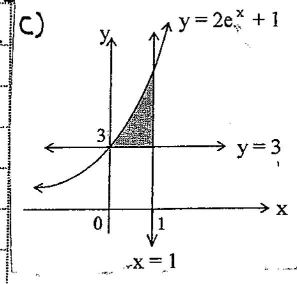
$\therefore A_1 - A_2 = 3 - 1\frac{2}{3} = 1\frac{1}{3}$ units^2. (3 marks)

b)

x	0	1	2
$\sin^2 x$	0	0.708	0.827
	y_0	y_1	y_2

$A \doteqdot \frac{1}{3}(0 + 4 \times 0.708 + 0.827)$

$\doteqdot 1.2196 \doteqdot 1.220$ (3dp) (2 marks)



The area under the curve $y = 2e^{x} + 1$ between $x=0$ and $x=1$ is A_1.

$A_1 = \int_0^1 (2e^{x} + 1)\,dx$

$= \left[2e^{x} + x\right]_0^1$

$= (2e + 1) - (2)$

$= (2e - 1)$ units^2

The area of the rectangle is A_2.

$A_2 = 1 \times 3 = 3$ units^2.

The shaded area can be found by subtracting A_2 from A_1.

$\therefore A_1 - A_2 = 2e - 1 - 3 = 2e - 4$
$= 2(e-2)$ units^2 (2 marks)

ii) The volume made when the curve $y = 2e^{x} + 1$ is rotated about the x-axis is $V_1 = \pi\int_0^1 (2e^{x}+1)^2\,dx$.

$= \pi\int_0^1 (4e^{2x} + 4e^{x} + 1)\,dx$

The volume made when the line $y=3$ is rotated about the x-axis is $V_2 = \pi\int_0^1 (3)^2\,dx$

$\therefore V = V_1 - V_2 = \pi\int_0^1 (4e^{2x} + 4e^{x} + 1 - 9)\,dx$

$= \pi\int_0^1 (4e^{2x} + 4e^{x} - 8)\,dx$

$= 4\pi\int_0^1 (e^{2x} + e^{x} - 2)\,dx$ (as required) (2 marks)

iii) $V = 4\pi\left[\frac{1}{2}e^{2x} + e^{x} - 2x\right]_0^1$

$= 4\pi\left[\left(\frac{1}{2}e^2 + e - 2\right) - \left(\frac{1}{2} + 1\right)\right]$

$= 4\pi\left[\frac{1}{2}e^2 + e - 2 - 1\frac{1}{2}\right]$

$= 4\pi\left[\frac{1}{2}e^2 + e - 3\frac{1}{2}\right]$

$= \pi\left[2e^2 + 4e - 14\right]$ units^3. (1 mark)

d) $2x^2+mx+8=0$

$\Delta=m^2-4\times2\times8$

$=m^2-64$

The quadratic equation will have two unequal real roots if $\Delta>0$.

$\therefore m^2-64>0 \ \therefore (m-8)(m+8)>0$

$\therefore m<-8,\ m>8$.

For the roots to be positive, the sum of the roots must be positive and the product must be also positive.

The sum is $\frac{-m}{2}$ and the product is 4. So, only the sum must be positive. That is, $\frac{-m}{2}>0 \ \therefore m<0$.

$\therefore$ The roots are unequal, real and positive when $m<-8$. (2 marks)

Question 8

a) i) $v=\frac{dx}{dt}=3t^2-18t+24$

$\int dx=\int(3t^2-18t+24)\,dt$

$x=\frac{t^3}{3}-9t^2+24t+C$

when $t=0$, $x=-216$

$\therefore C=-216$

$\therefore x=t^3-9t^2+24t-216$ (1 mark)

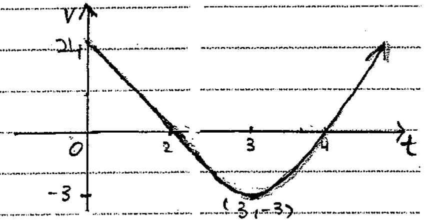
ii) $v=3t^2-18t+24$

The particle is at rest when $v=0$

$\therefore 3t^2-18t+24=0$

$t^2-6t+8=0,\ (t-4)(t-2)=0$

So, $t=2$ seconds, $t=4$ seconds.



From the graph we can see that the particle changes direction at $t=2$ seconds because it changes from a positive direction to a negative direction. This occurs at $x=-196$. It also changes direction when $t=4$ seconds because the particle changes direction from negative to positive. This occurs at $x=-200$. (2 marks)

iii) $x=t^3-9t^2+24t-216$

$x=t^2(t-9)+24(t-9)$

$x=(t^2+24)(t-9)$. The particle passes by the origin when $(t^2+24)(t-9)=0$

$t^2+24>0$ for all values of t (sum of two positive terms).

