SYDNEY GRAMMAR SCHOOL HALF-YEARLY EXAMINATION 2004

FORM VI MATHEMATICS EXTENSION 1

Time allowed: 2 hours

Exam date: 19th May 2004

Instructions

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the right margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

The writing booklets will be collected in one bundle.
Start each question in a new writing booklet.
If you use a second booklet for a question, place it inside the first. <u>Don't staple</u>.
Write your candidate number on each booklet.

Checklist

SGS writing booklets required — 7 booklets per boy.
Candidature: 121 boys.

Examiner

MLS

QUESTION ONE (14 marks) Use a separate writing booklet. **Marks**

(a) Find the remainder when $3x^4 - 4x^3 + 4x - 8$ is divided by $x - 2$. $\boxed{1}$

(b) Find the real zeroes of the polynomial $P(x) = x^4 - 5x^2 - 36$. $\boxed{2}$

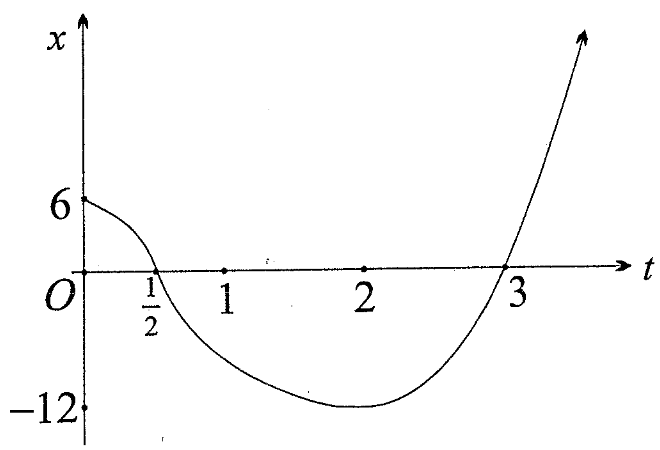
(c) Solve $\tan 2\theta = \sqrt{3}$, for $0 \le \theta \le \pi$. $\boxed{3}$

(d) (i) Write $\cos 2x$ in terms of $\sin^2 x$. $\boxed{1}$

(ii) Hence find $\displaystyle\int \sin^2 x \, dx$. $\boxed{2}$

(e) Find the exact value of $\displaystyle\int_0^2 \frac{1}{4 + x^2} \, dx$. $\boxed{2}$

(f)



The displacement of a particle moving along a horizontal line is described by the diagram above. The point $(\frac{1}{2}, 0)$ is the only point of inflection and there is a turning point at $(2, -12)$. The displacement x is in metres and the time t is in seconds.

(i) When is the particle stationary? $\boxed{1}$

(ii) What is the total distance travelled in the first 3 seconds? $\boxed{1}$

(iii) When is the acceleration of the particle positive? $\boxed{1}$

Exam continues next page ...

QUESTION TWO (14 marks) Use a separate writing booklet. **Marks**

(a) Consider the polynomial

$$P(x) = x^3 - 2x^2 - 5x + 6.$$

(i) Show that 1 is a zero of $P(x)$. $\boxed{1}$

(ii) Express $P(x)$ as a product of three factors. $\boxed{3}$

(iii) Sketch the graph of $y = P(x)$. Show clearly all the intercepts with axes. Do not calculate the coordinates of the turning points. $\boxed{1}$

(iv) Solve the inequality $P(x) \leq 0$. $\boxed{1}$

(b) The displacement of a particle moving in simple harmonic motion is given by

$$x = a \cos nt,$$

where x is the displacement in metres from the origin and t is the time in seconds.

(i) Write down expressions for the velocity and the acceleration of this particle in terms of t. $\boxed{2}$

(ii) Suppose now that the initial acceleration is $-12\,\text{m/s}^2$ and the initial displacement is 4 metres.

(α) Find the values of a and n. $\boxed{3}$

(β) Write down the period of the motion. $\boxed{1}$

(γ) In what interval is the particle confined? $\boxed{1}$

(δ) Find the maximum speed of the particle. $\boxed{1}$

Exam continues overleaf ...

QUESTION THREE (14 marks) Use a separate writing booklet. **Marks**

(a) Let the roots of the equation $x^3 + 2x^2 - 3x + 5 = 0$ be α, β and γ.

