St George Girls High School

Year 12

Mid-HSC Course Examination

2009

Mathematics

General Instructions

- Working time: 1½ hours
- Reading time: 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.

Total marks – 85

- Attempt Questions 1 – 5
- All questions are of equal value
- Begin each question in a new booklet
- All necessary working should be shown in every question.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} \, dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} \, dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

Question 1 – (17 marks) – (Start a new booklet) Marks

a) Differentiate with respect of x

(i) $y = \frac{x^8}{2} - 2x^3$ 2

(ii) $f(x) = \frac{2}{21x^7}$ 1

b) Show that the function $y = 2x^3 - 3x^2 + 5x - 10$ is increasing when $x = 2$ 3

c) (i) For what values of x is $f(x) = 5x^2 - 3x + 2$ a decreasing function? 3

(ii) Sketch the graph of the derivative of the function. 1

d) (i) Find any stationary points on the curve $y = (x+2)^5 - 3$ and determine their nature. 2

(ii) Find the equation of the tangent to the curve $y = (x+2)^5 - 3$ at the point $(0,\ 29)$ 2

e) A certain company caters for dinner parties. The profit is calculated according to the formula: $P = 300 + 75n - 2.5n^2$, where P is the net profit and n is the number of guests at the party.

Find:

(i) The number of guests that will maximize the profit. 2

(ii) What this maximum profit will be. 1

Question 2 – (17 marks) – (Start a new booklet) Marks

a) Find the primitive functions of:

(i) x^5 1

(ii) 7 1

(iii) $(3x+5)^4$ 2

(iv) $-\frac{3}{x^2}$ 1

(v) $-\frac{5}{2x\sqrt{x}}$ 2

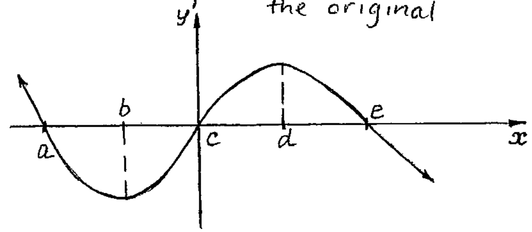
b) Find the equation of the curve $y = f(x)$ for which $f'(x) = 3x^2 - 4x - 1$ and $f(3) = -1$ 3

c) Integrate:

(i) $\int \frac{6x^2 - 3x}{x^4}\, dx$ 2

(ii) $\int (x^2+4)^2\, dx$ 2

~~d)~~ The graph of the derivative of the function is shown below. Copy it into your booklet and make a sketch of ~~a possible primitive~~ the original function. may look like on a seperate diagram 3



Question 3 – (17 marks) – (Start a new booklet) **Marks**

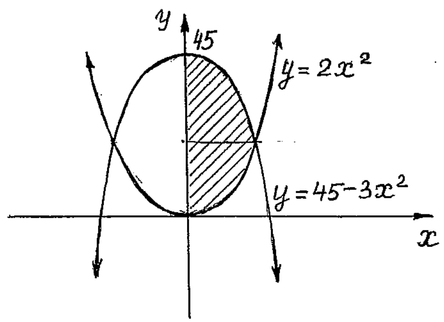
a) (i) Show that $f(x) = 5x^3 + 2x$ is an odd function. 1

(ii) Without integrating, find the value of $\int_{-3}^{3}(5x^3 + 2x)\ dx$ 1

b) (i) Sketch the curve $y = x^2 - 5$ 1

(ii) Find the area bounded by the curve and x-axis. Give the answer correct to four significant figures. 3

c) The graphs of the curves $y = 2x^2$ and $y = 45 - 3x^2$ are shown on the diagram.



(i) Find the points of intersection of the two curves. 1

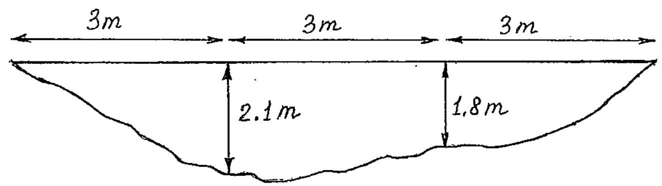
(ii) Find the area of the shaded region. 3

(iii) The shaded region between the curves and the y-axis is rotated about the y-axis. By splitting the shaded region in two parts or otherwise, find the volume of the solid formed. Give your answer in terms of π. 4

Question 3 d) on next page

Question 3 (cont'd) **Marks**

d) The diagram shows the cross-section of a creek with the depth of the creek shown in metres at 3 metre intervals. The creek is 9 metres in width.



