SYDNEY GIRLS HIGH SCHOOL

2006 HSC Assessment Task 2

March 15, 2006

MATHEMATICS Extension 1

Year 12

Time allowed: 75 minutes

Topics: Trigonometry I & II, Polynomials

DIRECTIONS TO CANDIDATES:

- Attempt all questions
- Questions are of equal value
- There are 4 questions with part marks shown in brackets
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used
- Write on one side of the paper only

Question 1.

a) Find $\frac{dy}{dx}$ in the following

i) $y = 2\sin(3x-1)$

ii) $y = \sin^2 3x$

iii) $y = e^{-x}\cos x$

iv) $y = \log_e(\cos x)$

v) $y = \frac{\tan x}{x}$ [8]

b) Find the following integrals

i) $\int \sin 2x.dx$

ii) $\int \cos(2-3x).dx$

iii) $\int \sec^2(\frac{x}{2}).dx$

iv) $\int \frac{\sin 3x}{2-\cos 3x}.dx$

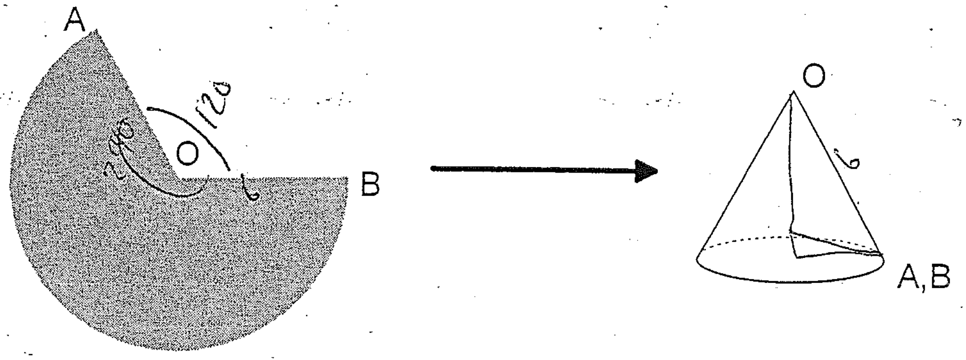
v) $\int \cot x.dx$ [8]

c) Find $\frac{d}{dx}\{\tan^2 x\}$ and hence evaluate $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x.dx$ [2]

d) Evaluate $\int_0^{\frac{\pi}{4}} \cos^2 x.dx$ [2]

Question 2:

a) A circular piece of metal of radius 6 cm is cut to leave a sector with angle 240^0 a the centre.



Find :

i) The area of the sector [2]

ii) The arc length of the sector [2]

iii) The base radius of a cone that would be formed if the sector was bent so that OA and OB were joined. [2]

b) Evaluate:

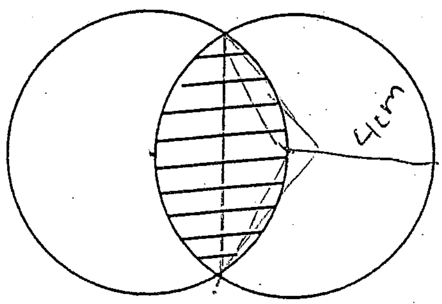
i) $\lim_{x \to 0}\left\{\frac{\sin 2x}{3x}\right\}$ ii) $\lim_{x \to 0}\left\{\frac{\sin^2 2x}{x^2}\right\}$ [4]

c) Sketch in the range $0 \le x \le 2\pi$

i) $y = 1 - \cos x$ [3]

ii) $y = 2\cos\left(x + \frac{\pi}{4}\right)$ [3]

d) Two circles of radius 4cm are drawn so that the centre of one is on the circumference of the other. Find to 3 significant figures, the common area of the two circles



[4]

Question 3:

a) i) Express $\sqrt{3}\sin x + \cos x$ in the form $R\sin(x+\alpha)$ [2]

ii) Hence or otherwise solve the equation:

$\sqrt{3}\sin x + \cos x = 1$ for $0 \le x \le 2\pi$ [2]

b) Find the acute angle between the lines $2x + y = 4$ and $3x - 2y = 6$. [3]

c) Prove the expression $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$ [3]

d) If $t = \tan\frac{\theta}{2}$,

i. Show how to obtain $\cos\theta$ and $\sin\theta$ in terms of t. [2]

ii. Express $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}$ in terms of t in the simplest form. [2]

e) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and hence or otherwise evaluate

$\int_0^{\frac{\pi}{6}} \cos^3\theta . d\theta$ [3]

f) Express 1+ cos x in terms of $\cos\frac{x}{2}$ and use the result to evaluate

$\int_0^{\frac{\pi}{3}} \dfrac{dx}{1+\cos x}$

[3]