(i) State the values of:

(α) $\alpha + \beta + \gamma$ $\boxed{1}$

(β) $\alpha\beta + \alpha\gamma + \beta\gamma$ $\boxed{1}$

(γ) $\alpha\beta\gamma$ $\boxed{1}$

(ii) Hence find the values of:

(α) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ $\boxed{1}$

(β) $(\alpha - 1)(\beta - 1)(\gamma - 1)$ $\boxed{2}$

(b) When the polynomials $\boxed{4}$

$$f(x) = x^4 + 5x^3 - ax + b \quad \text{and} \quad g(x) = ax^2 + bx - 1$$

are each divided by $x + 1$, the remainders are 7 and -6 respectively.

Find the values of a and b.

(c) Use the substitution $t = \tan\frac{\theta}{2}$ to find the general solution in degrees to the equation $\boxed{4}$

$$6\sin\theta - 5\cos\theta = 5.$$

Give your solution correct to the nearest degree.

Exam continues next page ...

QUESTION FOUR (14 marks) Use a separate writing booklet. **Marks**

(a) Suppose that $\sin\alpha = \frac{3}{5}$ where $0 < \alpha < \frac{\pi}{2}$, and $\sin\beta = \frac{5}{13}$ where $\frac{\pi}{2} < \beta < \pi$. Find the exact values of:

(i) $\tan\alpha$ **1**

(ii) $\tan\beta$ **1**

(iii) $\tan(\alpha + \beta)$ **1**

(b) Let $f(x) = \cos(\log_e x)$.

(i) Find $f'(x)$. **2**

(ii) Is the function increasing or decreasing at $x = e^{\frac{\pi}{2}}$? Give a reason for your answer. **2**

(c) Consider the equation $y = x\log_e x - x$.

(i) State the domain of the function. **1**

(ii) Find the first and second derivatives of the function. **2**

(iii) Explain why the graph of this function is concave up for all x in its domain. **1**

(iv) Solve the equation $x\log_e x - x = 0$. **1**

(v) By sketching, or otherwise, solve $x\log_e x - x < 0$. **2**

QUESTION FIVE (14 marks) Use a separate writing booklet. **Marks**

(a) Consider the function $y = \cos^{-1}(2x)$.

(i) Write down the domain of the function. **1**

(ii) Draw a neat sketch of the function, showing the coordinates of its endpoints. **2**

(iii) Find the gradient of the tangent to $y = \cos^{-1}(2x)$ at the point $\left(\frac{\sqrt{2}}{4}, \frac{\pi}{4}\right)$. **2**

(iv) Find the equation of the inverse function, stating its domain. **2**

(v) Find the area of the region in the first quadrant bounded by $y = \cos^{-1}(2x)$ and the coordinate axes. **2**

(b) The acceleration of a particle P moving in a straight line is given by $\ddot{x} = 2x(4 + x^2)$, where x is the displacement in metres from the origin at time t seconds. Initially the particle is at the origin and its velocity is 4 m/s.

(i) Using $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$, show that $v^2 = (4 + x^2)^2$, where v is the velocity of P. **1**

(ii) Explain why the velocity can never be negative. **1**

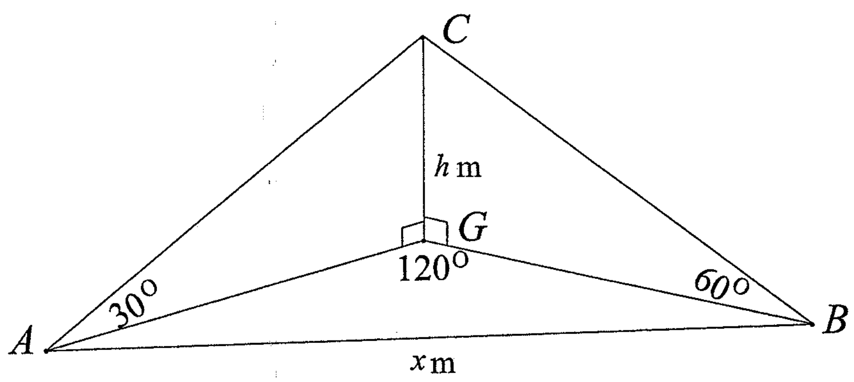
(iii) Find x as a function of t. **2**

(iv) When is the particle distant 2 metres from the origin? **1**

Exam continues overleaf ...