(i) Use the trapezoidal rule to find an approximate value for the area of the cross-section. 2

(ii) Water flows through this section of the creek at a speed of 0.75m/s. Calculate the approximate volume of the water that flows through this section in one hour. 1

Question 4 – (17 marks) – (Start a new booklet) Marks

a) $f(x) = (x+1)(x^2-1)$

(i) Find the coordinates of the points where the curve crosses the axes. 2

(ii) Find the coordinates of the stationary points and determine their nature. 4

(iii) Find the coordinates of any points of inflection. 2

(iv) Sketch the graph of $y = f(x)$, indicating clearly the intercepts, stationary points and points of inflection. 3

(v) What is the maximum value of $f(x)$ for $-1 \le x \le 3$? 1

b) (i) Differentiate $y = (1+x^2)^3$ using the chain rule. 3

(ii) Hence find $\int_0^1 6x\,(1+x^2)^2\,dx$ 2

Question 5 – (17 marks) – (Start a new booklet) Marks

a) Integrate $\int_1^5 \frac{7}{x^2}\,dx$ 2

b) Find the value of k if $k > 0$ and $\int_0^k 5x\,dx = 40$ 3

c) (i) Sketch the graph of the function $y = f(x) = \sqrt{25 - x^2}$ 1

(ii) Evaluate $\int_{-5}^{5} \sqrt{25 - x^2}\,dx$ giving your answer correct to three decimal places. 2

(iii) Copy and complete the table of values for $y = \sqrt{25 - x^2}$ 2

x	-5	-2.5	0	2.5	5
y					

(iv) Using all function values from the table, approximate 3

$\int_{-5}^{5} \sqrt{25 - x^2}\,dx$ by Simpson's Rule giving your answer correct to 3 decimal places.

d) (i) Find the area enclosed by the curve $y = \sqrt{x}$, x-axis and the lines $x = 4$ and $x = 9$. 2

(ii) Find the volume of the solid formed when this area is rotated around x-axis. 2

End of Paper

out of 85. Always test for ... for ...

Solutions to
Mid-HSC ...
Year 12
2009.

Q1.

a) (i) $\frac{d}{dx}\ \frac{x^8}{2} - 2x^3 = 4x^7 - 6x^2$ (2)

(ii) $f'(x) = \frac{2}{21} \cdot -7x^{-8}$ (1)

$= -\frac{2}{3}x^{-8}$

b) $y = 2x^3 - 3x^2 + 5x - 10$

$\therefore y' = 6x^2 - 6x + 5$ (3)

when $x = 2$, $y' = 6\times4 - 6\times2 + 5$

$= 17 > 0$

$\therefore$ function increasing when $x = 2$.

c) (i) $f(x) = 5x^2 - 3x + 2$

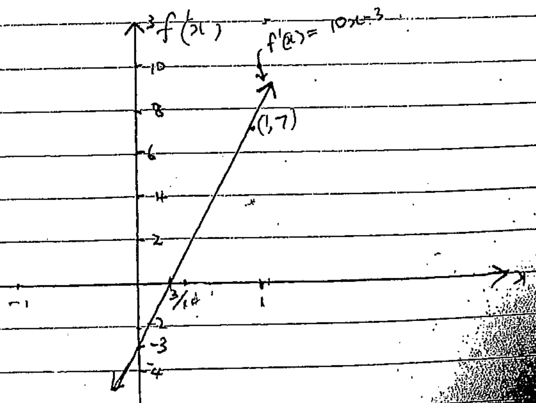
$f'(x) = 10x - 3$

for decreasing f'n, $f'(x) < 0$

ie $10x - 3 < 0$ (3)

ie $x < 3/10$

(ii)



(1)

d) (i) $y = (x+2)^5 - 3$

$\therefore \frac{dy}{dx} = 5(x+2)^4$

St. pts occur where $\frac{dy}{dx} = 0$.

$\Rightarrow$ st. pt at $x = -2$, $y = -3$.

$\frac{d^2y}{dx^2} = 20(x+2)^3$ (2)

x	-2^-	-2	-2^+
y''	$-$	0	$+$

y'' changes sign as we pass through $x = -2$

$\therefore$ horizontal point of inflection at $(-2, -3)$.

(ii) at $x = 0$, $y' = 5\times2^4$ from $*$.

