SYDNEY GIRLS HIGH SCHOOL

2007 HSC Assessment Task 2

March 5, 2007

MATHEMATICS Extension 2

Year 12

Reading Time 5 minutes
Time allowed: 90 minutes

Topics: Complex Numbers

DIRECTIONS TO CANDIDATES:

- Attempt all questions
- Questions are not of equal value
- There are 3 questions with part marks shown
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used
- Write on one side of the paper only

QUESTION 1 (Start a new page) 24 marks

a) Solve $2x^2 - x + 3 = 0$ and express your answer in the form x + iy [2]

b) If $z_1 = 2 + 3i$ and $z_2 = 3 - 5i$, find in x+iy form

i) $\overline{z_2}$ [1]

ii) $|z_1|$ [1]

iii) $3z_1 - 2z_2$ [2]

iv) $\frac{z_1}{z_2}$ [2]

c) Find $\sqrt{25 - 24i}$ and express your answer in x+iy form. [3]

d) If $z = -3 + i$

i) Express z in modulus-argument form [2]

ii) Evaluate z^6 and express your answer in x+iy form [2]

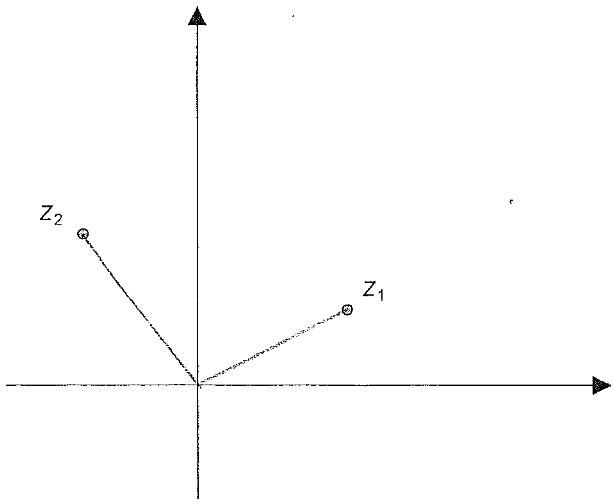
e) If $z_1 = a + ib$ and z_2=c+id, prove that $|z_1 z_2| = |z_1||z_2|$ [3]

f) If $z = rcis\theta$, what is the modulus and argument of $rkiz$ where k is a non zero real number. [2]

g) Solve for x if $ix^2 - (1+i)x + 2i - 3 = 0$ and express your solutions in x+iy form. [4]

QUESTION 2 (Start a new page) 24 marks

a) Copy (or trace) the diagram below



On your diagram mark the positions of

i) $z_1 + z_2$ [1]

ii) $z_2 - z_1$ [2]

iii) $-iz_2$ [2]

iv) $(1+i)z_2$ [2]

b) i) Use a diagram to show that for any complex numbers z and w that $|z+w| \le |z| + |w|$ [2]

ii) Give a numerical example so that $|z+w| = |z| + |w|$ [1]

c) Prove, by induction that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for positive n. [4]

d) i) Solve $z^5 + 1 = 0$ over the complex field [2]

ii) Sketch the solutions to $z^5 + 1 = 0$ on an Argand Diagram [1]

iii) Let α be the root with the smallest positive argument and write all roots in terms of α [1]

iv) Factorise $z^5 + 1$ over

α) The complex field [1]

β) The real field [2]

γ) the rational field [1]

v) Find the perimeter of the polygon whose vertices are represented by the roots of $z^5 + 1 = 0$ on the Argand Diagram. [2]

QUESTION 3 (Start a new page) 26 marks

a) If $z = x + iy$ express $\dfrac{z+i}{z-i}$ in the form $a + ib$ [3]

b) i) If $z = \cos\theta + i\sin\theta$,show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ [2]

ii) Express $\cos^4\theta$ in terms of cosines of multiples of θ [3]

iii) Find $\int \cos^4\theta . d\theta$ {given $\int \cos\theta . d\theta = \sin\theta + c$ } [2]

c) If w is a complex root of $z^3 - 1 = 0$,

i) Show that $1 + w + w^2 = 0$ [2]

ii) Find a quadratic equation whose roots are $1+w$ and $1+w^2$ [2]

iii) Evaluate $(1+w)(1+w^2)(1+w^3)(1+w^4)(1+w^5)$ [2]

d) Express $-1+\sqrt{3}i$ and $-1-\sqrt{3}i$ in modulus-argument form, and hence prove that if n is a multiple of 3 then $\left(-1+\sqrt{3}i\right)^n + \left(-1-\sqrt{3}i\right)^n = 2^{n+1}$ [3]

e) Find all the solutions of $z^2 = i\bar{z}$ where z is a complex number [3]

f) Express $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$ and hence express $\tan 4\theta$ in terms of $\tan\theta$ [4]

