St George Girls High School

Year 12

Mid-HSC Course Examination

2009

Mathematics Extension 1

General Instructions
- Working time – 1½ hours
- Reading time – 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question in a new booklet.

Total marks – 72
- Attempt Questions 1 – 6
- All questions are of equal value

Question 1 – (12 marks) – (Start a new booklet) Marks

a) Differentiate with respect to x 4

(i) e^{4x-3}

(ii) $\sin(2x+1)$

(iii) $\log_e(\cos x)$

b) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin\left(\frac{x}{5}\right)}{2x}$ 2

c) Find the equation of the tangent to $y = x\log_e x$ at the point $x = e$ 3

d) Show that the second derivative of e^{x^3} is $3x(2+3x^3)e^{x^3}$ 3

Question 2 – (12 marks) – (Start a new booklet) Marks

a) If $y = \log_e \frac{(x-4)^2}{3x+1}$ find $\frac{dy}{dx}$ 2

b) A radioactive material decomposes at a rate proportional to the mass at any time. The rate is given by the equation $\frac{dm}{dt} = km$ where m is the mass of the material present after t years, and k is a negative constant.

(i) Show that $m = m_0 e^{kt}$ is a solution of the equation $\frac{dm}{dt} = km$, where m_0 is the amount present at $t = 0$ 2

(ii) A mass of 60 grams of this material given in the data above decomposes to 50 grams in 10 years.

(α) Show that $k = \frac{1}{10} \log_e \frac{5}{6}$ 2

(β) Find, to the nearest year, the time it takes to reach one half of its mass (that is, the half-life). 2

(γ) Find the rate of decomposition at this time (in β), giving your answer correct to two decimal places. 1

c) If $\cos A = \frac{4}{5}$ and $\sin B = \frac{3}{7}$ and A and B are acute, find $\sin(A + B)$ giving your answer in simple surd form. 3

Question 3 – (12 marks) – (Start a new booklet) Marks

a) Express $300°$ in radians (in terms of π) 1

b) What is the exact value of $\sin \frac{5\pi}{4}$? 1

c) Find the exact value of $\tan 75°$ 2

d) Find:

(i) $\int \sin 5x \, dx$ 2

(ii) $\int e^{7x-5} \, dx$ 2

(iii) $\int \frac{x}{x^2+3} \, dx$ 2

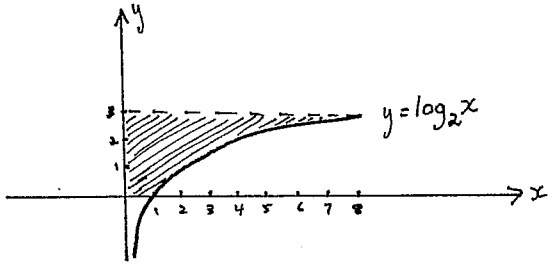
(iv) $\int \sec^2 \frac{x}{2} \, dx$ 2

Question 4 – (12 marks) – (Start a new booklet) Marks

a) Show that $\frac{\cos 2\theta}{\cos\theta - \sin\theta} = \cos\theta + \sin\theta$ 2

b) By multiplying top and bottom by $1 + \cos x$, show that $\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$ 3

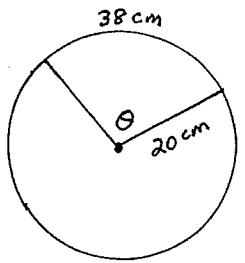
c) The diagram shows the graph of $y = \log_2 x$ between $x = 1$ and $x = 8$. The shaded region, bounded by $y = \log_2 x$, the line $y = 3$, and the x and y axes, is rotated about the y-axis to form a solid.



(i) Show that the volume of the solid is given by $V = \pi \int_0^3 e^{y \ln 4}\, dy$ 3

(ii) Hence find the volume of the solid. 2

d)



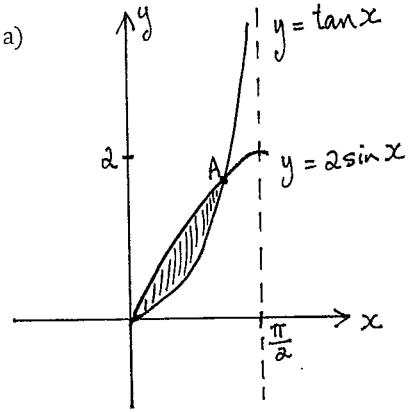
Not to scale

The length of the arc between two spokes on a car's steering wheel is 38cm. Each spoke is 20cm in length. 2

Calculate the angle θ between the spokes correct to the nearest degree.