$\therefore t-9=0,\ t=9$

$\therefore$ The particle passes through the origin only once, when $t=9$ seconds. (2 marks)

iv) At time $t=0$ seconds, the particle was initially 216 metres to the left of the origin. In between times $t=0$ seconds and $t=2$ seconds, velocity>0 and acceleration<0. This means that the particle was moving in the positive direction but slowing down from $x=-216$ to $x=-196$. At time $t=2$ seconds, $V=0$

and $a<0$. This means that the particle is changing direction, to move in the negative direction. This happens when $x=-196$. In between time $t=2$ seconds and $t=3$ seconds, velocity<0, acceleration<0. This means that the particle is accelerating in the negative direction. In between times $t=3$ seconds and $t=4$ seconds, velocity<0, acceleration>0. This means that the particle is slowing down in a negative direction from $x=-198$ to $x=-200$. At $t=4$ seconds, $v=0$ and $a>0$. This means that the particle is changing direction, to move in the positive direction, and this happens when $x=-200$. when $t>4$ seconds, velocity>0 and acceleration>0. This means that the particle was accelerating in the positive direction. As it continues, the particle crosses the origin at $t=9$ seconds. (2 marks)

b) i) Let P be the amount of her single deposit, by using the compound interest formula.

$P(1+\frac{9}{1200})^{48}=8000,\ P(1\cdot0075)^{48}=8000$

$P=\frac{8000}{(1\cdot0075)^{48}}=\$5588\cdot91$ (1 mark)

ii) Let M be her regular deposit. The value of her first deposit is $A_1=M(1\cdot0075)^{48}$. Her second is $A_2=M(1\cdot0075)^{47}$, and so on.

$\therefore$ Her last deposit is $M(1\cdot0075)$.

The sum of the e deposits is: $S=M(1\cdot0075)+M(1\cdot0075)^2+\ldots+M(1\cdot0075)^{48}$

$=M(1\cdot0075)[(1+1\cdot0075+\ldots+1\cdot0075^{47})]$

Let $(1+1\cdot0075+\cdots+1\cdot0075^{47})=K$. K is a geometric series where $a=1$, $r=1\cdot0075$

$S=M(1\cdot0075)\left[\frac{(1\cdot0075)^{48}-1}{1\cdot0075-1}\right]$, but $S=8000$

$\therefore M(1\cdot0075)\left[\frac{(1\cdot0075)^{48}-1}{0\cdot0075}\right]=8000$

$M=\frac{8000}{1\cdot0075\left[\frac{(1\cdot0075)^{48}-1}{0\cdot0075}\right]}=\$138\cdot05$ (3 marks)

iii) Method 1 will cost Vanessa \$5588·91. Method 2 will cost Vanessa \$138·05×48 = \$6626·40. Hence, the single deposit is \$1037·49 cheaper than the regular deposits. (1 mark)

Question 9

a) i) S is the speed at which the ship is travelling. Using the formula; speed $=\frac{\text{distance}}{\text{time}}$, that is, $s=\frac{d}{t}$. The time of the trip is $t=\frac{150}{s}$. $\therefore$ The cost of hiring the captain will be $C_1=50+(24\times\frac{150}{s})$

$=50+\frac{3600}{s}$. The cost of the ship and the fuel is $C_2=\frac{360+s^2}{9}\times\frac{150}{s}$

$=(40+\frac{s^2}{9})\times\frac{150}{s}=\frac{6000}{s}+\frac{50s}{3}$

$\therefore$ The cost of the trip is:

$C=50+\frac{3600}{s}+\frac{6000}{s}+\frac{50s}{3}$

$=\frac{9600}{s}+\frac{50s}{3}+50$. (2 marks)

ii) $\frac{dC}{ds}=\frac{-9600}{s^2}+\frac{50}{3}$ $\quad u=9600\ \ v=s$, $u'=0\ \ v'=1$

Let $\frac{dC}{ds}=0$ to find possible stationary turning points.

$-\frac{9600}{s^2}+\frac{50}{3}=0$, $\frac{9600}{s^2}=\frac{50}{3}$

$50s^2=28800$, $s^2=576$, $s=\pm 24$

As s represents speed, $s>0$.

$\therefore s=24$

s	23	24	25
$\frac{ds}{dC}$	-1.48	0	1.306
C	↘	min	↗

When $s<24$, $\frac{ds}{dc}<0$, C is decreasing.

