HIGHER SCHOOL CERTIFICATE EXAMINATION TRIAL PAPER

1999

MATHEMATICS

3/4 UNIT (COMMON)

Time Allowed – Two Hours
(Plus 5 minutes' reading time)

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Directions to Candidates
- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Board-approved calculators may be used.

YEAR 12 – TRIAL 1999 – 3 UNIT

QUESTION 1

MARKS

a) Given that $\log_a 2 = x$, find $\log_a (2\sqrt{a})$ in terms of x. 2

b) Evaluate $\lim\limits_{x \to 0} \dfrac{1 - \cos 2x}{x^2}$ 2

c) Find the acute angle between the lines $x + \sqrt{3}\,y = 3$ and $y = 3$ 2

d) Solve the inequation 2

$$\frac{1}{2x+1} < \frac{1}{2x-1}$$

e) Use the substitution $u = \tan 2x$ to show that 4

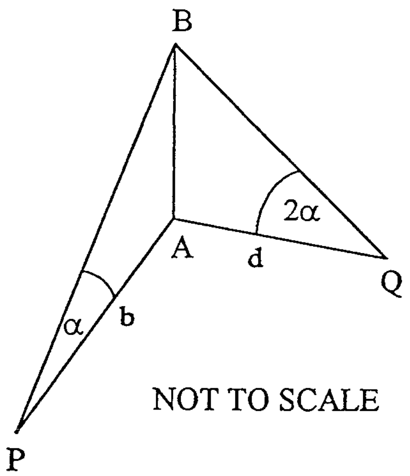
$$\int_0^{\frac{\pi}{8}} \frac{2\sec^2 2x}{\sqrt{2 - \tan^2 2x}}\,dx = \frac{\pi}{4}$$

<u>QUESTION 2</u> **MARKS**

a) Consider the equation $2x^3 + x^2 - 15x - 18 = 0$ **3**
One of the roots of this equation is positive and equals the product of the other 2 roots.
Find the roots of this equation.

b) When the polynomial P(x) is divided by $1 - x^2$ it gives $4 - x$ as a remainder. What is the remainder of P(x) when divided by $1 + x$? **2**

c)



From a point P, distant b metres south of a tower AB, the angle of elevation of the top of the tower B is α. From a point Q, distant d metres from a point due east of the tower, the angle of elevation to the top of the tower is 2α.

i) Show that $b \tan \alpha = d \tan 2\alpha$ **2**

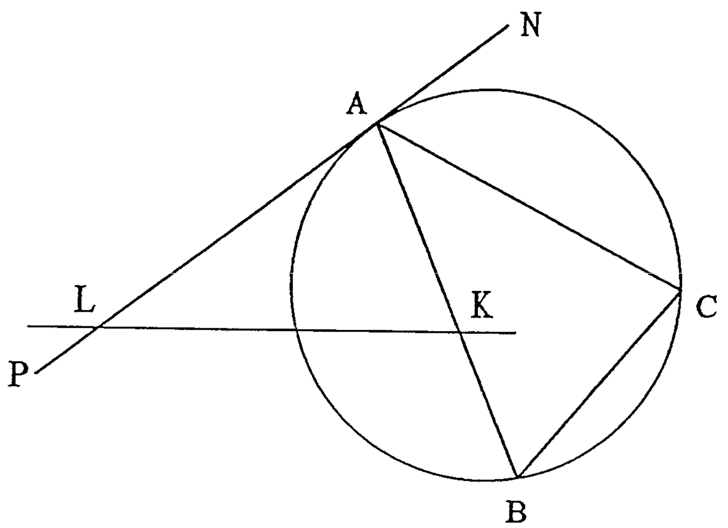
ii) Find the height of the tower in terms of d and b. **3**

iii) If the distance PQ is $d\sqrt{10}$ metres find α. **2**

QUESTION 3 | **MARKS**

a) Use the method of mathematical induction to show that $n^3 + 5n$ is divisible by 3 for all positive integers n. 3

b)



ABC is a triangle inscribed in a circle. PN is the tangent at A to the circle. From a point K on AB, the line KL is drawn to meet the tangent PN at L. Given that $\angle AKL = \angle NAC$

i) Prove that triangles ACB and AKL are similar. 2

ii) Hence, prove that $AL \times AB = KL \times AC$. 2

c) The displacement x metres of a particle moving along the x-axis is given by $x = 1 + 2\sin 3t$.

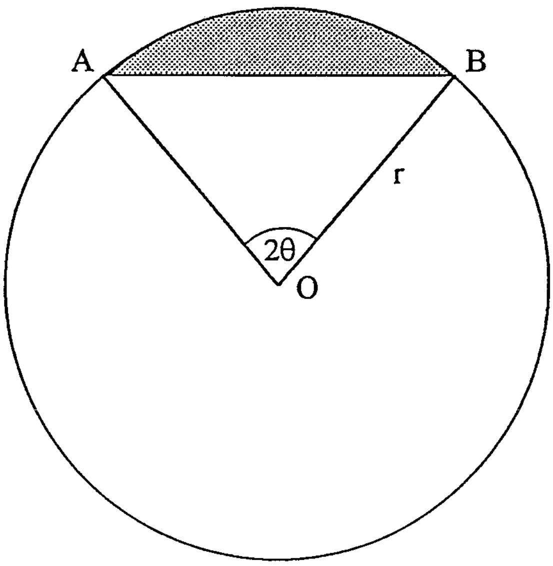
i) Show that the motion is simple harmonic. 2

ii) Find the centre of the motion. 1

iii) Find the greatest speed of the particle. 2

QUESTION 4 MARKS

a)



The diagram above shows a shaded segment which subtends an angle of 2θ radians at the centre O of a circle with radius r. Given that the perimeter of the shaded segment equals twice the diameter of the circle

i) Show that $\sin\theta = 2 - \theta$ 2

ii) Draw the graphs of $g(\theta) = \sin\theta$ and $k(\theta) = 2 - \theta$ on the same axes for $0 \le \theta \le \pi$ 1

iii) Use your graph to show that the equation $\sin\theta + \theta - 2 = 0$ has only one root near $\theta = 1.1$ 2

iv) Use two applications of Newton's method to find a better approximation of the root of the equation $\sin\theta + \theta - 2 = 0$ 3

v) Find to the nearest degree the size of $\angle$ AOB. 1

QUESTION 4 (continued) **MARKS**

b) A rugby league goal kicker finds that on the average he can kick 3 goals from every four attempts .