Question 5 – (12 marks) – (Start a new booklet) Marks

a)



The diagram shows the curves $y = \tan x$ and $y = 2\sin x$ for $0 \le x \le \frac{\pi}{2}$

(i) Show that the coordinates of A are $\left(\frac{\pi}{3}, \sqrt{3}\right)$ 2

(ii) Show that $\frac{d}{dx}(\log_e \cos x) = -\tan x$ 1

(iii) Hence find the area of the shaded area in the diagram. 3

b) Solve $\cos 2x = -\frac{1}{2} \quad 0 \le x \le 2\pi$ 3

c) Sketch the graph of $y = 3\sin 2x, \quad 0 \le x \le 2\pi$ 3

Question 6 – (12 marks) – (Start a new booklet) **Marks**

Consider the curve $y = xe^x$

(i) Find the first derivative. 2

(ii) Show that the second derivative is $(2 + x)\, e^x$ 2

(iii) Show that there is one stationary point and determine its coordinates and nature. 2

(iv) Find the coordinates of the point of inflexion. 2

(v) Given $y \to 0$ as $x \to -\infty$, sketch the curve, then write down its range. 2

(vi) Hence using a transformation sketch $y = -xe^{-x}$ 2

End of Paper

2009 Extension 1 HSC Marks

Question 1

a) i) $\frac{d}{dx}\left(e^{4x-3}\right) = \boxed{4e^{4x-3}}$ ✓

ii) $\frac{d}{dx}\left(\sin[2x+\pi\right) = \boxed{2\cos(2x+1)}$ ✓

iii) $\frac{d}{dx}\left(\log_e(\cos x)\right) = \frac{-\sin x}{\cos x}$ ✓

$= \boxed{-\tan x}$ ✓

b) Evaluate $\lim_{x\to 0} \frac{\sin\left(\frac{x}{5}\right)}{2x}$

$= \lim_{x\to 0} \frac{1}{10} \frac{\sin\left(\frac{x}{5}\right)}{\left(\frac{x}{5}\right)}$ ✓

$= \boxed{\frac{1}{10}}$ ✓ since $\lim_{x\to 0} \frac{\sin\left(\frac{x}{5}\right)}{\left(\frac{x}{5}\right)} = 1$

c. $y = x\log_e x$

use $y - y_1 = m(x - x_1)$

find m

$\frac{dy}{dx} = \log_e x + x \cdot \frac{1}{x}$ use product rule

$= \boxed{\log_e x + 1}$ ✓

at $x = e$, $m = \log_e e + 1$, $y = e\log_e e$

$= e\log_e e$

$\boxed{m = 2}$ ✓

$y - e = 2(x - e)$

$y = 2x - 2e + e$

$\boxed{y = 2x - e}$ ✓ equation of tangent.

Question 1

d) $y = e^{x^3}$

$\frac{dy}{dx} = \boxed{3x^2 e^{x^3}}$ ✓

$\frac{d^2y}{dx^2} = 6xe^{x^3} + 3x^2 \cdot 3x^2 \cdot e^{x^3}$

$= 6xe^{x^3} + 9x^4 e^{x^3}$ ✓

$= \boxed{3xe^{x^3}(2 + 3x^3)}$ ✓

Question 2

a) $y = \log_e \frac{(x-4)^2}{3x+1}$

$= \boxed{2\log_e(x-4) - \log_e(3x+1)}$ ✓ logarithm laws

$\frac{dy}{dx} = \frac{2}{x-4} - \frac{3}{3x+1}$

$= \frac{2(3x+1) - 3(x-4)}{(x-4)(3x+1)}$

$= \frac{6x + 2 - 3x + 12}{(x-4)(3x+1)}$

$= \boxed{\frac{3x+14}{(x-4)(3x+1)}}$ ✓

b) i) $M = M_0 e^{kt}$ at $t=0$ $M = M_0 e^0$ ✓ $M = M_0$

$\frac{dM}{dt} = kM_0 e^{kt}$ Now $M = M_0 e^{kt}$

$= kM$ ✓

ii) 2) $M = M_0 e^{kt}$ $M_0 = 60$

$50 = 60e^{10k}$ $M = 50$

$\frac{5}{6} = e^{10k}$ ✓ $t = 10$

$\ln\left(\frac{5}{6}\right) = 10k$ find k.