QUESTION SIX (14 marks) Use a separate writing booklet. **Marks**

(a)



In the diagram above, Crystabelle is standing at her window C, which is h metres above a point G on the ground. Two boys, Andrew and Bart, are standing in the garden below the window at the points A and B. The interval AB subtends an angle of $120°$ at G. From Andrew, the angle of elevation of Crystabelle is $30°$. From Bart, the angle of elevation of Crystabelle is $60°$. The distance between the boys is x metres.

(i) Show that the distance AG is $h\sqrt{3}$ metres and find a similar expression for the distance BG. **1**

(ii) Show that $3x^2 = 13h^2$. **2**

(b) Consider the curve $y = x^3 - 4x$.

(i) Show that the equation of the tangent to the curve at the point $P(p, p^3 - 4p)$ is **1**

$$y = (3p^2 - 4)x - 2p^3.$$

(ii) This tangent cuts the curve again at the point R. Explain why the x-coordinate of R is one of the roots of the equation $x^3 - 3p^2x + 2p^3 = 0$. **1**

(iii) Hence or otherwise find the coordinates of R. **2**

(c) A particle moves so that it satisfies the equation

$$\frac{d^2x}{dt^2} + 9x = 0.$$

(i) Show that $x = C\cos(3t + \alpha)$ is a solution to this equation, where C and α are constants with $C > 0$ and $0 \le \alpha < 2\pi$. **1**

(ii) Initially the particle is 2 metres on the positive side of the origin and has a velocity of -6 metres per second. Find the position of this particle when t is π seconds. **3**

(d) Two of the zeroes of the cubic

$$P(x) = x^3 + px^2 + qx + r$$

are equal in magnitude but opposite in sign.

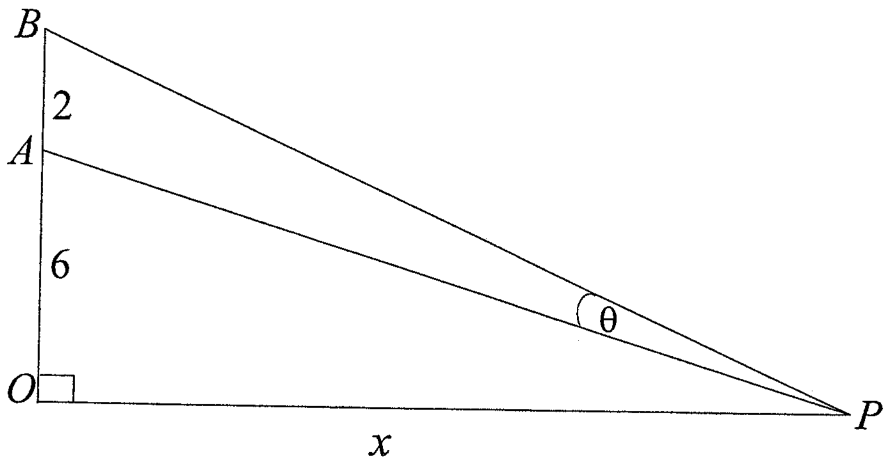
(i) Show that $x = -p$ is the third zero. **1**

(ii) Show that $r = pq$. **2**

Exam continues next page ...

QUESTION SEVEN (14 marks) Use a separate writing booklet. **Marks**

(a)



In the diagram above, $\triangle BOP$ has a right angle at O, OA is 6 units, AB is 2 units, OP is x units and $\angle BPA$ is θ.

(i) Show that $\theta = \tan^{-1}\dfrac{8}{x} - \tan^{-1}\dfrac{6}{x}$. **1**

(ii) Show that θ is a maximum when $x = 4\sqrt{3}$. **3**

(iii) Deduce that the maximum size of $\angle BPA$ is $\theta = \tan^{-1}\dfrac{1}{4\sqrt{3}}$. **1**