---------------------------------------end of paper---------------------------------------

1/5

1(a) $x = \dfrac{1 \pm \sqrt{1-24}}{4}$

$= \dfrac{1 \pm \sqrt{23}\,i}{4}$

(b) i) $3+5i$

ii) $\sqrt{2^2+3^2}$

$= \sqrt{13}$

iii) $6+9i-(6-10i)$

$= 19i$

iv) $\dfrac{2+3i}{3-5i} \times \dfrac{3+5i}{3+5i}$

$= \dfrac{6+10i+9i-15}{9+25}$

$= \dfrac{-9+19i}{34}$

(c) $(x+iy)^2 = 25-24i$

$x^2-y^2 = 25 \qquad 2xy = -24$

$y = -\dfrac{12}{x}$

$x^2 - \dfrac{144}{x^2} = 25$

$x^4 - 25x^2 - 144 = 0$

$x^2 = \dfrac{25 \pm \sqrt{625+576}}{2}$

$= \dfrac{25 \pm \sqrt{1201}}{2}$

$x^2 = \dfrac{25+\sqrt{1201}}{2}$

$x = \pm\sqrt{\dfrac{25+\sqrt{1201}}{2}} \qquad y = \mp 12\sqrt{\dfrac{2}{25+\sqrt{1201}}}$

$\therefore \sqrt{25-24i} = \pm\left(\sqrt{\dfrac{25+\sqrt{1201}}{2}} - i\,12\sqrt{\dfrac{2}{25+\sqrt{1201}}}\right)$

$= \pm(5.46\ldots - 2.19\ldots\, i)$

(d) i) $|z| = \sqrt{3^2+1^2}$

$= \sqrt{10}$

$\tan(\arg z) = \dfrac{1}{-3}$

$\therefore \arg z = 161.56505\ldots^\circ$

$\doteqdot 162^\circ$

ii) $z^6 = (\sqrt{10})^6 \operatorname{cis}(6 \times 162^\circ)$

$= 1000 \operatorname{cis}(972^\circ)$

$x = 1000 \cos(252^\circ)$

$= -309.0169944$

$y = 1000 \sin(252^\circ)$

$= -951.0565163$

$\therefore z^6 \doteqdot -309 - 951i$

(e) $|z_1 z_2|$

$= |ac + iad + ibc - bd|$

$= \sqrt{(ac-bd)^2 + (ad+bc)^2}$

$= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}$

$|z_1||z_2|$

$= \sqrt{a^2+b^2}\,.\,\sqrt{c^2+d^2}$

$= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$

$= |z_1 z_2| \quad$ Q.E.D.

(f) $rki \,.\, r\operatorname{cis}\theta$

$= r^2 k \operatorname{cis}(\theta + \frac{\pi}{2})$

$\therefore |rkiz| = r^2 k$

$\arg(rkiz) = \theta + \frac{\pi}{2}$

(g) $x = \dfrac{(1+i) \pm \sqrt{(1+i)^2 - 4 \times i(2i-3)}}{2i}$

$= \dfrac{(1+i) \pm \sqrt{1+2i-1+8+12i}}{2i}$

$= \dfrac{(1+i) \pm \sqrt{8+14i}}{2i}$

Let $8+14i = (a+ib)^2$

$a^2-b^2 = 8 \qquad 2ab = 14$

$b = \dfrac{7}{a}$

$a^2 - \dfrac{49}{a^2} = 8$

$a^4 - 8a^2 - 49 = 0$

$a^2 = \dfrac{8 \pm \sqrt{64+196}}{2}$

$= \dfrac{8 \pm \sqrt{260}}{2}$

$= 4 \pm \sqrt{65}$

$a = \pm\sqrt{4+\sqrt{65}}$

$= \pm 3.4730\ldots$

$b = \dfrac{7}{\pm\sqrt{4+\sqrt{65}}}$

$= \pm 2.0155\ldots$

$\therefore x = \dfrac{1 \pm \sqrt{4+\sqrt{65}}}{2i} + \dfrac{i \pm \frac{7}{\sqrt{4+\sqrt{65}}}\, i}{2i}$