If in a game he has 6 kicking attempts, find the probability that

i) he can kick 5 goals. 1

ii) he can kick at least one goal. 1

iii) he can kick goals only on the first, third and fifth attempts. 1

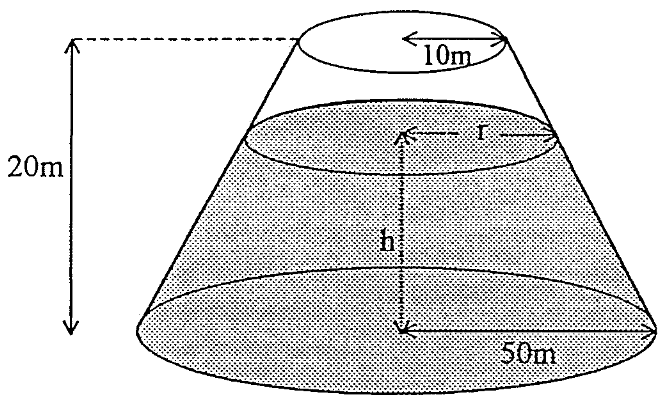
QUESTION 5 **MARKS**

a) A reservoir containing water is in the shape of a truncated right circular cone of height 20m. The radii of the top and the base circles are 10m and 50m respectively.

The water from the reservoir is being emptied at the rate of $\frac{\pi}{6}$ m^3/min. At any moment during this process, the volume of water remaining is given by

$$V = \frac{\pi}{3}\left[62500 - r^2(25 - h)\right]$$

where r is the radius of the upper surface of the water and h is its depth as shown in the diagram.



NOT TO SCALE

i) Show that the volume of the water remaining can be expressed by 2

$$V = \frac{4\pi}{3}\left[15625 - (25 - h)^3\right]$$

ii) Find the rate at which the water level is decreasing when the depth of the water is 15m. 2

iii) Find the rate at which the area of the surface of the water is increasing when the depth of the water is 10m. 2

QUESTION 5 (continued) **MARKS**

b) The velocity of a particle moving on the x axis starting at t = 0 sec from x = 1.8 is given by $V = e^{-2x}\sqrt{2x^2 - 6}$, $x \geq 1.8$ where x is the displacement of the particle from the origin.

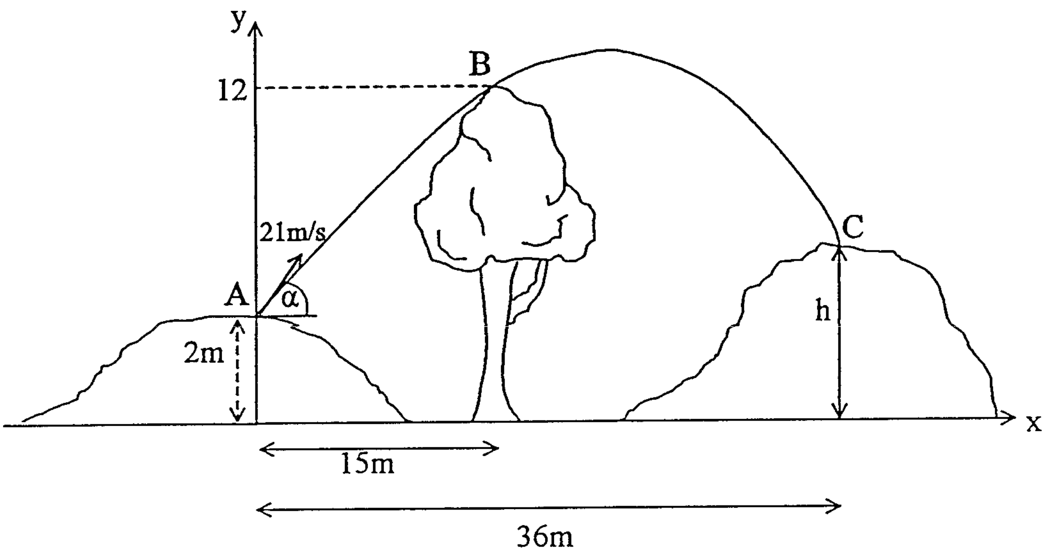
i) Show that the acceleration of the particle in terms of its displacement can be expressed by $a = -2\, e^{-4x}(2x^2 - x - 6)$ 2

ii) Hence, find the displacement of the particle at which the maximum speed occurs. 1

iii) Show that the time T in seconds taken by the particle to move from x = 2 to x = 3 can be expressed as $T = \int_2^3 \frac{e^{2x}}{\sqrt{2x^2 - 6}}\, dx$ 1

iv) Use Simpson's rule with three function values to obtain an approximate value for T. 2

QUESTION 6 **MARKS**



a) A golf ball is projected from point A on the top of elevated ground 2 metres high with a speed of 21m/s and at an angle $\alpha < 50°$ to the horizontal, aiming to reach a point C on top of a hill.
The horizontal distance separating A and C is 36metres.
In the course of its trajectory the ball just clears a point B which is the top of a tree 12 metres high and 15 metres away from A.