QUESTION SEVEN CONTINUES ON THE NEXT PAGE

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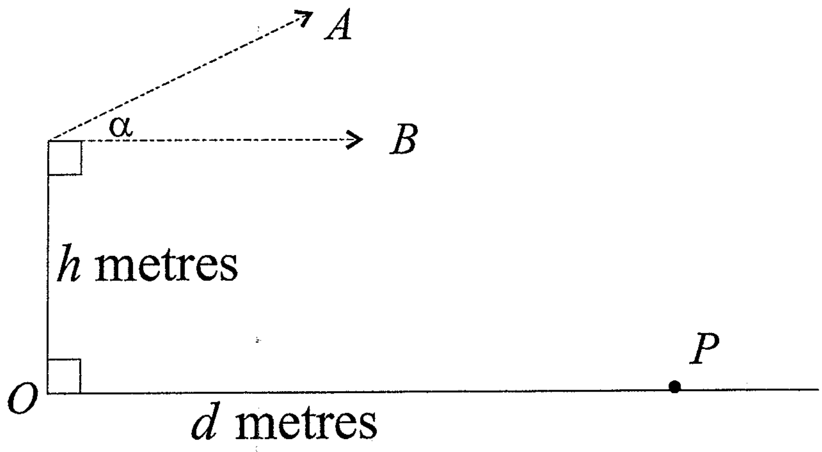
(b)



In the diagram above, two projectiles are fired simultaneously from the top of a hill that is h metres high. Projectile A is fired at an angle of α to the horizontal and projectile B is fired horizontally. Both have an initial velocity V m/s.

The equations of motion of both projectiles are $\ddot{x} = 0$ and $\ddot{y} = -g$, and O is the origin of motion. The trajectories of both projectiles lie in the same vertical plane.

(i) Using calculus, and beginning with the acceleration equations, show that the position of A at time t is given by **2**

$$x = Vt\cos\alpha$$

$$y = -\tfrac{1}{2}gt^2 + Vt\sin\alpha + h.$$

(ii) Hence show that the trajectory of A is given by **1**

$$y = h - \frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha.$$

(iii) Use the results from part (ii) to show that the trajectory of B is given by **1**

$$y = h - \frac{gx^2}{2V^2}.$$

(iv) Show that if both projectiles fall to the ground at a point P that lies d metres from O, then **3**

$$\tan\alpha = \frac{d}{h}.$$

(v) Suppose now that B lands at P, but A goes further to land at a point $3d$ metres from O. Show that $d \geq 4\sqrt{2}h$ metres. **2**

END OF EXAMINATION

Q1. Form 6 3U HY 2004 May

(a) $P(2) = 48 - 32 + 8 - 8$
$= 16$

(b) $P(x) = x^4 - 5x^2 - 36$
$= (x^2 - 9)(x^2 + 4)$
$= (x-3)(x+3)(x^2+4)$ (✓ for some sensible method)
zeroes are 3 or -3.

(c) $\tan 2\theta = \sqrt{3}$, $0 \le \theta \le \pi$
$0 \le 2\theta \le 2\pi$
$2\theta = \frac{\pi}{3}$ or $\frac{4\pi}{3}$ (for $\frac{\pi}{3}$).
$\theta = \frac{\pi}{6}$ or $\frac{2\pi}{3}$

(d) (i) $\cos 2x = 1 - 2\sin^2 x$

(ii) $\int \sin^2 x \, dx = \frac{1}{2}\int (1 - \cos 2x) \, dx$
$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$
$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

(e) $\int_0^2 \frac{1}{4+x^2} \, dx = \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2$
$= \frac{1}{2}\left[\tan^{-1} 1 - \tan^{-1} 0\right]$
$= \frac{1}{2} \times \frac{\pi}{4}$
$= \frac{\pi}{8}$

(f) (i) at $t = 2$ seconds

(ii) distance $= 6 + 12 + 12$
$= 30$ m

(iii) when $t > \frac{1}{2}$ second.

Q2.

(a) (i) $P(1) = 1 - 2 - 5 + 6$

$= 0$

so 1 is a zero of $P(x)$.

(ii) $P(-2) = 0$ so $(x+2)$ is a factor

$P(3) = 0$ so $(x-3)$ is a factor

so $P(x) = (x-1)(x-3)(x+2)$

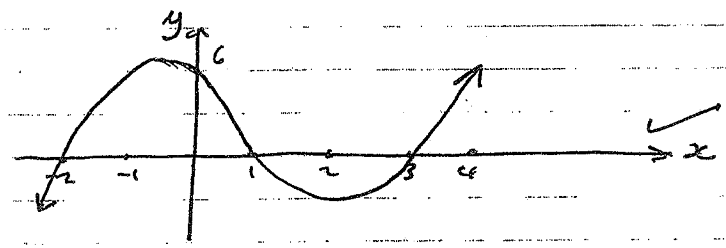
or/ divide

$$\begin{array}{r} x^2 - x - 6 \\ x-1 \,\big)\, x^3 - 2x^2 - 5x + 6 \\ \underline{x^3 - x^2} \qquad\qquad \\ -x^2 - 5x + 6 \\ \underline{-x^2 + x} \qquad \\ -6x + 6 \\ \underline{-6x + 6} \end{array}$$