$= 80$

$\therefore$ eqn of tangent is (2)

$y - 29 = 80(x - 0)$

ie $y = 80x + 29$.

e) $P = 300 + 75n - 2.5n^2$

(i) Max profit when $\frac{dP}{dn} = 0$ and $\frac{d^2P}{dn^2} < 0$

$\frac{dP}{dn} = 75 - 5n = 0$

ie $n = 15$ (2)

$\frac{d^2P}{dn^2} = -5 < 0$

$\therefore$ Max profit when $n = 15$.

ie 15 guests to max. profit.

(ii) when $n = 15$, $P = 300 + 75\times15 - 2.5\times15^2$

$= \$862.50$. (1)

Q2 a) (i) $\int x^5\,dx = \frac{1}{6}x^6 + C$ ①

(ii) $\int 7\,dx = 7x + C$ ~~②~~ ①

(iii) $\int (3x+5)^4 = \frac{1}{(4+1)3}(3x+5)^{4+1} + C$

$= \frac{1}{15}(3x+5)^5 + C$ ②

(iv) $\int -3x^{-2}\,dx = -3 \cdot \frac{1}{-1} x^{-1} + C$

$= 3/x + C$ ①

(v) $\int -5/2\, x^{-3/2}\,dx = -5/2 \cdot \frac{1}{-\frac{1}{2}} x^{-\frac{1}{2}} + C$ ②

$= \frac{5}{\sqrt{x}} + C$

b) $f'(x) = 3x^2 - 4x - 1$

$\therefore f(x) = x^3 - 2x^2 - x + K$

$f(3) = -1$

$\therefore -1 = 27 - 18 - 3 + K$

$K = -1 - 6$

$= -7$ ③

$\therefore f(x) = x^3 - 2x^2 - x - 7$

c) (i) $\int \frac{6x^2 - 3x}{x^4}\,dx$

$= \int 6x^{-2} - 3x^{-3}\,dx$ ②

$= -6x^{-1} + 3/2\, x^{-2} + C$

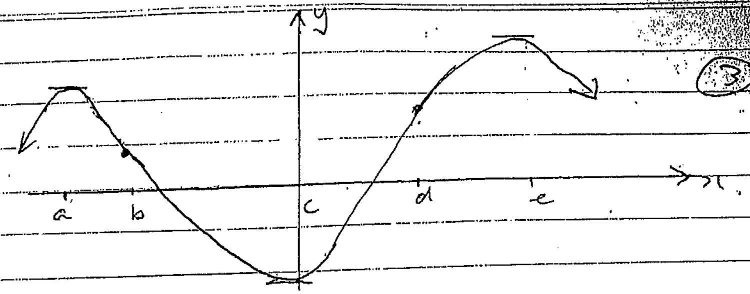
(ii) $\int (x^2+4)^2\,dx$ ②

$= \int x^4 + 8x^2 + 16\,dx$

$= \frac{1}{5}x^5 + 8/3\, x^3 + 16x + K$

d)

d)



③

Q3 a) (i) $f(x) = 5x^3 + 2x$

$f(-x) = 5(-x)^3 + 2(-x)$

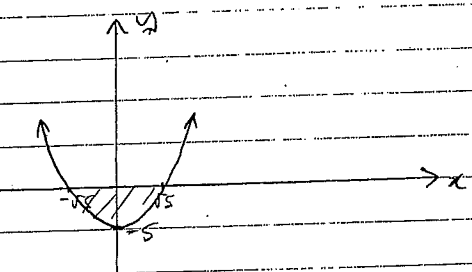
$= -(5x^3 + 2x)$

$= -f(x)$

$\therefore$ odd. ①

(ii) 0 ①

b) (i)



①

(ii) $A = \left| \int_{-\sqrt{5}}^{\sqrt{5}} x^2 - 5\,dx \right|$

$= 2\int_{\sqrt{5}}^{0} x^2 - 5\,dx$

$= 2\left[x^3/3 - 5x\right]_{\sqrt{5}}^{0}$

$= 2\left(5\sqrt{5} - \frac{5\sqrt{5}}{3}\right)$ ③

$= 2 \times 2/3 \times 5\sqrt{5}$

$= \frac{20}{3}\sqrt{5}\ u^2$

$= 14.91\ u^2$ (4 sig figs)

c) (i) $2x^2 = 45 - 3x^2$ at pt. of X'n

ie $5x^2 = 45$

$x = \pm 3$. (1)

$\therefore (3, 18)$, $(-3, 18)$ are pts of X'n.