$= \dfrac{1 \pm \frac{7}{\sqrt{4+\sqrt{65}}}}{2} + \dfrac{1 \mp \sqrt{4+\sqrt{65}}}{2}\, i$

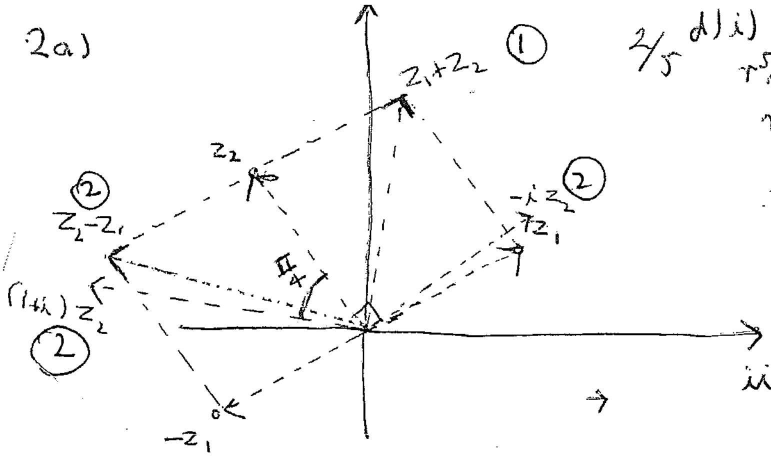
$= 1.57\ldots - 2.23\ldots\, i$

or

$-0.50\ldots + 1.23\ldots\, i$

S.G.H.S.
EXT 2
TASK 2
2007

2a) ①

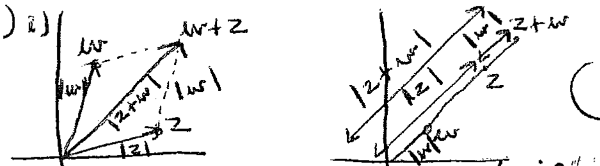


ii) $|1+i| = \sqrt{1^2+1^2}$
$= \sqrt{2}$

$\tan[\arg(1+i)] = \frac{1}{1}$

$\therefore \arg(1+i) = \frac{\pi}{4}$

b) i)



②

Since one side of a triangle is less than the sum of the other two sides.

$|z+w| < |z|+|w|$ $\quad$ $|z+w| = |z|+|w|$

$\therefore |z+w| \le |z|+|w|$

ii) let $z = w = 0$ ①
$|0+0| = |0|+|0|$

c) 1. Prove for $n=1$
$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$
$\cos 1\theta + i\sin 1\theta = \cos\theta + i\sin\theta$
$\therefore$ true for $n=1$

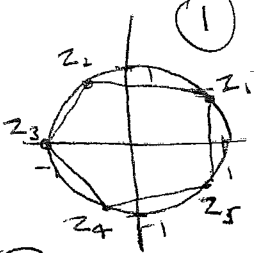
2. Assume true for $n=k$
$\therefore (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

3. Prove for $n=k+1$
$(\cos\theta + i\sin\theta)^{k+1}$
$= (\cos\theta+i\sin\theta)^k . (\cos\theta + i\sin\theta)$
$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$
$= \cos k\theta . \cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta$
$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \sin\theta\cos k\theta)$
$= \cos[(k+1)\theta] + i\sin[(k+1)\theta]$
$\therefore$ true for $n=k+1$

4. Since true for $n=1$ and $n=k+1$ ④
$\therefore$ true for $n \ge 1$

2/5 d) i) $z^5 = -1$
$r^5 \text{cis}\, 5\theta = -1$
$r = \sqrt{1^2+0^2}$ $\quad \cos 5\theta = -1 \quad \sin 5\theta = 0$
$= 1$
$\therefore 5\theta = \pi, 3\pi, 5\pi, 7\pi, 9\pi$
$\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$
$z = \text{cis}\frac{\pi}{5}, \text{cis}\frac{3\pi}{5}, \text{cis}\,\pi, \text{cis}\frac{7\pi}{5}, \text{cis}\frac{9\pi}{5}$ ②

ii) ①



iii) $\alpha = \text{cis}\frac{\pi}{5}$ $\quad \text{cis}\frac{3\pi}{5} = \alpha^3 \quad \text{cis}\,\pi = \alpha^5 \quad \text{cis}\frac{7\pi}{5} = \alpha^7$
$= -\alpha^2$
$\text{cis}\frac{9\pi}{5} = \alpha^9 = -\alpha^4$ ①

iv) α) $(z+1)(z - \cos\frac{\pi}{5} - i\sin\frac{\pi}{5})(z - \cos\frac{\pi}{5} + i\sin\frac{\pi}{5})$
$(z - \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5})(z - \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5})$ ①