$k = \frac{1}{10}\ln\left(\frac{5}{6}\right)$ ✓

Question 2

b) i) α) $M = \frac{1}{2}M_0$ ✓

$\frac{1}{2}M_0 = M_0 e^{kt}$

$\frac{1}{2} = e^{kt}$

$\ln\left(\frac{1}{2}\right) = kt$

$t = \frac{\ln(1/2)}{k}$

$= \frac{10\ln(1/2)}{\ln(5/6)}$ ✓

$= 38.017840 17$

$\doteq \boxed{38 \text{ years}}$

β) $\frac{dM}{dt} = kM$ ✓ $M = M_0 e^{kt}$

$= \frac{1}{10}\ln\left(\frac{5}{6}\right) \cdot M_0 e^{kt}$

$= \frac{1}{10}\ln\left(\frac{5}{6}\right) \cdot 60 \cdot e^{38 \times \frac{1}{10}\ln(5/6)}$

$= 6\ln\left(\frac{5}{6}\right) e^{3.8\ln(5/6)}$

$= -1.093928$ -0.092821915 ✓

$= \boxed{-0.0847}$

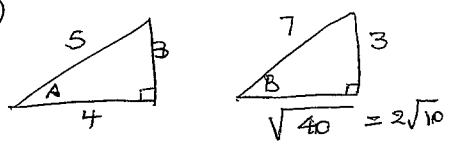
$\frac{dM}{dt} = -0.55$ rate of decomposition

or $\frac{dm}{dt} = 2k \times 30$

$\frac{dm}{dt} = -0.55$

Question 2

c)



$\sin(A+B) = \sin A\cos B + \sin B\cos A$ ✓

$= \frac{3}{5}\cdot\frac{2\sqrt{10}}{7} + \frac{3}{7}\cdot\frac{4}{5}$ ✓

$= \frac{6\sqrt{10}}{35} + \frac{12}{35}$

$= \frac{6\sqrt{10}+12}{35}$

$\sin(A+B) = \frac{6(\sqrt{10}+2)}{35}$ 1

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Question 3

a) $180^\circ = \pi$

$1^\circ = \frac{\pi}{180}$

$300^\circ = \frac{300\pi}{180}$

$= \boxed{\frac{5\pi}{3}}$ ✓

b) $\sin\frac{5\pi}{4} = -\sin\frac{\pi}{4}$ sin is negative in 4th quad.

$= \boxed{\frac{-1}{\sqrt{2}}}$ ✓

c) $\tan 75^\circ = \tan(30^\circ + 45^\circ)$

$= \frac{\tan 30^\circ + \tan 45^\circ}{1 - \tan 30^\circ \tan 45^\circ}$ ✓

$\tan 45^\circ = 1$

$\tan 30^\circ = \frac{1}{\sqrt{3}}$

$= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}}$

$= \boxed{\frac{1+\sqrt{3}}{\sqrt{3}-1}}$ ✓

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Question 8.

d) i) $\int \sin 5x\,dx = \boxed{\frac{-1}{5}\cos 5x + C}$ ✓

ii) $\int e^{7x-5}\,dx = \boxed{\frac{1}{7}e^{7x-5} + C}$ ✓

iii) $\int \frac{x}{x^2+3}\,dx = \boxed{\frac{1}{2}\ln(x^2+3) + C}$ ✓

iv) $\int \sin^2\frac{x}{2}\,dx = \boxed{\frac{2\tan\frac{x}{2}}{2} + C}$ ✓

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Question 4

a) Show $\frac{\cos 2\theta}{\cos\theta - \sin\theta} = \cos\theta + \sin\theta$

LHS

$= \frac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta}$ ✓

$= \frac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{(\cos\theta - \sin\theta)}$ ✓

$= \cos\theta + \sin\theta$

$= RHS$

b) $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}$

$= \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$ ✓

$= \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}$

$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \cdot \frac{1}{(1+\cos x)}$ ✓