(Assuming there is no air resistance and $g = 9.8\text{m/s}^2$)

i) Using axes, as shown, show that the cartesian equation of the path in terms of α is 3

$$y = \frac{-x^2}{90}(1 + \tan^2\alpha) + x\tan\alpha + 2$$

ii) Find the value of the angle of projection α. 3

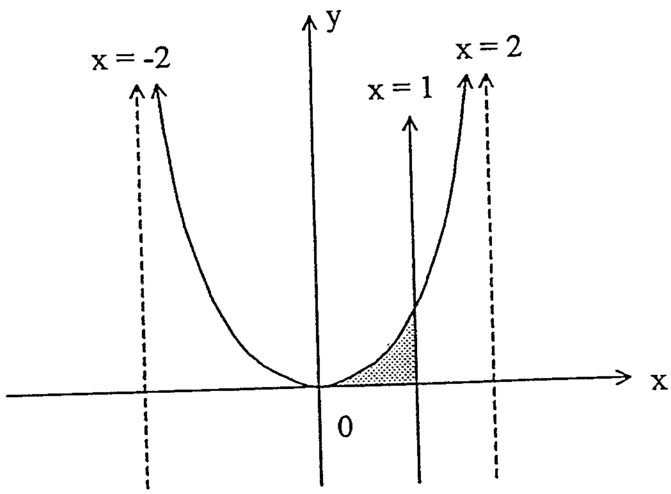
iii) Find the height in metres of the point C. 1

iv) Find the maximum height reached by the ball. 2

QUESTION 6 (continued)

MARKS

b) The shaded area in the diagram below represents the area bounded by the curve $y = \dfrac{x^2}{\sqrt{4-x^2}}$, the x axis and the line $x = 1$. Using the substitution $x = 2\sin\theta$, find the shaded area. 3



QUESTION 7 MARKS

a) i) Use the binomial theorem to obtain an expansion for $1 + (1 + 2x)^n - 2(1 + x)^n$ where n is a positive integer. 2

ii) Hence, show that

$${}^nC_2 + 3\,{}^nC_3 + 7\,{}^nC_4 + \ldots + \left(2^{n-1} - 1\right){}^nC_n = \frac{1}{2}\left(1 + 3^n - 2^{n+1}\right)$$ 2

b) Consider the function

$$g(x) = 12\cos^{-1}\left(\frac{x}{2}\right) + 4\sqrt{3}\,x - 8\sqrt{3}$$

i) What is the domain of g (x) 1

ii) Find the stationary points and determine their nature. 2

iii) Find the end points of g(x) in its domain and discuss its behaviour at those points. 2

iv) Sketch a neat graph of the curve y = g (x). 2

v) Use your graph, or otherwise, to solve the ineqaution 1

$$2 \le \sqrt{3}\cos^{-1}\left(\frac{x}{2}\right) + x \le \pi\sqrt{3} - 2$$

year 12

Trial 3-unit 1999 Solutions

Question 1

a) $\log_a(a\sqrt{2}) = \log_a a + \log_a \sqrt{2}$

$= 1 + \frac{1}{2}\log_a 2 = 1 + \frac{1}{2}x$ (2 marks)

b) $\lim_{x \to 0} \frac{1-\cos 2x}{x^2} = \lim_{x \to 0} \frac{2\sin^2 x}{x^2}$

$= 2 \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 = 2\left[\lim_{x \to 0} \frac{\sin x}{x}\right]^2$

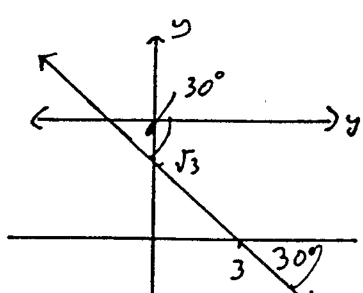
$= 2 \times 1 = 2$ (2 marks)

(c) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ $m_1 = -\frac{1}{\sqrt{3}}, m_2 = 0$

$\tan\theta = \left|\frac{-\frac{1}{\sqrt{3}} - 0}{1 + 0}\right| = \left|-\frac{1}{\sqrt{3}}\right|$

$\therefore \theta = \tan^{-1}\frac{1}{\sqrt{3}} = 30°$

Alternative method



(2 marks)

$m = -\frac{1}{\sqrt{3}} \Rightarrow \theta = -30°$ with positive x-axis

$\therefore$ acute angle $= 30°$ (corresponding angles, parallel lines)

(d) $\frac{1}{2x+1} < \frac{1}{2x-1}$ $\therefore \frac{1}{2x+1} - \frac{1}{2x-1} < 0$

$\frac{2x-1-(2x+1)}{(2x+1)(2x-1)} < 0$ $\therefore \frac{-2}{(2x+1)(2x-1)} < 0$

$\therefore \frac{2}{(2x+1)(2x-1)} > 0$

x		$-\frac{1}{2}$		$\frac{1}{2}$	
$2x+1$	$-$		$+$		$+$
$2x-1$	$-$		$-$		$+$
Result	$+$		$-$		$+$

$\therefore$ solutions are

$x < -\frac{1}{2}$, $x > \frac{1}{2}$ (2 marks)

(e) $\int_0^{\pi/8} \frac{2\sec^2 2x}{\sqrt{2 - \tan^2 2x}}\,dx$

let $u = \tan 2x$ $\frac{du}{dx} = 2\sec^2 2x$

$\therefore du = 2\sec^2 2x\,dx$

$x = 0,\ u = 0$, $x = \frac{\pi}{8}$ $u = 1$

$\therefore \int_0^1 \frac{du}{\sqrt{2-u^2}} = \left[\sin^{-1}\left(\frac{u}{\sqrt{2}}\right)\right]_0^1$