Question 4:

a) Find the remainder when $P(x) = 2x^3 - 3x^2 + 4x - 5$ is divided by $(x - 2)$ [1]

b) Find the value of k, if $P(x) = 2x^3 + 5x^2 + kx - 6$ has a remainder of 4 when divided by (x+2) [2]

c) Show that (x-2) is a factor of $P(x) = x^3 + x^2 - 2x - 8$ and explain why there are no other factors [2]

d) Fully factorise the expression: $P(x) = 8x^3 + 3x^2 - 8x - 3$ [2]

e) If the polynomial $P(x) = x^3 + kx + 54$ has two equal roots, find the value of k, and all of the solutions to P(x) = 0 [2]

f) If the roots of $8x^3 + 4x^2 - 2x - 1 = 0$ are α, β, γ, find the value of:

i) $\alpha + \beta + \gamma$, ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$, iii) $\alpha^2 + \beta^2 + \gamma^2$ [4]

g) Solve $24x^3 - 14x^2 - 7x + 3 = 0$ given that its roots are in geometric progression [3]

h) The polynomial $3x^3 + 4x^2 - 5x - 1 = 0$ has roots: α, β, γ. Find the equation of the polynomial with roots:

i) $2\alpha, 2\beta, 2\gamma$ ii) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ [4]

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1 a) i) $y = 2\sin(3x-1)$

$\frac{dy}{dx} = 6\cos(3x-1)$

ii) $y = \sin^2 3x$

$\frac{dy}{dx} = 2.3\cos 3x \sin 3x$

$= 6\cos 3x \sin 3x$

[OR $3\sin 6x$]

iii) $y = e^{-x}\cos x$

$\frac{dy}{dx} = e^{-x}(-\sin x) + \cos x(-e^{-x})$

$= e^{-x}(-\sin x - \cos x)$

$= -e^{-x}(\sin x + \cos x)$

iv) $y = \log_e(\cos x)$

$\frac{dy}{dx} = \frac{-\sin x}{\cos x}$

$= -\tan x$

v) $y = \frac{\tan x}{x}$

$\frac{dy}{dx} = \frac{x.\sec^2 x - \tan x.1}{x^2}$

$= \frac{x\sec^2 x - \tan x}{x^2}$

b) i) $\int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$

ii) $\int \cos(2-3x)\,dx = \frac{-1}{3}\sin(2-3x) + C$

iii) $\int \sec^2\left(\frac{x}{2}\right)dx = 2\tan\frac{x}{2} + C$

iv) $\int \frac{\sin 3x}{2-\cos 3x} = \frac{\ln(2-\cos 3x)}{3} + C$

OR $= \frac{1}{3}\ln(2-\cos 3x) + C$

v) $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$

$= \ln(\sin x) + C$

c) $\frac{d}{dx}(\tan^2 x)$

$= 2\sec^2 x \tan x$.

$\therefore \int_0^{\pi/4} \tan x \sec^2 x\,dx = \frac{1}{2}\Big[\tan^2 x\Big]_0^{\pi/4}$

$= \frac{1}{2}(1-0)$

$= \frac{1}{2}$

d) $\int_0^{\pi/4} \cos^2 x\,dx = \frac{1}{2}\int_0^{\pi/4}(1+\cos 2x)\,dx$

$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/4}$

$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}.1\right] - 0$

$= \frac{\pi}{8} + \frac{1}{4}$ OR $\frac{1}{8}(\pi+2)$

2 a) i) $A = \frac{1}{2}r^2\theta$

$240° = 4\pi/3$

$\therefore A = \frac{1}{2} \times 36 \times \frac{4\pi}{3}$

$= 24\pi$

$\therefore$ area $= 24\pi\ cm^2$

ii) $l = r\theta$

$= 8 \times 4\pi/3$

$= 8\pi$

$\therefore$ length $= 8\pi$ cm.

iii) If 8π cm is the base radius of a cone

$2\pi r = 8\pi$

$\therefore r = 4$

i.e. radius $= 4$ cm.

b) $\lim_{x\to 0}\left\{\frac{\sin 2x}{3x}\right\} \to \frac{2x}{3x} = \frac{2}{3}$

($\sin 2x \doteqdot 2x$ as $x \to 0$)