$= 1 \cdot \frac{1}{1+1}$ $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1.$

$= \frac{1}{2}$ ✓ at $x=0$ $\frac{1}{1+\cos 0} = \frac{1}{2}.$

Question 5

$y = \tan x$, $y = 2\sin x$.

i) $A\left(\frac{\pi}{3}, \sqrt{3}\right)$ at $x = \frac{\pi}{3}$

$y = \tan\frac{\pi}{3}$, $y = 2\sin\frac{\pi}{6}$

$= \sqrt{3}$ $= 2 \times \frac{\sqrt{3}}{2}$

$= \sqrt{3}$

Hence $A\left(\frac{\pi}{3}, \sqrt{3}\right)$ satisfy both and the point of intersection. ✓

ii) $\frac{d}{dx}\left(\log_e \cos x\right) = \frac{-\sin x}{\cos x}$

$= -\tan x$ ✓

iii) $\text{Area} = \int_0^{\frac{\pi}{3}} 2\sin x\,dx - \int_0^{\frac{\pi}{3}} \tan x$

$= -2\cos x \Big|_0^{\pi/3} + \log_e \cos x \Big|_0^{\pi/3}$

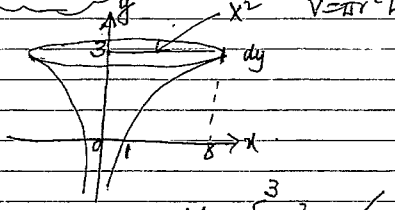
$= -2\cos\frac{\pi}{3} + 2\cos 0 + \log_e \cos\left(\frac{\pi}{3}\right) - \log_e \cos 0$

$= -1 + 2 - \ln 2$

$= 1 - \ln 2$ sq units ✓

Question 4

c) x^2 $V = \pi r^2 h$



$V = \pi \int_0^3 x^2\,dy$ ✓

i) $V = \pi \int_0^3 e^{\ln 4y}\,dy$

$y = \log_2 x$

$= \frac{\log_e x}{\log_e 2}$ ✓

$\ln 2^y = \ln x$

$x = e^{\ln 2^y}$

$x^2 = e^{2\ln 2^y}$

$= e^{\ln 2^{2y}}$

$x^2 = e^{\ln 4y}$ ✓

ii) $V = \pi\left[\frac{e^{\ln 4y}}{\ln 4}\right]_0^3$ ✓

$= \pi\left[\frac{e^{3\ln 4}}{\ln 4} - \frac{e^0}{\ln 4}\right]$

$= \frac{\pi}{\ln 4}\left[e^{\ln 4^3} - 1\right]$

$= \frac{\pi}{\ln 4}\left(4^3 - 1\right)$

$= \frac{63\pi}{\ln 4}$ cubic units ✓

d) $l = r\theta_{rad.}$ $\theta = \frac{38}{20}$ $\pi = 180$

$= 1.9$ ✓ $l = 180$

$= 108°52'$ $1.9 = \frac{180 \times 1.9}{\pi}$

$\theta = 109°$ ✓ π

Question 6

$y = xe^x$

i) $\frac{dy}{dx} = e^x + xe^x$ ✓

$= \boxed{e^x(1+x)}$

ii) $\frac{d^2y}{dx^2} = e^x + e^x + xe^x$

$= 2e^x + xe^x$ ✓

$= \boxed{e^x(2+x)}$

iii) Stationary point occurs when $\frac{dy}{dx} = 0$

$e^x(1+x) = 0$

$\boxed{x = -1}$ at $x = -1$, $y = -e^{-1}$

$= -\frac{1}{e}$ $\left(-1, -\frac{1}{e}\right)$

Determine nature

at $x = -1$, $\frac{d^2y}{dx^2} = e^{-1}(2+-1)$

$= \frac{1}{e}$ > 0 for $x = -1$.

Minimum turning point at $\left(x = -1, y = \frac{-1}{e}\right)$

Question 5

b) $\cos 2x = -\frac{1}{2}$

$\cos\frac{\pi}{3} = \frac{1}{2}$ and cos is negative in 2nd and 3rd quad.