$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1} 0$

$= \frac{\pi}{4}$ (4 marks)

Question 2

(a) $2x^3 + x^2 - 15x - 18 = 0$

let the roots be $\alpha, \beta, \alpha\beta$

product of roots: $\alpha^2\beta^2 = 9$

$\therefore \alpha\beta = 3$ since $\alpha\beta$ is positive.

sum of roots: $\alpha + \beta + \alpha\beta = -\frac{1}{2}$

$\therefore \alpha + \beta = -3\frac{1}{2}$

forming a new quadratic equation

$x^2 + 3\frac{1}{2}x + 3 = 0$

$2x^2 + 7x + 6 = 0$

$(2x+3)(x+2) = 0$ $\therefore x = -\frac{1}{2}$ or $x = -2$

$\therefore$ The roots are $-2, -1\frac{1}{2}, 3$

(3 marks)

(b) $P(x) = Q(x)(1-x^2) + (4-x)$

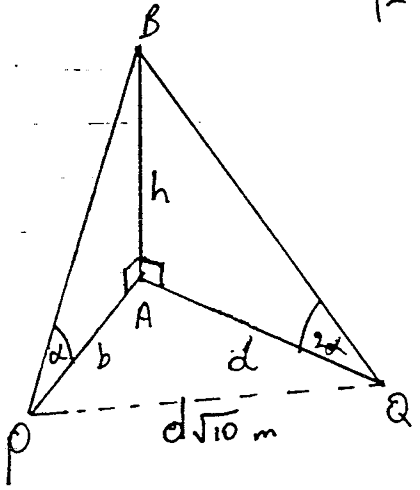
if $P(x)$ is divided by $x+1$ then by the remainder theorem, $x=-1$ gives the remainder

$\therefore P(-1) = Q(-1)(1-1) + (4-(-1))$

$P(-1) = 5$

$\therefore$ The remainder is 5 (2 marks)

(c)



1) let the height $AB = h$

using the tan ratio on ΔPAB

$\tan\alpha = \frac{h}{b} \quad \therefore h = b\tan\alpha$ ①

Using the tan ratio on ΔQAB

$\tan 2\alpha = \frac{h}{d} \quad \therefore h = d\tan 2\alpha$ ②

$h = h$

$\therefore b\tan\alpha = d\tan 2\alpha$ (2 marks)

2) $b\tan\alpha = d\tan 2\alpha$

$b\tan\alpha = d\left(\frac{2\tan\alpha}{1-\tan^2\alpha}\right)$

Since $\alpha \neq 0$ $\tan\alpha \neq 0$

$\therefore b = \frac{2d}{1-\tan^2\alpha}$

$1-\tan^2\alpha = \frac{2d}{b}$

$\tan^2\alpha = \frac{b-2d}{b}$

$\tan\alpha = \pm\sqrt{\frac{b-2d}{b}}$

Since α is acute, $\tan\alpha > 0$

$\therefore \tan\alpha = \sqrt{\frac{b-2d}{b}} \times \frac{\sqrt{b}}{\sqrt{b}}$

$\tan\alpha = \frac{\sqrt{b^2-2db}}{b}$

$\therefore b\tan\alpha = \sqrt{b^2-2db}$

but from ① $h = b\tan\alpha$

$\therefore h = \sqrt{b^2-2db}$ (3 marks)

3) using pythagoras theorem on ΔAPQ

$d^2+b^2 = (d\sqrt{10})^2 \quad \therefore 9d^2 = b^2$

$\therefore b = 3d$ (b is a positive distance)

$\therefore 3d\tan\alpha = \sqrt{9d^2-6d^2}$

$\tan\alpha = \frac{\sqrt{3d^2}}{3d} = \frac{d\sqrt{3}}{3d} = \frac{\sqrt{3}}{3}$

$\therefore \alpha = 30^\circ$ (2 marks)

Question 3

a) To prove n^3+5n is divisible by 3

Step 1: for $n=1$

$n^3+5n = 1+5 = 6$ which is true

Step 2: assume statement is true for $n=k$

ie $k^3+5k = 3m$ ① (m is a positive integers)

for $n=k+1$

$(k+1)^3+5(k+1) = k^3+3k^2+3k+1+5k+5$

$= k^3+5k+3k^3+3k+6$

from ① $k^3+5k = 3m$

$\therefore = 3m+3k^2+3k+6$

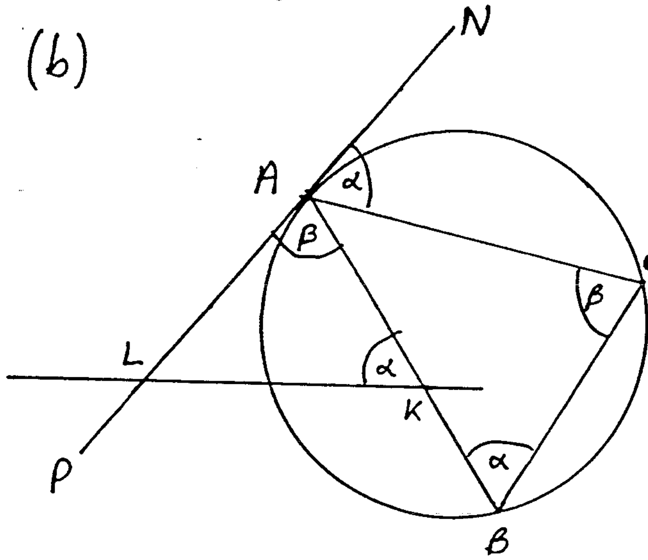
$= 3[m+k^2+k+2]$ where m, k are positive integers

$= 3H$ (H is a positive integer)

$\therefore$ the statement is true for $n=k$ and $n=k+1$.