When $s>24$, $\frac{ds}{dc}>0$, C increasing.

$\therefore$ When $s=24$, $\frac{ds}{dc}=0$, cost is a minimum.

$\therefore$ The speed that will result in the smallest cost occurs at 24 km/h. (3 marks)

b) i) For the first two seconds, the acceleration is $5\,m/s^2$. $\therefore v=\int 5\,dt$

$v=5t+C$, when $t=0$, $v=0$, $\therefore c=0$

$\therefore v=5t$, $0\le t\le 2$ (1 mark)

ii) $\frac{dv}{dt}=t^2-10t+21$ $\quad 2\le t\le 4$

$\int dv=\int(t^2-10t+21)\,dt$

$v=t^3-5t^2+21t+C$

$v=\frac{t^3}{3}-5t^2+21t+C$

when $t=2$, $v=5\times2=10$

$10=\frac{8}{3}-(5\times4)+(21\times2)+C$

$\therefore C=-14\frac{2}{3}$

$\therefore v=\frac{t^3}{3}-5t^2+21t-14\frac{2}{3}$ $\quad 2\le t\le 7$ (3 marks)

iii) $\frac{dv}{dt}=t^2-10t+21=(t-3)(t-7)$

Let $\frac{dv}{dt}=0$ to find possible stationary turning points.

$\therefore (t-3)(t-7)=0$

$\therefore t=3$ seconds, $t=7$ seconds

t	2	2.5	3	5	7	8
$\frac{dv}{dt}$		2.25	0	-4	0	5
v		↗	max	↘	min	↗

When $2\le t\le 3$, $\frac{dv}{dt}>0$, v increasing.

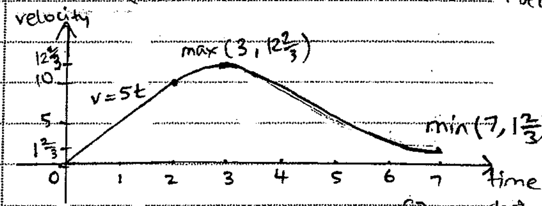
When $3<t\le 7$, $\frac{dv}{dt}<0$, v decreasing

$\therefore$ when $t=3$, $\frac{dv}{dt}=0$, v is a maximum.

When $t>7$, $\frac{dv}{dt}>0$, v is increasing.

$\therefore$ when $t=7$, $\frac{dv}{dt}=0$, v is a minimum.

To sketch the velocity time graph, we have to draw the graph of the line between $t=0$ seconds and $t=2$ seconds. We also have to draw the curve found in (ii) between $t=2$ seconds, $t=7$ seconds.



(2 marks)

iv) The distance travelled by the particle in a velocity time graph can be found by calculating the area under the graph. The distance the particle travelled between $t=0$ seconds and $t=2$ seconds is

$A_1=\int_0^2 5t\,dt=\left[\frac{5t^2}{2}\right]_0^2=\frac{(5\times4)}{2}-0$

$=10$ m. The distance the particle travelled between $t=2$ seconds and $t=7$ seconds is $A_2=\int_2^7\left(\frac{t^3}{3}-5t^2+21t-14\frac{2}{3}\right)dt$

$=\left[\frac{t^4}{12}-\frac{5t^3}{3}+\frac{21t^2}{2}-14\frac{2}{3}t\right]_2^7$

$=40\frac{1}{4}-\frac{2}{3}=39\frac{7}{12}$ m.

$\therefore$ Total distance is $10+39\frac{7}{12}$

$=49\frac{7}{12}$ m (1 mark)

Question 10

a) i) In ΔOFE, $\cos\alpha=\frac{OF}{r}$, $\therefore OF=r\cos\alpha$

$\sin\alpha=\frac{EF}{r}$, $EF=r\sin\alpha$.

$EF=DC$ (opposite sides of a rectangle are equal).

$\therefore DC=r\sin\alpha$.