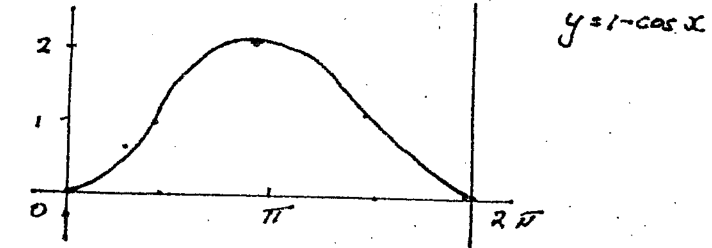
$\lim_{x\to 0}\left\{\frac{\sin^2 2x}{x^2}\right\} = \lim_{x\to 0}\left\{\frac{\sin 2x}{x}\cdot\frac{\sin 2x}{x}\right\}$

as $x \to 0$ $\sin 2x \to 2x$

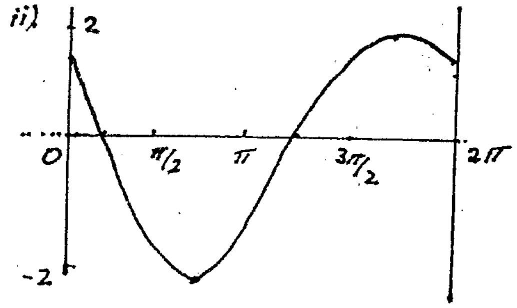
$\therefore \lim_{x\to 0}\left(\frac{\sin^2 2x}{x^2}\right) \to \frac{2x}{x}\cdot\frac{2x}{x}$

$= 4$

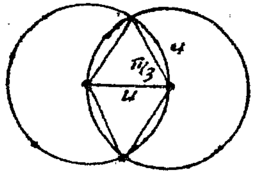
2c) i)



ii)



d)



$A_\Delta = \frac{1}{2}ab\sin C$

$A_{seg} = \frac{1}{2}r^2(\theta - \sin\theta)$

$A = 2\ \Delta's + 4$ segments

$= 2 \times \frac{1}{2}.4.4.\frac{\sqrt{3}}{2} + 4 \times \frac{1}{2}.16\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= 8\sqrt{3} + 32\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= \frac{32\pi}{3} - 8\sqrt{3}$

Area $\doteqdot 19.7\ cm^2$

5a) $\sqrt{3}\sin x + \cos x = R\sin(x+\alpha)$

$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

$\therefore R\cos\alpha = \sqrt{3}, \quad R\sin\alpha = 1$

$\therefore R^2 = 3+1, \quad R = 2$

$\tan\alpha = \frac{1}{\sqrt{3}} \quad \alpha = \frac{\pi}{6}$

$\therefore \sqrt{3}\sin x + \cos x = 2\sin(x + \pi/6)$

ii) $2\sin(x + \frac{\pi}{6}) = 1$

$\therefore \sin(x + \pi/6) = \frac{1}{2}$

$\therefore x + \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$

& $x = 0, 2\pi/3, 2\pi$

b) $2x + y = 4, \quad y = -2x + 4$ & $m_1 = -2$

$3x - 2y = 6 \quad y = \frac{3}{2}x - 3$, & $m_2 = 3/2$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{-2 - 3/2}{1 - 2(3/2)}\right|$

$= \left|\frac{-3\frac{1}{2}}{-2}\right|$

$= 7/4$

$\therefore \theta \doteqdot 60°15'$

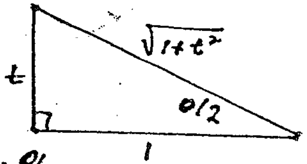
c) $\frac{\sin 2x}{1 + \cos 2x} = \tan x.$

LHS $= \frac{2\sin x\cos x}{1 + \cos^2 x - \sin^2 x}$

$= \frac{2\sin x\cos x}{\cos^2 x + \cos^2 x}$

$= \frac{2\sin x\cos x}{2\cos^2 x} = \tan x =$ RHS

d) $t = \tan\theta/2$



$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\theta/2$

$= \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$

$= \frac{1-t^2}{1+t^2}$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= 2 . \frac{t}{\sqrt{1+t^2}} . \frac{1}{\sqrt{1+t^2}}$

$= \frac{2t}{1+t^2}$

ii) $\frac{1+\sin\theta - \cos\theta}{1+\sin\theta+\cos\theta} = \frac{1 + \frac{2t}{1+t^2} - \left(\frac{1-t^2}{1+t^2}\right)}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}$

$= \frac{1+t^2+2t-1+t^2}{1+t^2+2t+1-t^2}$

$= \frac{2t+2t^2}{2+2t}$

$= \frac{2t(1+t)}{2(1+t)}$

$= t.$

e) $\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= \cos\theta(\cos^2\theta - \sin^2\theta) - 2\sin^2\theta\cos\theta$

$= \cos^3\theta - \cos\theta.\sin^2\theta - 2\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\therefore 4\cos^3\theta = 3\cos\theta + \cos 3\theta$

$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$

$\therefore \int_0^{\pi/6} \cos^3\theta \, d\theta = \left[\frac{3}{4}\sin\theta + \frac{1}{12}\sin 3\theta\right]_0^{\pi/6}$