Now $x^2 - x - 6 = (x+2)(x-3)$

So $P(x) = (x-1)(x+2)(x-3)$

(iii)



(iv) $x \leq -2$ or $1 \leq x \leq 3$

(b) (i) $x = a\cos nt$

velocity, $\dot{x} = -na\sin nt$

acceleration, $\ddot{x} = -n^2 a\cos nt$

(ii) (α) when $t = 0$, $x = 4$ so $4 = a\cos 0$

so $a = 4$

when $t = 0$, $\ddot{x} = -12$ so $-12 = -4n^2\cos 0$

$n^2 = 3$

$n = \sqrt{3}$

(β) $\frac{2\pi}{\sqrt{3}}$ second

(γ) $-4 \leq x \leq 4$

(δ) $4\sqrt{3}$ ms^{-1}

Q3.

(a)(i) (α) $\alpha+\beta+\gamma = -2$

(β) $\alpha\beta+\alpha\gamma+\beta\gamma = -3$

(γ) $\alpha\beta\gamma = -5$

(ii) (α) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$

$= \frac{3}{5}$

(β) $(\alpha-1)(\beta-1)(\gamma-1) = \alpha\beta\gamma - (\alpha\beta+\alpha\gamma+\gamma\beta)$
$+ (\alpha+\beta+\gamma) - 1$
$= -5+3-2-1$
$= -5$

(b) $f(x) = x^4+5x^3-ax+b$
$f(-1) = 1-5+a+b = 7$
$a+b = 11$

$g(x) = ax^2+bx-1$
$g(-1) = a-b-1 = -6$
$a-b = -5$

So we have $a+b=11$
$a-b=-5$

add $2a = 6$
$a = 3$
and $b = 8$

(c) $6\sin\theta - 5\cos\theta = 5$

$\frac{6\times 2t}{1+t^2} - \frac{5(1-t^2)}{1+t^2} = 5, \quad \theta \neq 180°$

$12t - 5(1-t^2) = 5(1+t^2)$
$12t-5+5t^2 = 5+5t^2$
$12t = 10$
$t = \frac{5}{6}$

so, $\tan\frac{\theta}{2} = \frac{5}{6}$

related angle is $40°$ (to nearest degree).

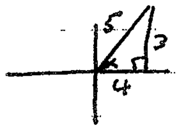
$\frac{\theta}{2} \doteqdot 40° + 180n°$, n an integer

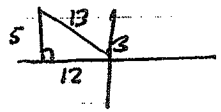
$\theta \doteqdot 80° + 360n°$

and check $\theta = 180°$, is it a solution?
$LHS = 6\sin 180° - 5\cos 180°$
$= 5$
$= RHS$
so $\theta = 180°$ is a solution.
so θ = odd multiples of $180°$ are solution
$= (2n-1)180°$

Solutions are $\theta = (2n-1)180°$
~~or~~ $\theta \doteqdot 80° + 360n°$, n an integer

Q4.





(a)

(i) $\tan\alpha = \frac{3}{4}$

(ii) $\tan\beta = \frac{-5}{12}$

(iii) $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$$= \frac{\frac{3}{4} + -\frac{5}{12}}{1 - (\frac{3}{4})(-\frac{5}{12})}$$

$$= \frac{\frac{1}{3}}{\frac{21}{16}}$$

$$= \frac{16}{63}$$

(b) $f(x) = \cos(\log_e x)$

(i) $f'(x) = \frac{-\sin(\log_e x)}{x}$

(ii) $f'(e^{\pi/2}) = \frac{-\sin(\log e^{\frac{\pi}{2}})}{e^{\frac{\pi}{2}}}$

$$= \frac{-\sin\frac{\pi}{2}}{e^{\frac{\pi}{2}}}$$

$= -e^{-\frac{\pi}{2}}$ which is negative

so $f(x)$ is decreasing at $x = e^{\pi/2}$

(e) $y = x\log x - x$

(i) Domain: $x > 0$

(ii) $\frac{dy}{dx} = x\frac{1}{x} + \ln x - 1$

$= \log_e x$

$\frac{d^2y}{dx^2} = \frac{1}{x}$

(iii) $\frac{1}{x} > 0$ for all $x > 0$

so $\frac{d^2y}{dx^2} > 0$ for all $x > 0$ and the curve is concave up.