Step 3: this statement is true for $n=1, n=2 \ldots$ and so on. Hence it is true for all n is a positive integer (3 marks)

(b)



Data: $\angle AKL = \angle NAC$

PN Tangent to the circle at A

Aim: to prove that triangles ACB and AKL are similar

Proof

(i) let $\angle NAC = \alpha$

$\therefore \angle AKL = \alpha$ (Data)

also $\angle ABC = \alpha$ (angle in alternate segment, cut by chord AB and the tangent PN)

let $\angle LAK = \beta$

$\therefore \angle ACB = \beta$ (angle in alternate segment cut by chord AC and tangent PN)

$\angle ALK = \angle BAC$ (remaining angles)

$\therefore \Delta LAK ||| \Delta ACB$ (equiangular) (2 marks)

ii) Since $\Delta LAK ||| \Delta ACB$ $\therefore$ using ratio of sides

$$\frac{AC}{LA} = \frac{BA}{KL}$$

$\therefore AC \times KL = AL \times AB$ (2 marks)

(C) i) $x = 1 + 2\sin 3t$ ①

$\dot{x} = 6\cos 3t$ ②

$\ddot{x} = -18\sin 3t = -9(2\sin 3t)$

$\ddot{x} = -9(x-1)$ ③

since the accelaration is proportional to the displacement and always directe towards the centre of motion, the motion is S.H.M. (2 marks)

ii) from ③ centre of motion is at $\ddot{x}=0$ $\therefore$ at $x = 1$ m (1 mark)

iii) squaring ②

$v^2 = 36\cos^2 3t = 36(1-\sin^2 3t)$

$V^2 = 36 - 36\sin^2 3t$

$V^2 = 36 - 9[4\sin^2 3t]$

$V^2 = 36 - 9(x-1)^2 = 9[4-(x-1)^2]$

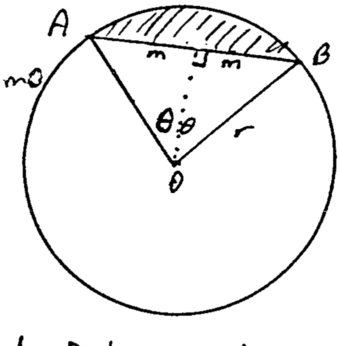
greatest speed occurs at the centre of motion $\therefore$ at $x=1$

$\therefore v^2 = 9[4-0] = 36$

$V_{max} = 6$ m/s (2 marks)

Question 4

a) i) constructing the perpendicular from O to AB. This ⊥ bisects $\angle AOB$.



$\sin\theta = \frac{m}{r}$ $m = r\sin\theta$

$\therefore AB_{chord} = 2r\sin\theta$

$AB_{arc} = 2r\theta$ and Data we have

$AB_{chord} + AB_{arc} = 2d = 4r$

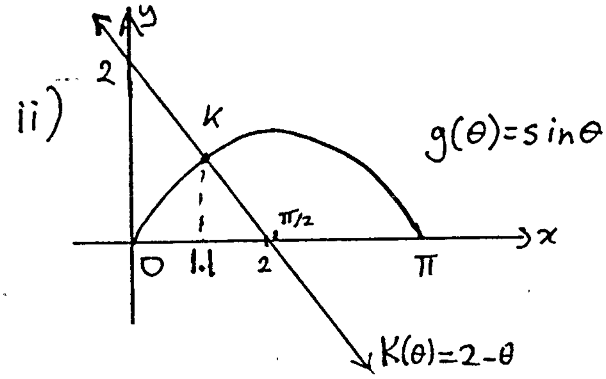
$\therefore 2r\sin\theta + 2r\theta = 4r$

$2r[\sin\theta + \theta] = 4r$

$\sin\theta + \theta = 2$

$\sin\theta = 2 - \theta$ (2 marks)

ii)



(1 mark)

iii) at point K on the graph

$\sin\theta = 2-\theta \quad \therefore \sin\theta + \theta - 2 = 0$

The solution to this equation is the point of intersection of the two curves $K(\theta)$ and $g(\theta)$

$\therefore$ from the graph $\theta \doteqdot 1\cdot1$

(2 marks)

iv) let $f(\theta) = \sin\theta + \theta - 2$

$f'(\theta) = \cos\theta + 1 \qquad \theta_1 = 1\cdot1$

By Newtons method

$$\therefore \theta_2 = \theta_1 - \frac{f(\theta_1)}{f'(\theta_1)} = 1\cdot1 - \left(\frac{\sin(1\cdot1)+1\cdot1-2}{\cos(1\cdot1)+1}\right)$$

$\theta_2 = 1\cdot106048887$

$$\theta_3 = \theta_2 - \frac{f(\theta_2)}{f(\theta_2)}$$

$$\theta_3 = 1\cdot106048887 - \left(\frac{f(1\cdot106048887)}{f'(1\cdot106048887)}\right)$$

$\theta_3 = 1.106060157$

$f(\theta_3) = -0\cdot000000001$

$\therefore \theta = 1\cdot106060157$ is a better approximation of the root.

