St George Girls High School

Year 12

Common Test 3

May 2009

Mathematics

General Instructions

- Time allowed – 70 minutes
- Write using blue or black pen
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question on a new page.

Total marks – 65

- Attempt Questions 1 – 5
- All questions are of equal value

Question 1 – Start a New Page – (13 marks) Marks

a) Find the value of e^4, correct to 3 significant figures. 2

b) Convert $\frac{7\pi}{4}$ radians to degrees. 1

c) Simplify 6

(i) $cosec\ x\ .\ tan\,x$

(ii) $(1 - cos^2\theta)cot^2\theta$

(iii) $\frac{cosec^2x - cot^2x}{cos^2x}$

d) Differentiate 4

(i) x^2e^x

(ii) $\frac{x}{e^{3x}}$

Question 2 – Start a New Page – (13 marks) Marks

a) Differentiate 6

(i) $(e^{2x}+1)^4$

(ii) $2ln(1-4x)$

(iii) $ln\left(\frac{2x}{x-5}\right)$

b) Find the primitive function of e^{4x+1} 1

c) Evaluate $\int_2^3 \frac{dx}{2x-3}$ 2

d) Solve for x: $2xlog_b 4 + 2xlog_b 8 = 5log_b 2$ 2

e) Given that $log_a c = 0.641$, find the value of $log_a ac^2$ 2

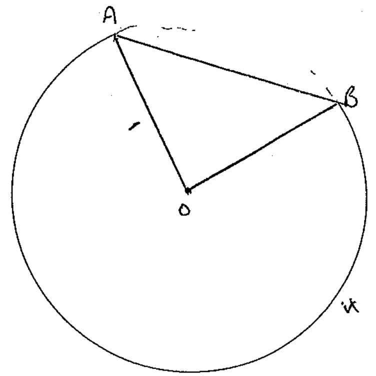
Question 3 – Start a New Page – (13 marks) Marks

a) A major sector of a circle, radius 1 unit, has an arc length of 4 units. 6

(i) Find the angle of the major sector, at the centre of the circle, in radians.

(ii) Find the area of the major sector.

(iii) Calculate the area of the triangle OAB cut off by the chord joining the endpoints of the two radii.



b) Calculate the area enclosed by the curve $y = log_e x$, the y-axis, and the lines $y = 1$ and $y = 2$. 3

c) Copy the following table into your answer booklet and complete, correct to two decimal places where necessary, for the function $f(x) = x^2 lnx$ 4

x	1	1.5	2	2.5	3
y					

Use Simpson's rule with five function values to get an approximate answer to $\int_1^3 x^2 lnx\, dx$ correct to three significant figures.

Question 4 – Start a New Page – (13 marks)

Marks

a) Using the function $y = 2\cos 2x$

(i) Sketch the graph for $0 \le \overset{x}{\cancel{\theta}} \le 2\pi$ 3

(ii) State the period and amplitude 2

(iii) Solve $\cos 2x = -1$, $0 \le \overset{x}{\cancel{\theta}} \le 2\pi$ 2

b) Find the equation of the tangent to the curve $y = 3\ln x$ at the point $(e, 3)$. 3

c) Sketch the curve $y = \log_e(x - 2)$ without using calculus. Show all essential features. 3

Question 5 – Start a New Page – (13 marks)

Marks

a) Show that $\frac{3x-1}{x+1} = 3 - \frac{4}{x+1}$. Hence evaluate $\int_0^2 \frac{3x-1}{x+1}\,dx$. 3

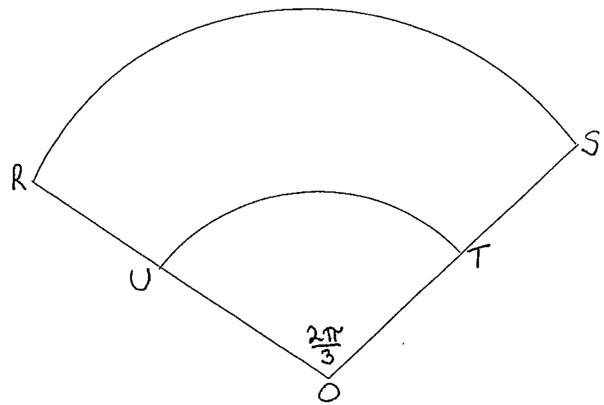
Leave your answer in exact form.

b) Find the second derivative of $y = x\log_e x$ 2

c) The area under the curve $y = e^{3x}$ between $x = 0$ and $x = 2$ is rotated about the x-axis. Find the volume of the solid so generated. 3

d) A car windscreen wiper traces out the area $RSTU$ where RS and UT are arcs of circles centre O, radius 40cm and 20 cm respectively, as shown in the figure.