$= \left[\frac{3}{4}\sin\frac{\pi}{6} + \frac{1}{12}\sin\frac{\pi}{2}\right]$

$= \frac{3}{4}\times\frac{1}{2} + \frac{1}{12}\times 1$

$= \frac{3}{8} + \frac{1}{12}$

$= \frac{11}{24}$

f) $1+\cos x = 1 + \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$= \cos^2\frac{x}{2} + \cos^2 x/2$

$= 2\cos^2 x/2$

$\therefore \int_0^{\pi/3} \frac{dx}{1+\cos x} = \int_0^{\pi/3} \frac{dx}{2\cos^2 x/2}$

$= \int_0^{\pi/3} \frac{1}{2}\sec^2\frac{x}{2}\, dx$

$= \left[\tan\frac{x}{2}\right]_0^{\pi/3} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.$

Q4. a) $P(x) = 2x^3 - 3x^2 + 4x - 5$

$P(2) = 16 - 12 + 8 - 5$

$= 24 - 17$

$= 7$

b) $P(x) = 2x^3 + 5x^2 + kx - 6$

$P(-2) = 4$

$\therefore 4 = -16 + 20 - 2k - 6$

$4 = -2k - 2$

$2k = -6, \quad k = -3$

c) $P(x) = x^3 + x^2 - 2x - 8$

$P(2) = 8 + 4 - 4 - 8$

$= 12 - 12 = 0$

$\therefore (x-2)$ is a factor.

$\therefore P(x) = (x-2)(x^2+3x+4)$

For x^2+3x+4 $\quad b^2-4ac = 9-16$

$= -7$

< 0

$\therefore$ there are no other factors.

d) $P(x) = 8x^3 + 3x^2 - 8x - 3$

$P(1) = 8 + 3 - 8 - 3 = 0$

$\therefore P(x) = (x-1)(8x^2+11x+3)$

$= (x-1)(x+1)(8x+3)$

e) $P(x) = x^3 + kx + 54$

Let the roots be α, α, β

$2\alpha + \beta = 0 \Rightarrow \beta = -2\alpha$

$\alpha^2\beta = -54$

$\therefore -2\alpha^3 = -54$

$\alpha^3 = 27$

$\alpha = 3$

$\therefore P(3) = 27 + 3k + 54 = 0$

$3k = -81$

$k = -27$

$\therefore P(x) = x^3 - 27x + 54$

& Roots are $3, 3, -6$

f) $8x^3 + 4x^2 - 2x - 1 = 0$

i) $\alpha+\beta+\gamma = -b/a = -1/2$

ii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\ldots}{\alpha\beta\gamma}$

$= -\frac{1}{4} \div \frac{1}{8} = -2$

iii) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2\Sigma\alpha\beta$

$= \frac{1}{4} - 2(-\frac{1}{4})$

$= 3/4$

g) $24x^3 - 14x^2 - 7x + 3 = 0$

Let roots be $\frac{a}{r}, a, ar$

Product. $a^3 = \frac{-3}{24}$, $\therefore a = -\frac{1}{2}$

$\therefore P(x) = (2x+1)(12x^2 - 13x + 3)$

$= (2x+1)(4x-3)(3x-1)$

& If $P(x) = 0$, $x = \frac{1}{3}, -\frac{1}{2}, \frac{3}{4}$

h) $3x^3 + 4x^2 - 5x - 1 = 0$ has roots α, β, γ

i.e. P'nomial is $(x-\alpha)(x-\beta)(x-\gamma) = 0$

i) If roots are $2\alpha, 2\beta, 2\gamma$

then $(x-2\alpha)(x-2\beta)(x-2\gamma) = 0$

$\therefore (\frac{x}{2}-\alpha)(\frac{x}{2}-\beta)(\frac{x}{2}-\gamma) = 0$

$\Rightarrow 3(\frac{x}{2})^3 + 4(\frac{x}{2})^2 - 5(\frac{x}{2}) - 1 = 0$

$\therefore 3x^3 + 8x^2 - 20x - 8 = 0$

ii) If roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ then

$(x-\frac{1}{\alpha})(x-\frac{1}{\beta})(x-\frac{1}{\gamma}) = 0$

OR $(\frac{1}{x}-\alpha)(\frac{1}{x}-\beta)(\frac{1}{x}-\gamma) = 0$

$\therefore 3(\frac{1}{x})^3 + 4(\frac{1}{x})^2 - 5(\frac{1}{x}) - 1 = 0$

$\therefore 3 + 4x - 5x^2 - x^3 = 0$

OR $x^3 + 5x^2 - 4x - 3 = 0$