(3 marks)

v) $\angle AOB = 2\theta = 2\cdot212120314$ rad

$\therefore \angle AOB = 127^\circ$ (nearest degree)

(1 mark)

(b) success $= \frac{3}{4}$ (goals kicked), failure $= \frac{1}{4}$ (goals missed)

6 attempts are taken

i) To kick 5 goals

$P = {}^6C_5 \left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right) = 0\cdot355957031$

(1 mark)

ii) To kick at least one goal means he will kick either 1, 2, 3, 4, 5 or 6 goals

$\therefore P = 1 - P(\text{zero success})$

$P = 1 - {}^6C_0\left(\frac{1}{4}\right)^6 = 0\cdot999755859$

(1 mark)

(iii) To have a goal on the 1^{st}, 3^{rd} and 5^{th} attempts means to take one branch of the tree diagram

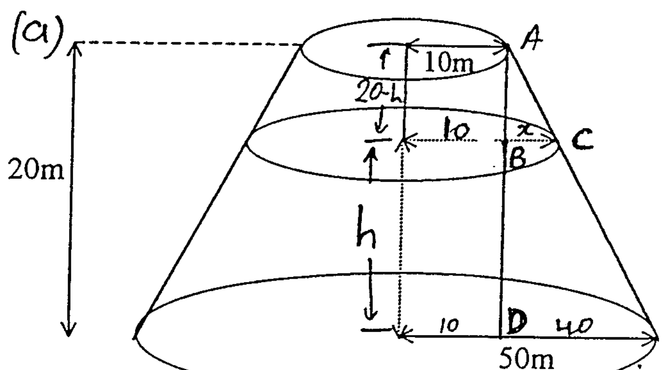
$\therefore P = \frac{3}{4}\times\frac{1}{4}\times\frac{3}{4}\times\frac{1}{4}\times\frac{3}{4}\times\frac{1}{4}$

$P = \left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^3 = 0\cdot006591796$

(1 mark)

Question 5

(a)



(i) the radius r can be written as $r = 10 + x$. using the ratio of sides in the similar Δ's ABC and ADE we get

$$\frac{20-h}{20} = \frac{x}{40}$$

$\therefore x = 2(20-h) = 40-2h$

$\therefore r = 50-2h = 2(25-h)$

sub r into V

$\therefore V = \frac{\pi}{3}\left[62500 - 4(25-h)^2(25-h)\right]$

$V = \frac{\pi}{3}\left[62500 - 4(25-h)^3\right]$

$V = \frac{4\pi}{3}\left[15625 - (25-h)^3\right]$

(2 marks)

(ii) $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$ $\quad \frac{dV}{dt} = -\frac{\pi}{6}$ m³/min

$\frac{dV}{dh} = \frac{4\pi}{3}\left[-3(25-h)^2 \times -1\right]$

$\frac{dV}{dh} = 4\pi(25-h)^2$

$\therefore \frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{dh}{dV} = \frac{-\pi}{6} \times \frac{1}{4\pi(25-h)^2}$

$\frac{dh}{dt} = \frac{-1}{24(25-h)^2}$ (which is the rate at any height)

at $h = 15$m

$\frac{dh}{dt} = \frac{-1}{24(25-15)^2} = \frac{-1}{2400}$ m/min

∴ The height is decreasing at a rate of $1/2400$ m/min at a height 15m

(2 marks)

(iii) $\frac{dA}{dt} = \frac{dA}{dh} \cdot \frac{dh}{dt}$

$A = \pi r^2 = 4\pi(25-h)^2$ since $r = 2(25-h)$

$\therefore \frac{dA}{dh} = -8\pi(25-h)$

$\therefore \frac{dA}{dt} = -8\pi(25-h) \cdot \frac{-1}{24(25-h)^2}$

$\frac{dA}{dt} = \frac{\pi}{3(25-h)}$ (the rate at any height)

at $h = 10$m

$\frac{dA}{dt} = \frac{\pi}{3(25-10)} = \frac{\pi}{45}$ m²/min

∴ The area is increasing at a rate $\frac{\pi}{45}$ m²/min at a height of 10m.

(2 marks)

(b)

i) $v = \sqrt{2x^2-6} \cdot e^{-2x}$

$v^2 = (2x^2-6) \cdot e^{-4x}$

$\frac{1}{2}v^2 = (x^2-3) \cdot e^{-4x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x \cdot e^{-4x} + (x^2-3) \cdot -4e^{-4x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = a$

$\therefore a = 2xe^{-4x} + (-4x^2+12)e^{-4x}$

$a = e^{-4x}\left[2x - 4x^2 + 12\right]$

$a = -2e^{-4x}\left[2x^2 - x - 6\right]$

(2 marks)

ii) The maximum speed occurs when $a=0$, as $e^{-4x}>0$ for all x

$\therefore 2x^2-x-6=0$

$(2x+3)(x-2)=0$

x		$-1\frac{1}{2}$		2	
a	$-$	0	$+$	0	$-$
V	↘	$-$	↗	max	↘

$x \geq 1.8$ condition

$\therefore$ The maximum speed occurs at $x=2m$ (1 mark)

(iii) $V=\sqrt{2x^2-6}\,.\,e^{-2x}$

$v=\frac{dx}{dt}$

$\therefore \frac{dx}{dt}=\sqrt{2x^2-6}\,.\,e^{-2x}$

$\frac{dt}{dx}=\frac{1}{\sqrt{2x^2-6}\,.\,e^{-2x}}=\frac{e^{2x}}{\sqrt{2x^2-6}}$

$dt=\frac{e^{2x}.dx}{\sqrt{2x^2-6}}$ integrating between $x=2$ and $x=3$

$$T=\int_2^3\frac{e^{2x}.dx}{\sqrt{2x^2-6}}$$

(1 mark)

iv) By Simpsons Rule

x	2	2.5	3
y	$e^4/\sqrt{2}$	$e^5/\sqrt{6.5}$	$e^6/\sqrt{12}$

$h=\frac{b-a}{n}=\frac{3-2}{2}=\frac{1}{2}$

$$T=\frac{1}{6}\left[\frac{e^6}{\sqrt{12}}+\frac{e^4}{\sqrt{2}}+\frac{4e^5}{\sqrt{6.5}}\right]\simeq 64.6\text{ sec}$$

or 1 min 5 sec.