Calculate the exact perimeter of RSTU. 2



e) Find the derivative of e^{x^2} and hence evaluate $\int_0^1 xe^{x^2}\,dx$. 3

End of Paper

Year 12 Mathematics Common Test 3 May 2009 Solutions

Question 1

(a) $e^4 = 54.59815003\ldots$
$\doteqdot 54.6$

(b) $\frac{7\pi}{4} \times \frac{180^\circ}{\pi} = 315^\circ$

(c) (i) $\operatorname{cosec} x \cdot \tan x$
$= \frac{1}{\cancel{\sin x}} \cdot \frac{\cancel{\sin x}}{\cos x}$
$= \sec x$

(ii) $(1-\cos^2\theta)\cot^2\theta$
$= \sin^2\theta \cot^2\theta$
$= \frac{\cancel{\sin^2\theta}\cos^2\theta}{\cancel{\sin^2\theta}}$
$= \cos^2\theta$

(iii) $\frac{\operatorname{cosec}^2 x - \cot^2 x}{\cos^2 x}$
$= \frac{\frac{1}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x}}{\cos^2 x}$
$= \frac{1-\cos^2 x}{\sin^2 x \cos^2 x}$
$= \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x}\cos^2 x} = \frac{1}{\cos^2 x}$

(d) (i) Let $y = x^2 e^x$
$u = x^2, \quad \frac{du}{dx} = 2x$
$v = e^x, \quad \frac{dv}{dx} = e^x$
product rule:
$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$
$\frac{dy}{dx} = e^x \times 2x + x^2 e^x$
or $= 2xe^x + x^2e^x$
or $= xe^x(2+x)$

(ii) Let $y = \frac{x}{e^{3x}}$
$u = x \quad u' = 1$
$v = e^{3x} \quad v' = 3e^{3x}$
Quotient rule:
$y' = \frac{vu' - uv'}{v^2}$
$\frac{dy}{dx} = \frac{e^{3x} - 3xe^{3x}}{e^{6x}}$
or $= \frac{e^{3x}(1-3x)}{e^{6x}}$
or $= \frac{1-3x}{e^{3x}}$

Question 2

(a) (i) $y = (e^{2x}+1)^4$ Let $u = e^{2x}+1$
Then $y = u^4$
$\frac{du}{dx} = 2e^{2x} \quad \frac{dy}{du} = 4u^3 = 4(e^{2x}+1)^3$
$\frac{dy}{dx} = 8e^{2x}(e^{2x}+1)^3$

(ii) $y = 2\ln(1-4x)$
$\frac{dy}{dx} = 2 \times \frac{-4}{1-4x} = \frac{-8}{1-4x}$

(iii) $y = \ln\left(\frac{2x}{x-5}\right) = \ln 2x - \ln(x-5)$
$\frac{dy}{dx} = \frac{\cancel{2}}{\cancel{2}x} - \frac{1}{x-5}$

Question 2 Continues

(b) If $\frac{dy}{dx} = e^{4x+1}$
$y = \frac{e^{4x+1}}{4} + C$

(c) $\int_2^3 \frac{dx}{2x-3} = \frac{1}{2}\int_2^3 \frac{2\,dx}{2x-3}$
$= \frac{1}{2}\left[\log(2x-3)\right]_2^3$
$= \frac{1}{2}(\log 3 - \log 1)$
$= \frac{1}{2}\log 3$

(d) $2x\log_b 4 + 2x\log_b 8 = 5\log_b 2$
$2x(\log_b 4 + \log_b 8) = 5\log_b 2$
$2x\log_b 32 = \log_b 32$
$2x = 1$
$x = \frac{1}{2}$

(e) $\log_a ac^2 = \log_a a + 2\log_a c$
$= 1 + 2 \times 0.641$
$= 2.282$

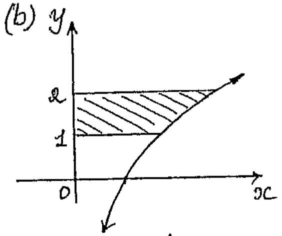
Question 3

(a) (i) $l = r\theta \qquad 4 = 1 \times \theta$
$\theta = 4$ radians

(ii) Area of sector $= \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 1^2 \times 4$
$= 2$ units^2

(iii) $A_\Delta = \frac{1}{2}ab\sin C$
$= \frac{1}{2}r^2\sin(2\pi - \theta)$
$= \frac{1}{2} \times 1^2 \times \sin(2\pi - 4)$
$\doteqdot 0.3784$ units^2
(to 4d.p.)