(ii) $A = \int_0^3 45 - 3x^2 - 2x^2 \, dx$

$= \int_0^3 45 - 5x^2 \, dx$

$= \left[45x - \frac{5x^3}{3}\right]_0^3$

$= 45 \times 3 - 45 - 0$ (3)

$= 90 u^2$

(iii) $V = \pi \int_0^{18} x_1^2 \, dy + \pi \int_{18}^{45} x_2^2 \, dy$

$y = 2x_1^2$

$y = 45 - 3x_2^2$

$x_2^2 = \frac{1}{3}(45 - y)$

$= \pi \int_0^{18} \frac{y}{2} \, dy + \frac{\pi}{3} \int_{18}^{45} 45 - y \, dy$

$= \pi \left[\frac{y^2}{4}\right]_0^{18} + \frac{\pi}{3}\left[45y - \frac{y^2}{2}\right]_{18}^{45}$

$= \pi(81) + \frac{\pi}{3}\left(\frac{2025}{2} - \left(45 \times 18 - \frac{18^2}{2}\right)\right)$ (4)

$= \pi\left(81 + \frac{2025}{6} - 15 \times 18 + \frac{18^2}{6}\right)$

$= 202.5\pi \ u^3$.

d) (i) $A \doteq \frac{h}{2}\left[y_1 + 2(y_2 + y_3) + y_4\right]$ (2)

$\doteq 3/2 \left[0 + 2(2.1 + 1.8) + 0\right]$

$\doteq 3/2 \times 2 \times 3.9$

$= 3 \times 3.9$

$= 11.7 \ m^2$.

(ii) $0.75 \ m/s \Rightarrow m/h$?

0.75×3600 m in 1h

ie 2700 m in 1h

$\therefore$ Vol. of H_2O in 1h $\doteq 11.7 \times 2700 \ m^3$

$\doteq 31590 \ m^3$. (1)

Q4

a) (i) $f(x) = (x+1)(x^2-1) = 0$ for x-intercepts.

ie $x = -1, +1, -1$. Y intercept: $x = 0 \Rightarrow -1$ (2)

(ii) $f'(x) = 0$ for st. pts.

$f(x) = x^3 - x + x^2 - 1$

$\therefore f'(x) = 3x^2 + 2x - 1 = 0$

$3x \quad -1$

$x \quad +1$

$(3x - 1)(x + 1) = 0$

$\therefore x = \frac{1}{3}, -1$ (4)

at $x = \frac{1}{3}$, $y = \frac{1}{27} - \frac{1}{3} + \frac{1}{9} - 1 = -1\frac{5}{27}$

at $x = -1$, $y = -1 + 1 + 1 - 1 = 0$

$f''(x) = 6x + 2$.

$f''(\frac{1}{3}) = 4 > 0$ $\therefore$ min. t. pt at $(\frac{1}{3}, -1\frac{5}{27})$

$f''(-1) = -4 < 0$ concave $\therefore$ max. t. pt $(-1, 0)$

(iii) For pts. of inflection, $y'' = 0$ and changes sign

$\therefore 6x + 2 = 0$

$x = -\frac{1}{3}$.

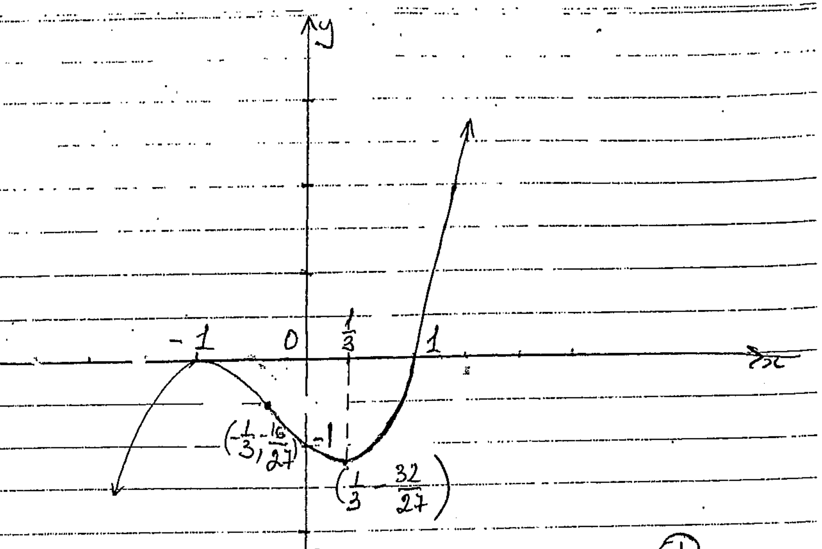
$y = -\frac{1}{27} + \frac{1}{3} + \frac{1}{9} - 1$