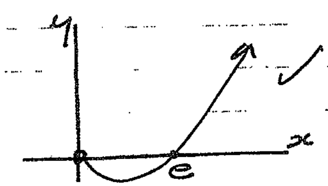
(iv) $x\log x - x = 0$

$x(\log x - 1) = 0$

$\log x = 1$ (since $x > 0$)

and $x = e$

(v)



$0 < x < e$

or// $x(\log x - 1) < 0$

$x > 0$, so want $\log x - 1 < 0$

$\log x < 1$

so $0 < x < e$

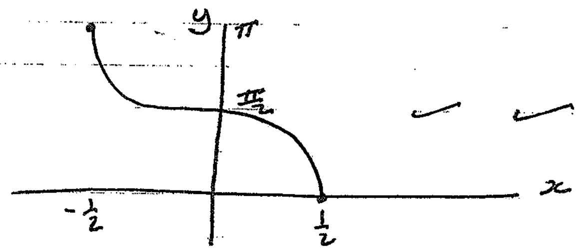
Q5.

(a) $y = \cos^{-1}(2x)$

(i) $-1 \le 2x \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$

(ii)



(iii) $\frac{dy}{dx} = \left(\frac{-1}{\sqrt{\frac{1}{4}-x^2}}\right) = \frac{-2}{\sqrt{1-4x^2}}$

at $x = \frac{\sqrt{2}}{4}$, gradient $= \frac{-2}{\sqrt{1-\frac{1}{2}}}$

$= \frac{-2}{\sqrt{\frac{1}{2}}}$

$= -2\sqrt{2}$

(iv) $x = \cos^{-1} 2y \qquad 0 \le x \le \pi$

$\cos x = 2y$

$y = \frac{1}{2}\cos x, \qquad 0 \le x \le \pi$

(v) area $= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos x \, dx$

$= \frac{1}{2}\left[\sin x\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin 0\right)$

$= \frac{1}{2} \ u^2$

b) $\ddot{x} = 2x(4+x^2)$

(i) $\frac{d(\frac{1}{2}v^2)}{dx} = 2x(4+x^2)$

$\frac{1}{2}v^2 = \int(8x+2x^3)dx$

$= 4x^2 + \frac{1}{2}x^4 + c$

at $x=0$, $v=4$, so $8 = c$

and $\frac{1}{2}v^2 = 4x^2 + \frac{1}{2}x^4 + 8$

$v^2 = x^4 + 8x^2 + 16$

$= (x^2+4)^2$

(ii) Initially P is moving in the positive direction from the origin. The acceleration of P is positive when $x>0$, so P continues to accelerate in the positive direction, that is, the velocity of P will always be greater than its initial velocity of 4 ms^{-1}.

OR $(x^2+4)^2 \ge 4$ for all x so v cannot be in the interval $-2 < v < 2$.

(iii) $\frac{dx}{dt} = x^2+4$

$\frac{dt}{dx} = \frac{1}{x^2+4}$

$t = \frac{1}{2}\tan^{-1}\frac{x}{2} + c$

when $t=0$, $x=0$, so $c=0$

and $t = \frac{1}{2}\tan^{-1}\frac{x}{2}$

so $\frac{x}{2} = \tan 2t$

and $x = 2\tan 2t \qquad 0 \le t < \frac{\pi}{4}$

(iv) when $x=2$, $2 = 2\tan 2t$

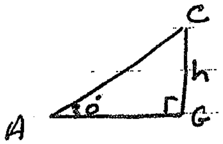
$\tan 2t = 1 \qquad 0 \le t < \frac{\pi}{4}$

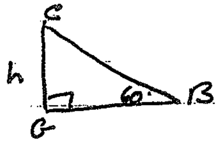
$2t = \frac{\pi}{4}$

$t = \frac{\pi}{8}$ seconds

Q6.