β) $(z+1)(z^2 - 2z\cos\frac{\pi}{5} + \cos^2\frac{\pi}{5} + \sin^2\frac{\pi}{5})$
$(z - 2z\cos\frac{3\pi}{5} + \sin^2\frac{3\pi}{5} + \cos^2\frac{3\pi}{5})$ ②
$= (z+1)(z^2 - 2z\cos\frac{\pi}{5} + 1)(z^2 - 2z\cos\frac{3\pi}{5}$

γ)
$$
\begin{array}{r}
z^4 - z^3 + z^2 - z + 1 \\
z+1 \,\big)\, z^5 \qquad\qquad\quad + 1 \\
z^5 + z^4 \qquad\qquad\quad \\
\hline
-z^4 \qquad\quad + 1 \\
-z^4 - z^3 \qquad\quad \\
\hline
z^3 \quad + 1 \\
z^3 + z^2 \quad \\
\hline
-z^2 \; + 1 \\
-z^2 - z \\
\hline
z + 1 \\
z + 1 \\
\hline
0
\end{array}
$$

$\therefore z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$ ①

v) let s be a side of the regular pentagon
$s^2 = 1^2 + 1^2 - 2\times 1\times 1\times\cos\frac{2\pi}{5}$
$= 2 - 2\cos\frac{2\pi}{5}$
$s = \sqrt{2 - 2\cos\frac{2\pi}{5}}$

$P = 5\sqrt{2 - 2\cos\frac{2\pi}{5}}$ ②
$\doteqdot 5.88$

Question 3.

a) $z = \dfrac{x+iy+i}{x+iy-i}$

conjugate of denom is (in terms of z) $(\bar{z}+i)$

$= \dfrac{x+i(y+1)}{x+i(y-1)} \times \dfrac{x-i(y-1)}{x-i(y-1)}$ ✓

$= \dfrac{x^2 - i(y-1)x + i(y+1)x - i^2(y+1)(y-1)}{x^2 - i^2(y-1)^2}$

$= \dfrac{x^2 - iyx + ix + iyx + ix + y^2 - 1}{x^2 + y^2 - 2y + 1}$ ✓

(3)

$= \dfrac{x^2+y^2-1+2ix}{x^2+y^2-2y+1}$ ✓ $\left[\dfrac{x^2+y^2-1}{x^2+y^2-2y+1} + \dfrac{2ix}{x^2+y^2-2y+1}\right]$

b) i) $z = \cos\theta + i\sin\theta$

$z^n = \cos(n\theta) + i\sin(n\theta)$ ✓ ①, $z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$z^{-n} = \cos(n\theta) - i\sin(n\theta)$ ✓ ② (cos θ even sin θ odd) (2)

①+② $z^n + \dfrac{1}{z^n} = 2\cos(n\theta)$

ii) put $n=1$ in above

$z + \dfrac{1}{z} = 2\cos(\theta)$

$\left(z+\dfrac{1}{z}\right)^4 = 2^4\cos^4\theta$ ✓

$16\cos^4\theta = z^4 + 4z^2 + 6 + \dfrac{4}{z^2} + \dfrac{1}{z^4}$ ✓

$\cos^4\theta = \dfrac{1}{16}\left(z^4 + \dfrac{1}{z^4} + 4\left(z^2 + \dfrac{1}{z^2}\right) + 6\right]$ (3)

put $n = 4, 2$

$\cos^4\theta = \dfrac{1}{16}\left(2\cos 4\theta + 8\cos 2\theta + 6\right]$ ✓

iii) $\int \cos^4\theta \, d\theta = \dfrac{1}{16}\int 2\cos 4\theta + 8\cos 2\theta + 6 \, d\theta$

$= \dfrac{1}{16}\left(\dfrac{1}{2}\sin 4\theta + 4\sin 2\theta + 6\theta\right) + C$ (2) ✓✓

c) i) $1, \omega, \omega^2$ are roots of $z^3 - 1$ and sum of roots is given by $\dfrac{-b}{a} = \dfrac{0}{1}$