(2 marks)

Question 6

a)

i) when the particle is in the air, the only force applicable is $m\vec{g}$ (downwards)

$\therefore$ By applying Newtons Law:

$m\vec{g}=m\vec{a} \quad \therefore \vec{a}=\vec{g}$

By projection we get

$\ddot{x}=0 \qquad \ddot{y}=-g=-9.8$

by integrating with respect to t

$\dot{x}=V\cos\alpha \qquad \dot{y}=-gt+V\sin\alpha$

by integrating with respect to t

$x=V\cos\alpha\, t \qquad y=-\frac{1}{2}gt^2+V\sin\alpha\, t+2$

↑ Since $t=0, y=2$

$t=\frac{x}{V\cos\alpha}$ sub into y

$$y=-\frac{1}{2}.\frac{gx^2}{V^2\cos^2\alpha}+\frac{Vx\sin\alpha}{V\cos\alpha}+2$$

$$y=x\tan\alpha-\frac{gx^2}{2V^2\cos^2\alpha}+2$$

$$y=x\tan\alpha-\frac{gx^2}{2V^2}(1+\tan^2\alpha)+2$$

$V=21\ m/s \qquad g=9.8 \qquad \frac{1}{\cos^2\alpha}=\sec^2\alpha=1+\tan^2\alpha$

$$y=x\tan\alpha-\frac{9.8}{2\times21^2}.x^2(1+\tan^2\alpha)+2$$

$$y=x\tan\alpha-\frac{x^2}{90}(1+\tan^2\alpha)+2$$

(3 marks)

ii) The ball just clears the tree

$\therefore$ the point $(15,12)$ lies on the trajectory.

sub $x=15$ and $y=12$ into the equation of path.

$$12 = \frac{-15^2}{90}(1+\tan^2\alpha) + 15\tan\alpha + 2$$

$$900 = -225 - 225\tan^2\alpha + 1350\tan\alpha$$

$$1125 = -225\tan^2\alpha + 1350\tan\alpha$$

$$225\tan^2\alpha - 1350\tan\alpha + 1125 = 0$$

$$225[\tan^2\alpha - 6\tan\alpha + 5] = 0$$

$$\therefore \tan^2\alpha - 6\tan\alpha + 5 = 0$$

$$(\tan\alpha - 1)(\tan\alpha - 5) = 0$$

$\therefore \tan\alpha = 1$ or $\tan\alpha = 5$

$\therefore \alpha = 45°$ or $\alpha = 78°41'$

but $\alpha < 50°$ (condition)

$\therefore \alpha = 45°$

(3 marks)

(iii) for $\alpha = 45°$

$$y = -\frac{x^2}{90}(1+1) + x + 2$$

$$y = -\frac{x^2}{45} + x + 2 \quad \text{(equation of path)}$$

for $x = 36$ m

$$y = -\frac{36^2}{45} + 36 + 2 = 9.2 \text{ m}$$

$\therefore C$ is 9.2 m above the ground

(1 mark)

iv) maximum height reached by the ball is when velocity vertically $= 0$ ie $\frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = \frac{-2x}{45} + 1 \quad \text{(gradient function to the curve)}$$

let $\frac{dy}{dx} = 0$

$$\therefore \frac{-2x}{45} + 1 = 0 \qquad x = \frac{45}{2} = 22\tfrac{1}{2} \text{ m}$$

x		$22\frac{1}{2}$	
$\frac{dy}{dx}$	+	0	−
y	↗	max	↘

for $x = 22\frac{1}{2}$

$$y = -\frac{(22\frac{1}{2})^2}{45} + 22\tfrac{1}{2} + 2$$

$$y = 13.25 \text{ m.}$$

$\therefore$ The maximum height reached is 13.25 m.

(2 marks)

(b) $$\text{Area} = \int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$$

let $x = 2\sin\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\frac{dx}{d\theta} = 2\cos\theta \qquad \therefore dx = 2\cos\theta\,d\theta$$

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = \sqrt{4\cos^2\theta}$$

$= |2\cos\theta|$ but since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$= 2\cos\theta$

$x = 0 \quad \theta = 0, \qquad x = 1, \ \theta = \frac{\pi}{6}$

$$\therefore \text{Area} = \int_0^{\pi/6} \frac{4\sin^2\theta \cdot 2\cos\theta\,d\theta}{2\cos\theta}$$

$$= \int_0^{\frac{\pi}{6}} 4\sin^2\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{6}} (1-\cos 2\theta)\,d\theta$$

$$= 2\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= 2\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

$$= \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ units}^2.$$

(3 marks)

Question 7

(a)(i) $1+(1+2x)^n-2(1+x)^n$

$(1+2x)^n = {}^nC_0 + {}^nC_1(2x) + {}^nC_2(2x)^2 + \dots + {}^nC_n(2x)^n$

$-2(1+x)^n = -2[{}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots {}^nC_n x^n]$

$= -2\,{}^nC_0 - 2\,{}^nC_1 x - 2\,{}^nC_2 x^2 - 2\,{}^nC_3 x^3 \dots - 2\,{}^nC_n x^n$

$\therefore 1+(1+2x)^n-2(1+x)^n = 1 + {}^nC_0 + {}^nC_1(2x) + {}^nC_2(2x)^2 + \dots {}^nC_n(2x)^n - 2\,{}^nC_0 - 2\,{}^nC_1 x - 2\,{}^nC_2 x^2 - 2\,{}^nC_3 x^3 \dots - 2\,{}^nC_n x^n$