(b)



$y = \log_e x$
or
$x = e^y$

Area $= \int_1^2 x\,dy = \int_1^2 e^y\,dy$
$= [e^y]_1^2 = (e^2 - e)$ units^2

(c) $y = f(x) = x^2\ln x$

x	1	1.5	2	2.5	3
y	0	0.91	2.77	5.73	9.89

$\int_1^3 x^2 \ln x\,dx =$

$= \frac{2-1}{6}(0 + 4 \times 0.91 + 2.77) +$

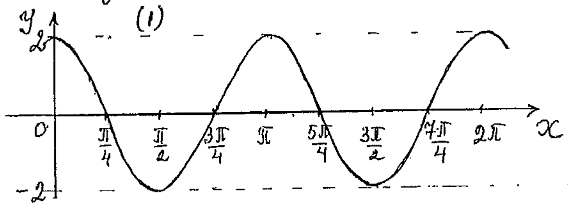
$+ \frac{3-2}{6}(2.77 + 4 \times 5.73 + 9.89)$

$= 1.068\dot{3} + 5.93$
$= 6.998\dot{3}$
≈ 7.00 (to three sign. fig.)

Question 4

(a) $y = 2\cos 2x$

(I)



(II) Period is $\frac{2\pi}{2} = \pi$

Amplitude is 2

(III) $\cos 2x = -1$

the amplitude is 1, the period is the same as for $y = 2\cos 2x$, π.

Reading from the graph:

$x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

(b) $y = 3\ln x$

$\frac{dy}{dx} = \frac{3}{x}$

at the point $(e, 3)$

$\frac{dy}{dx} = \frac{3}{e}$

$\therefore$ gradient of the tangent

$m = \frac{3}{e}$

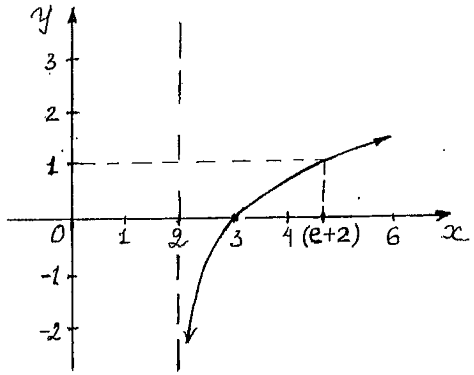
Using point-gradient form:

$(y-3) = \frac{3}{e}(x-e)$

$y = \frac{3x}{e} - \frac{3e}{e} + 3$

$y = \frac{3x}{e}$

(c) The graph of $y = \log(x-2)$ is the graph of $y = \log x$ shifted to the right 2 units.



Question 5

(a) RHS $= 3 - \frac{4}{x+1}$

$= \frac{3(x+1) - 4}{x+1}$

$= \frac{3x+3-4}{x+1}$

$= \frac{3x-1}{x+1}$

$=$ LHS.

$\int_0^2 \frac{3x-1}{x+1}\,dx = \int_0^2 \left(3 - \frac{4}{x+1}\right)dx$

$= \left[3x - 4\ln(x+1)\right]_0^2$

$= 6 - 4\ln 3$

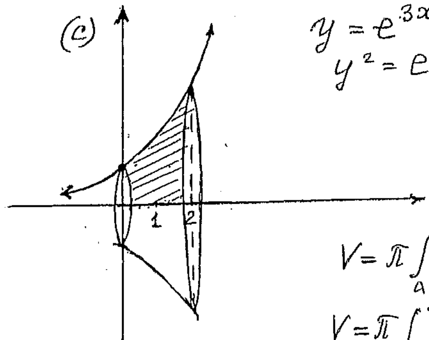
(b) $y = x\log x$

$\frac{dy}{dx} = \log x + 1$

$\frac{d^2y}{dx^2} = \frac{1}{x}$

(c) $y = e^{3x}$

$y^2 = e^{6x}$



$V = \pi\int_a^b y^2\,dx$

$V = \pi\int_0^2 e^{6x}\,dx$

$= \pi\left[\frac{e^{6x}}{6}\right]_0^2$

$= \frac{\pi}{6}(e^{12} - 1)\,u^3$

(d) $R = 40\text{cm}$ $r = 20\text{ cm}$

$\theta = \frac{2\pi}{3}$

Arc $RS = 40 \times \frac{2\pi}{3}$

Arc $UT = 20 \times \frac{2\pi}{3}$

$RU = ST = 40 - 20 = 20$

$\therefore$ Perimeter $= \frac{2\pi}{3}(40+20) + 2\times 20$

$= \frac{120\pi}{3} + 40$

$= 40\pi + 40$

$= 40(\pi + 1)$

(e) $\frac{d}{dx} e^{x^2} = 2x e^{x^2}$

$\int_0^1 x e^{x^2} = \frac{1}{2}\int_0^1 2x e^{x^2}$

$= \frac{1}{2}\left[e^{x^2}\right]_0^1$

$= \frac{1}{2}(e - 1)$

or $= \frac{1}{2}e - \frac{1}{2}$

The End