$= -\frac{16}{27}$

x	$-\frac{1}{3}^-$	$-\frac{1}{3}$	$-\frac{1}{3}^+$
$f''(x)$	$-$	0	$+$

$\therefore$ pt. of inflection at $(-\frac{1}{3}, -\frac{16}{27})$

(iv)



(v) When $x=3$, $y = 32$ ①

(from graph, maximum value occurs at $x=3$)

b) (i) $y = (1+x^2)^3$

$y' = 3(1+x^2)^2(2x)$ ③

$= 6x(1+x^2)^2$

(ii) $\therefore \int_0^1 6x(1+x^2)^2\,dx = \left[(1+x^2)^3\right]_0^1$

$= (1+1^2)^3 - (1+0)^3$

$= 8-1$

$= 7$ ②

Q5

a) $\int_1^5 \frac{7}{x^2}\,dx = \int_1^5 7x^{-2}\,dx$

$= \left[-7x^{-1}\right]_1^5$

$= \frac{-7}{5} + \frac{7}{1}$

$= 7 - 1\frac{2}{5}$

$= 5\frac{3}{5}$ ②

b) $\int_0^K 5x\,dx = 40$

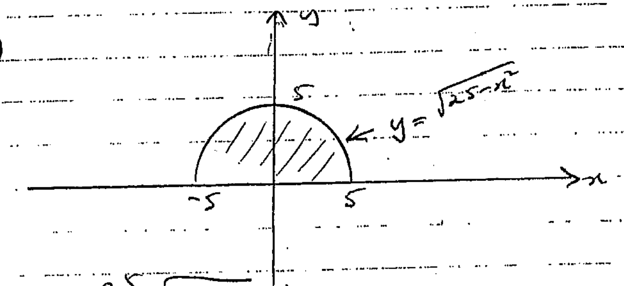
$\therefore \left[\frac{5}{2}x^2\right]_0^K = 40$

$\therefore \frac{5}{2}K^2 = 40$

$K^2 = 16$

$\therefore K = 4 \quad (K>0)$ ③

c) (i)



①

(ii) $\int_{-5}^{5} \sqrt{25-x^2}\,dx$ = area under semi-circle

$= \frac{1}{2}\pi r^2 \quad (r=5)$

$= \frac{25}{2}\pi$

$= 39.270$ (to 3 decpl) ②

(iii)

x	-5	-2.5	0	2.5	5
y	0	4.3301	5	4.3301	0

②

$\sqrt{25-(2.5)^2} = \sqrt{18.75} \doteq 4.3301$

(iv) $A \doteq \frac{1}{3}h\left[y_1 + y_5 + 2y_3 + 4(y_2+y_4)\right]$

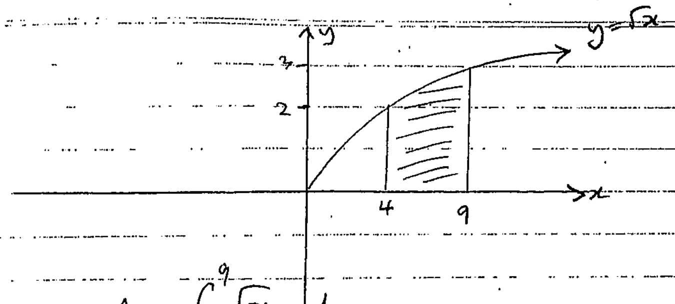
[or $\frac{1}{6}(b-a)\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]$ used twice]

$\doteq \frac{2.5}{3}\left[0+0+2\times5+4(8.6602)\right]$

$\doteq \frac{2.5}{3}(44.6408)$ ③

$\doteq 37.201$ (to 3 dec. pl.)

a) (i)



$$A=\int_4^9 \sqrt{x}\,dx$$

$$=\int_4^9 x^{\frac{1}{2}}\,dx$$

$$=\left[\frac{2}{3}x^{3/2}\right]_4^9$$

$$=2/3\left(9^{3/2}-4^{3/2}\right)$$

$$=2/3\left(9\times3-\cancel{4}\times2\right)$$

2

$$=2/3\left(27-8\right)$$

$$=2/3\times19$$

$$=\frac{38}{3}\ u^2$$

(ii) $$V=\pi\int_4^9 y^2\,dx$$

$$=\pi\int_4^9 x\,dx$$

$$=\pi\left[\frac{x^2}{2}\right]_4^9$$

$$=\pi\left(\frac{81}{2}-\frac{16}{2}\right)$$

$$=\frac{65\pi}{2}\ u^3$$

2