${}^nC_0 = 1$

$\therefore 1+(1+2x)^n-2(1+x)^n = (2^2-2)\,{}^nC_2 x^2 + (2^3-2)\,{}^nC_3 x^3 + (2^4-2)\,{}^nC_4 x^4 + \dots + (2^n-2)\,{}^nC_n x^n$

$= 2\,{}^nC_2 x^2 + 6\,{}^nC_3 x^3 + 14\,{}^nC_4 x^4 + \dots + (2^n-2)\,{}^nC_n x^n.$

(2 marks)

ii) let $x=1$

$1+(1+2)^n - 2(1+1)^n = 2\,{}^nC_2 + 6\,{}^nC_3 + 14\,{}^nC_4 + \dots + (2^n-2)\,{}^nC_n$

$1+3^n - 2.2^n = 2[{}^nC_2 + 3\,{}^nC_3 + 7\,{}^nC_4 + \dots (2^{n-1}-1)\,{}^nC_n]$

$\frac{1}{2}(1+3^n-2^{n+1}) = {}^nC_2 + 3\,{}^nC_3 + 7\,{}^nC_4 + \dots (2^{n-1}-1)\,{}^nC_n.$

(2 marks)

(b) $g(x) = 12\cos^{-1}\left(\frac{x}{2}\right) + 4\sqrt{3}x - 8\sqrt{3}$

(i) The domain of $g(x)$ is when $-1 \le \frac{x}{2} \le 1$.

$\therefore D: -2 \le x \le 2$

(1 mark)

(ii) $g'(x) = -\frac{12}{\sqrt{4-x^2}} + 4\sqrt{3}$ (gradient function of the curve)

let $g'(x)=0$ to find the stationary points

$\therefore -\frac{12}{\sqrt{4-x^2}} + 4\sqrt{3} = 0$

$\frac{12}{\sqrt{4-x^2}} = 4\sqrt{3}$

$\frac{12}{\sqrt{4-x^2}} = \frac{12}{\sqrt{3}}$

$\therefore 4-x^2 = 3$

$x^2 = 1$

$x = \pm 1$

x	-2		-1		1		2
$g'(x)$	$-\infty$	$-$	0	$+$	0	$-$	$-\infty$
$g(x)$	$12\pi-16\sqrt{3}$	↘	$8\pi-12\sqrt{3}$	↗	$4\pi-4\sqrt{3}$	↘	0

$\therefore (-1, 8\pi-12\sqrt{3})$ is a minimum turning point

$(1, 4\pi-4\sqrt{3})$ is a maximum turning point.

(2 marks)

(iii) at $x=-2$ at $x=2$

$g(-2) = 12\pi - 16\sqrt{3}$ $g(2) = 0$

also

as $x \to -2$ as $x \to 2$

$g'(-2) \to -\infty$ $g'(2) \to -\infty$

$\therefore$ at the endpoints $(-2, 12\pi - 16\sqrt{3})$ and $(2, 0)$ there are vertical tangents to these points

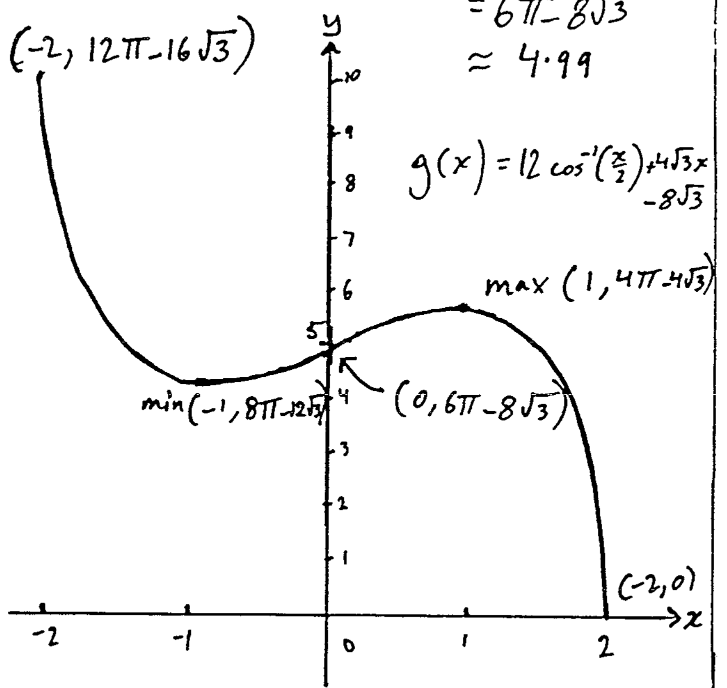
(2 marks)

(iv) at $x=0$ $g(x) = 12 \times \frac{\pi}{2} - 8\sqrt{3}$

$= 6\pi - 8\sqrt{3}$

$\approx 4 \cdot 99$

$g(x) = 12\cos^{-1}\left(\frac{x}{2}\right) + 4\sqrt{3}x - 8\sqrt{3}$



(2 marks)

(V) from the graph

$0 \leq g(x) \leq 12\pi - 16\sqrt{3}$

$0 \leq 12\cos^{-1}\left(\frac{x}{2}\right) + 4\sqrt{3}x - 8\sqrt{3} \leq 12\pi - 16\sqrt{3}$

$8\sqrt{3} \leq 12\cos^{-1}\left(\frac{x}{2}\right) + 4\sqrt{3}x \leq 12\pi - 8\sqrt{3}$

$\div$ by $4\sqrt{3}$

$2 \leq \sqrt{3}\cos^{-1}\left(\frac{x}{2}\right) + x \leq \sqrt{3}\pi - 2$

(1 mark)