In ΔODC, $\tan\frac{\pi}{3}=\frac{r\sin\alpha}{OC}$

$OC\sqrt{3}=r\sin\alpha$, $OC=\frac{r\sin\alpha}{\sqrt{3}}$

$\therefore CF=OF-OC=r\cos\alpha-\frac{r\sin\alpha}{\sqrt{3}}$ (3 marks)

ii) Area of rectangle CDEF is

$A=CF\times FE=r\sin\alpha\left(r\cos\alpha-\frac{r\sin\alpha}{\sqrt{3}}\right)$

$=r^2\cos\alpha\sin\alpha-\frac{r^2\sin^2\alpha}{\sqrt{3}}$

$=r^2\times\frac{1}{2}\sin2\alpha-\frac{r^2\sin^2\alpha}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$

$=\frac{r^2\sin2\alpha}{2}-\frac{\sqrt{3}r^2\sin^2\alpha}{3}=r^2\left(\frac{1}{2}\sin2\alpha-\frac{\sqrt{3}}{3}\sin^2\alpha\right)$ (2 marks)

iii) $A=r^2\left(\frac{1}{2}\sin2\alpha-\frac{\sqrt{3}}{3}\sin^2\alpha\right)$

$\frac{dA}{d\alpha}=r^2\left(\frac{1}{2}\times2\times\cos2\alpha-\frac{\sqrt{3}}{3}\times2\sin\alpha\cos\alpha\right)$

$\frac{dA}{d\alpha}=r^2\left(\cos2\alpha-\frac{\sqrt{3}}{3}\sin2\alpha\right)$

Let $\frac{dA}{d\alpha}=0$ to find possible stationary turning points.

$r^2\left(\cos2\alpha-\frac{\sqrt{3}}{3}\sin2\alpha\right)=0$

$\cos2\alpha-\frac{\sqrt{3}}{3}\sin2\alpha=0$

$\frac{\sqrt{3}}{3}\sin2\alpha=\cos2\alpha$

$\frac{\sqrt{3}}{3}\tan2\alpha=1$

$\tan2\alpha=\frac{3}{\sqrt{3}}$

$\tan2\alpha=\sqrt{3}$

$\tan2\alpha=\tan\frac{\pi}{3}$, $2\alpha=\frac{\pi}{3}$ (α is an acute angle, $\alpha\le\frac{\pi}{3}$) $\therefore \alpha=\frac{\pi}{6}$

α	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$
$\frac{dA}{d\alpha}$	$0.366\ldots r^2$	0	$-(r)^2$
A	↗	max	↘

When $\alpha<\frac{\pi}{6}$, $\frac{dA}{d\alpha}>0$, Area is increasing.

When $\alpha>\frac{\pi}{6}$, $\frac{dA}{d\alpha}<0$,

$\therefore$ when $\alpha=\frac{\pi}{6}$, $\frac{dA}{d\alpha}=0$

$\therefore$ Area is a maximum.

(2 marks)

b) i) The amount owing after one month is $A_1=12000\times1.01-\ln P$. The amount owing after two months is $A_2=(12000\times1.01-\ln P)\times1.01-2\ln P$

$=12000\times1.01^2-\ln P\times1.01-2\ln P$

$=12000\times1.01^2-\ln P(1.01+2)$. Amount owing after three months is $A_3=(12000\times1.01^2-\ln P\times1.01-2\ln P)\times1.01-4\ln P=12000\times1.01^3-\ln P\times1.01^2-2\ln P\times1.01-4\ln P$

$=12000\times1.01^3-\ln P(1.01^2+2\times1.01+2^2)$ (2 marks)

ii) $A_{15}=12000\times1.01^{15}-\ln P(1.01^{14}+2\times1.01^{13}+\ldots+2^{14})$

$A_{15}=0$, as the loan is paid off,

$12000\times1.01^{15}-\ln P(1.01^{14}+2\times1.01^{13}+\ldots+2^{14})=0$

$(1.01^{14}+2\times1.01^{13}+\ldots+2^{14})=K$, where K is a geometric series, and $a=1.01^{14}$, $r=\frac{2}{1.01}$,

$12000\times1.01^{15}-\ln P\left[\frac{1.01^{14}\left(\left(\frac{2}{1.01}\right)^{15}-1\right)}{\frac{2}{1.01}-1}\right]=0$

$\ln P=\frac{12000\times1.01^{15}}{\left[\frac{1.01^{14}\left(\left(\frac{2}{1.01}\right)^{15}-1\right)}{\frac{2}{1.01}-1}\right]}=0.4209\ldots$

$P=e^{0.4209\ldots}=1.52336\ldots=1.5234$ (4 dp) (2 marks)

Note: $e^{\ln P}=P$

iii) Total paid by Yvonne: $(1+2+4+8+\ldots+2^{14})\ln P$. $(1+2+4+8+\ldots+2^{14})$ is a geometric series, where $a=1$, $r=2$.

$P=1.5234$ $\therefore \ln P=0.4209$ (4dp)

$\therefore$ Total $=0.4209\left[\frac{1(2^{15}-1)}{2-1}\right]$

$=\$13791.63$ (1 mark)