$= 0$ ✓✓ (2)

OR any other soln

ii) Sum of roots $= (1+\omega+\omega^2) + 1$

$= 1$ ✓ (2)

Product of roots $= (1+\omega)(1+\omega^2)$ ie $x^2 - x + 1 = 0$

$= 1 + \omega + \omega^2 + \omega^3$

$= 1$ ✓

c) iii) $(1+\omega)(1+\omega^2)(1+\omega^3)(1+\omega^4)(1+\omega^5)$

$= (1+\omega)(1+\omega^2)(2)(1+\omega)(1+\omega^2)$ ✓

$= 2(1+\omega+\omega^2+\omega^3)(1+\omega+\omega^2+\omega^3)$ ✓ ②

$= 2(1)(1)$

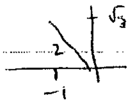
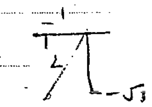
$= 2$

[Note $2\omega^3$ award 2 marks]

Quick Soln

$= (-\omega^2)(-\omega)\,2\,(-\omega^2)(-\omega)$

$= \omega^3 \times 2 \times \omega^3 = 2$

d) $-1+\sqrt{3}i$  , $-1-\sqrt{3}i$ 

$= 2\,\text{cis}\,\frac{2\pi}{3}$ $\qquad = 2\,\text{cis}\left(-\frac{2\pi}{3}\right)$ ✓

then $(-1+\sqrt{3}i)^n + (-1-\sqrt{3}i)^n = \left(2\,\text{cis}\,\frac{2\pi}{3}\right)^n + \left(2\,\text{cis}\,\frac{-2\pi}{3}\right)^n$

$= 2^n\,\text{cis}\,\frac{2\pi n}{3} + 2^n\,\text{cis}\,\frac{-2\pi n}{3}$

$= 2^n\left(\text{cis}\,\frac{2\pi n}{3} + \text{cis}\,\frac{-2\pi n}{3}\right)$ ✓

$= 2^n(\cos 2\pi + i\sin 2\pi + \cos -2\pi + i\sin -2\pi)$

$= 2^n(1+0+1+0)$ ✓ ③

$= 2^n(2)$

$= 2^{n+1}$

since n a multiple of 3 [or use $k=3n$]

Note: proving true for $n=3$ (only) is awarded 2 (max) marks only

e) $z^2 = i\bar{z}$

$(x+iy)^2 = i\,\overline{(x+iy)}$

$x^2-y^2+2iyx = \overline{ix-y}$

$x^2-y^2+2iyx = -y-ix$ ✓

$x^2-y^2+y+i(2yx+x) = 0$

$x^2-y^2+y=0 \qquad 2yx+x=0 \qquad$ equate Re, Im

$x(2y+1)=0$

$x=0, \quad y=-\frac{1}{2}$ ✓

<u>when $x=0$</u> $\qquad -y^2+y=0$

$y(1-y)=0 \qquad \underline{y=0,\ y=1}$ ③

<u>when $y=-\frac{1}{2}$</u> $\qquad x^2-\frac{1}{4}-\frac{1}{2}=0$

$x^2=\frac{3}{4}$ ✓

$\underline{x=\pm\frac{\sqrt{3}}{2}}$

i.e. **OR** $\left.\begin{matrix} x=0 \\ y=0 \end{matrix}\right\}$ $\left.\begin{matrix} x=0 \\ y=1 \end{matrix}\right\}$ (1 mark) $\quad \left.\begin{matrix} x=\frac{\sqrt{3}}{2} \\ y=-\frac{1}{2} \end{matrix}\right\}$ $\left.\begin{matrix} x=-\frac{\sqrt{3}}{2} \\ y=-\frac{1}{2} \end{matrix}\right\}$ (1 mark)

f) $(\cos\theta + i\sin\theta)^4 = (\cos 4\theta + i\sin 4\theta)$ [De Moivre Th]

$= \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta + 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4$

$= \cos^4\theta + 4\cos^3\theta\, i\sin\theta - 6\cos^2\theta\sin^2\theta - 4\cos\theta\, i\sin^3\theta + \sin^4\theta$ ✓

Equate Re

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ ✓

Equate Im

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ ✓

$\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta}$ (4)

$= \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$ $\begin{matrix}\div \cos^4\theta \\ \div \cos^4\theta\end{matrix}$

$= \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ ✓