(a) (i)





$\tan 30° = \frac{h}{AG}$

$AG = h \div \frac{1}{\sqrt{3}}$

$= \sqrt{3}\,h$

$\tan 60° = \frac{h}{GB}$

$GB = h \div \sqrt{3}$

$= \frac{h}{\sqrt{3}}$

(ii) In $\triangle ABG$

$x^2 = AG^2 + BG^2 - 2 \times AG \times BG \times \cos 120°$

$= 3h^2 + \frac{h^2}{3} - 2 \times \sqrt{3}\,h \times \frac{h}{\sqrt{3}} \times (-\frac{1}{2})$

$= 3h^2 + \frac{h^2}{3} + h^2$

$= \frac{13h^2}{3}$

so $3x^2 = 13h^2$

(b) $y = x^3 - 4x$

(i) $\frac{dy}{dx} = 3x^2 - 4$

at $x = p$, gradient $= 3p^2 - 4$

tangent is $y - (p^3 - 4p) = (3p^2 - 4)(x - p)$

$y - p^3 + 4p = 3p^2x - 3p^3 - 4x + 4p$

$y = (3p^2 - 4)x - 2p^3$

(ii) At R, $(3p^2 - 4)x - 2p^3 = x^3 - 4x$ and the x coordinate of R is the solution of this equation.

$3p^2x - 4x - 2p^3 = x^3 - 4x$

$x^3 - 3p^2x + 2p^3 = 0$

(iii) $x^3 - 3p^2x + 2p^3 = 0$ has 3 roots.
Two of them are p, let the third be α.
Sum of roots is $p + p + \alpha = 0$

$\alpha = -2p$

The x coordinate is $-2p$

y coordinate is $(-2p)^3 - 4(-2p)$

$= -8p^3 + 8p$

$= 8p(1 - p^2)$

R is $(-2p, 8p(1-p^2))$

(c) $\frac{d^2x}{dt^2} + 9x = 0$

(i) $x = C\cos(3t + \alpha)$

$\dot{x} = -3C\sin(3t + \alpha)$

$\ddot{x} = -9C\cos(3t + \alpha)$

$= -9x$ since $C\cos(3t + \alpha) = x$

so $\ddot{x} + 9x = 0$

(ii) When $t = 0$, $x = 2$ and $v = -6$

so $2 = C\cos\alpha$ ... ①

and $-6 = -3C\sin\alpha$ i.e. $2 = \sin\alpha$ ... ②

② ÷ ① gives $\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

Now $\cos\frac{3\pi}{4} < 0$ and $C > 0$ so using ①, $\alpha = \frac{\pi}{4}$

and $C = 2\sqrt{2}$

so we have $x = 2\sqrt{2}\cos(3t + \frac{\pi}{4})$

when $t = \pi$, $x = 2\sqrt{2}\cos(\frac{13\pi}{4})$

$= 2\sqrt{2} \times (-\frac{1}{\sqrt{2}})$

$= -2$m or 2m on negative side of

d) $P(x) = x^3 + px^2 + qx + r$

(i) Let the roots be $\alpha, -\alpha, \beta$
then sum of roots $= -p$

$= \alpha - \alpha + \beta$

$-p = \beta$

so third root is $x = -p$

(ii) the roots are now $\alpha, -\alpha, -p$
then product of roots $= -r$

$= -\alpha^2(-p)$

$-r = p\alpha^2$

and sum of roots 2 at a time $= -\alpha^2 + \alpha x - \alpha x$

$q = -\alpha^2$

so $-r = p(-q)$
and $r = pq$.

or $P(-p) = -p^3 + p^3 - pq + r = 0$ since $x = -p$ is a zero
so $r = pq$

Q2

(a)(i) $\angle BPO = \angle BPA + \angle APO$

so $\angle BPA = \angle BPO - \angle APO$

$\theta = \tan^{-1}\frac{8}{x} - \tan^{-1}\frac{6}{x}$

(ii) $\theta = \tan^{-1}\frac{8}{x} - \tan^{-1}\frac{6}{x}$

$$\frac{d\theta}{dx} = \frac{-8x^{-2}}{1+(\frac{8}{x})^2} - \frac{-6x^{-2}}{1+(\frac{6}{x})^2}$$

$$= \frac{-8}{x^2+8^2} + \frac{6}{x^2+6^2}$$

$$= \frac{-8(x^2+36) + 6(x^2+64)}{(x^2+64)(x^2+36)}$$

$= 0$ at stationary point.