$2x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}, 2\pi + \frac{2\pi}{3}, 2\pi + \frac{4\pi}{3}$ ✓

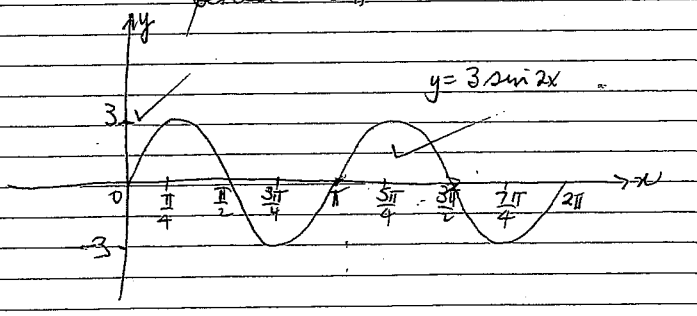
$2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$

$x = \boxed{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$ $0 \le x \le 2\pi$

c) $y = 3\sin 2x$

amplitude $= 3$

period $= \pi$ ✓



Jorg Model Exam HSC

Question 6

iv) Inflexion may occur when $\frac{d^2y}{dx^2} = 0$

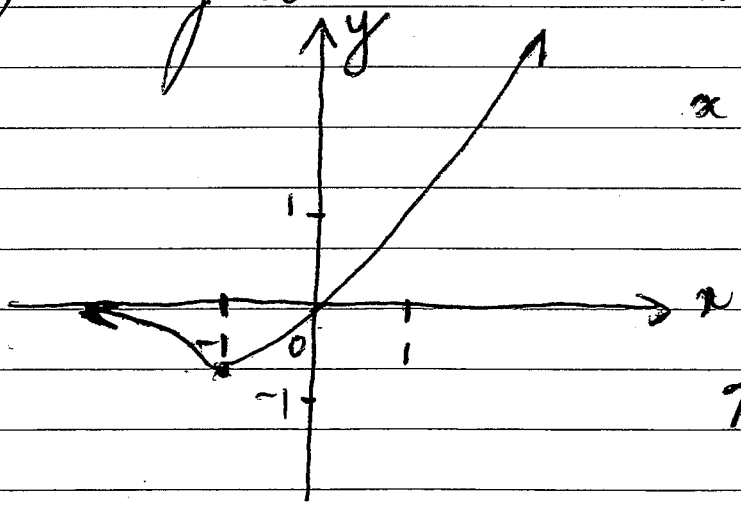
$e^x(2+x) = 0$ when $\boxed{x = -2}$ ✓

Check Change in Concavity

x	-3	-2	0
$\frac{d^2y}{dx^2}$	-ve	0	+ve

✓ Hence point of inflexion at $\left(-2, \frac{-2}{e^2}\right)$

v) $y = xe^x$



$x \to -\infty \quad y \to 0$

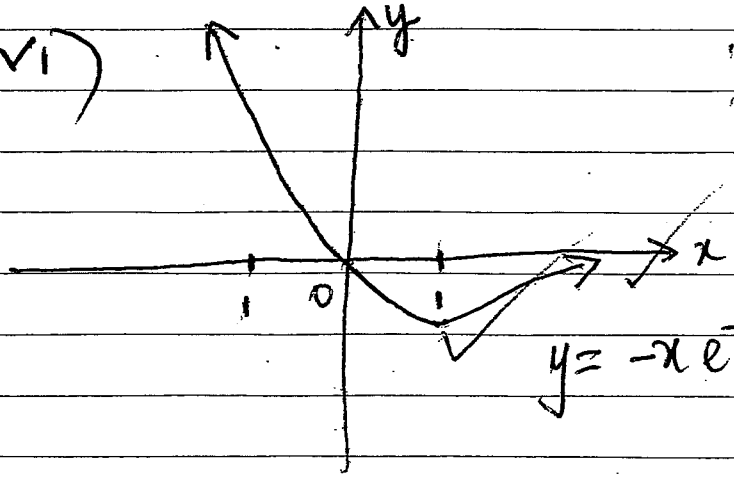
$x \to \infty \quad y \to \infty$

asymptotic to x axis for $x < 0$

Range $y \geq \frac{-1}{e}$ for $x \geq -1$

$\frac{-1}{e} \leq y < 0$ for $x < -1$

vi)



$y = -xe^{-x}$