$8(x^2+36) = 6(x^2+64)$

$4x^2 + 144 = 3x^2 + 192$

$x^2 = 48$

$x = 4\sqrt{3}$

Show this is maximum:

θ	6	$4\sqrt{3}$	7
$\frac{d\theta}{dx}$	$\frac{-576+600}{+ve}$ $\frac{24}{+ve}$	0	$\frac{-680+678}{+ve}$ $\frac{-2}{+ve}$
	/	—	\

$\rightarrow$ max. t. pt.

so θ is a maximum when $x = 4\sqrt{3}$

(iii) Now $0 < \theta < \frac{\pi}{2}$ so maximum $\tan\theta$ implies maximum θ.

when $x = 4\sqrt{3}$,

$$\tan\theta = \tan\left(\tan^{-1}\frac{8}{4\sqrt{3}} - \tan^{-1}\frac{6}{4\sqrt{3}}\right)$$

$$= \tan\left(\tan^{-1}\frac{2}{\sqrt{3}} - \tan^{-1}\frac{3}{2\sqrt{3}}\right)$$

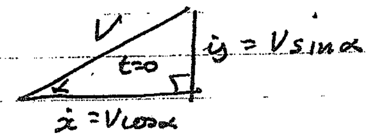
$$= \frac{\frac{2}{\sqrt{3}} - \frac{3}{2\sqrt{3}}}{1 + \frac{2}{\sqrt{3}}\frac{3}{2\sqrt{3}}}$$

$$= \frac{\frac{1}{2\sqrt{3}}}{2}$$

$$= \frac{1}{4\sqrt{3}}$$

so $\theta = \tan^{-1}\frac{1}{4\sqrt{3}}$

(b)



(i)

$\ddot{x} = 0$
$\dot{x} = c_1$
$t = 0,\ \dot{x} = V\cos\alpha$ so $c_1 = 0$
$\dot{x} = V\cos\alpha$
$x = Vt\cos\alpha + C_2$
$t = 0,\ x = 0,$ so $C_2 = 0$
$x = Vt\cos\alpha$ ①

$\ddot{y} = -g$
$\dot{y} = -gt + c_3$
$t = 0,\ \dot{y} = V\sin\alpha$ so $c_3 = V\sin\alpha$
$\dot{y} = -gt + V\sin\alpha$
$y = -\frac{1}{2}gt^2 + Vt\sin\alpha + c_4$
$t = 0,\ y = h,$ so $c_4 = h$
$y = -\frac{1}{2}gt^2 + Vt\sin\alpha + h$ ②

(ii) from ①, $t = \frac{x}{V\cos\alpha}$

substitute into ②

$$y = -\frac{1}{2}g\frac{x^2}{V^2\cos^2\alpha} + \frac{Vx\sin\alpha}{V\cos\alpha} + h$$

$$= h - \frac{gx^2\sec^2\alpha}{2V^2} + x\tan\alpha$$

(iii) For B, $\alpha = 0$ so $\sec\alpha = 1$ and $\tan\alpha = 0$

$$y = h - \frac{gx^2}{2V^2}$$

iv) P is $(d, 0)$

for B: $0 = h - \frac{gd^2}{2V^2}$

$h = \frac{gd^2}{2V^2}$

for A: $0 = h - \frac{gd^2 \sec^2\alpha}{2V^2} + d\tan\alpha$

so $0 = h - h\sec^2\alpha + d\tan\alpha$

$h - h(1 + \tan^2\alpha) + d\tan\alpha = 0$

$-h\tan^2\alpha + d\tan\alpha = 0$

$\tan\alpha(d - h\tan\alpha) = 0$, $\tan\alpha \neq 0$

$h\tan\alpha = d$

$\tan\alpha = \frac{d}{h}$

v) B lands at P so $h = \frac{gd^2}{2V^2}$

A lands at $(3d, 0)$

so $0 = h - \frac{g\,9d^2}{2V^2}\sec^2\alpha + 3d\tan\alpha$

$0 = h - 9h\sec^2\alpha + 3d\tan\alpha$.

$0 = h - 9h(1 + \tan^2\alpha) + 3d\tan\alpha$

we have, $9h\tan^2\alpha - 3d\tan\alpha + 8h = 0$

for $\tan\alpha$ to be real we need $\Delta \geq 0$

$9d^2 - 288h^2 \geq 0$

$d^2 \geq 32h^2$

$d \geq 4\sqrt{2}\,h$.