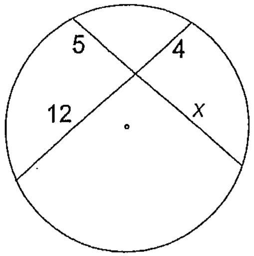
Year 12 : Extension 1 Mathematics
ASSESSMENT TASK 3
2009 [June 2009]
Time Allowed: 75 minutes

Topics: Integration by Substitution, Inverse Functions and Inverse Trigonometric Functions, Parametric Equations of the Parabola and Locus, Circle Geometry, Mathematical Induction (Divisibility & Inequalities)

Question 1 [13 marks]

(a) Find the value of x in the diagram. 2



(b) Differentiate with respect to x :

(i) $\sin^{-1}(3x)$ 2

(ii) $x^2 \tan^{-1} x$ 2

(c) Given $f(x) = 3 - 2x^2$ and $g(x) = 4x - 3$:

(i) find $g^{-1}(x)$ 2

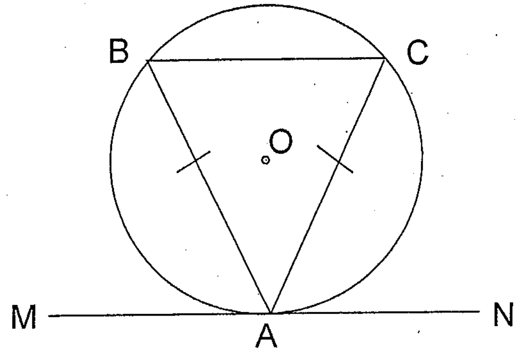
(ii) evaluate $f\left(g^{-1}(-5)\right)$ 2

(d) Find $\int (1-\sin x)^4 \cos x \, dx$ by using the substitution $u = 1 - \sin x$. 3

Question 2 [13 marks]

(a) In the diagram; $AB = AC$ and MN is a tangent to the circle centre O at A. 3

Prove that $MN \parallel BC$.



(b) Evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$ 3

(c) Find $\int x\sqrt{3+x^2}\, dx$ using the substitution $u = 3 + x^2$. 3

(d) (i) For the function $H(x) = x^2 - 8x$, state the largest domain containing the value $x = 5$ for which the function has an inverse function $H^{-1}(x)$. 1

(ii) Hence find this inverse function $H^{-1}(x)$. 3

Question 3 [13 marks]

(a) (i) State the formula used to find the equation of the chord of contact from the external point (x_1, y_1) to the parabola $x^2 = 4ay$. 1

(ii) Hence find the equation of the chord of contact of the tangents to the parabola $x^2 = y$ from the external point $(-2,-7)$. Express your answer in general form. 2

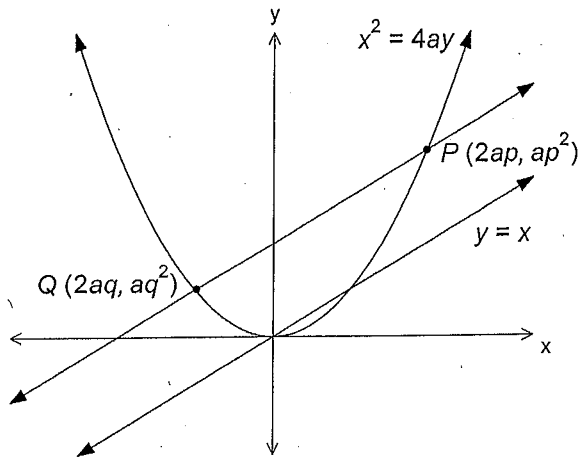
(b) Evaluate $\sin\left(2\cos^{-1}\left(\frac{4}{7}\right)\right)$. Express your answer in simplest exact form. 3

(c) (i) Show that $\frac{1}{x^2+1} - \frac{1}{x^2+3} = \frac{2}{(x^2+1)(x^2+3)}$. 1

(ii) Hence determine the value of $\int_{-1}^{\sqrt{3}} \frac{dx}{(x^2+1)(x^2+3)}$. 3

(d) Use mathematical induction to prove that $3^{2n} + 7$ is divisible by 8, for all integers $n \geq 1$. 3

Question 4 [13 marks]



A variable chord PQ of the parabola $x^2 = 4ay$ is such that it is always parallel to $y = x$. P and Q have coordinates $(2ap, ap^2)$ and $(2aq, aq^2)$ respectively $(p \neq q)$.

(i) Show that the gradient of PQ is $\frac{p+q}{2}$. 2

(ii) Hence show that $p + q = 2$. 1

(iii) Show that the gradient of the tangent at P is p. 1

(iv) Show that the equation of the normal at P is $x + py = 2ap + ap^3$. 2

(v) Write down the equation of the normal at Q. 1

(vi) Derive the coordinates of R, the point of intersection of these normals. 3

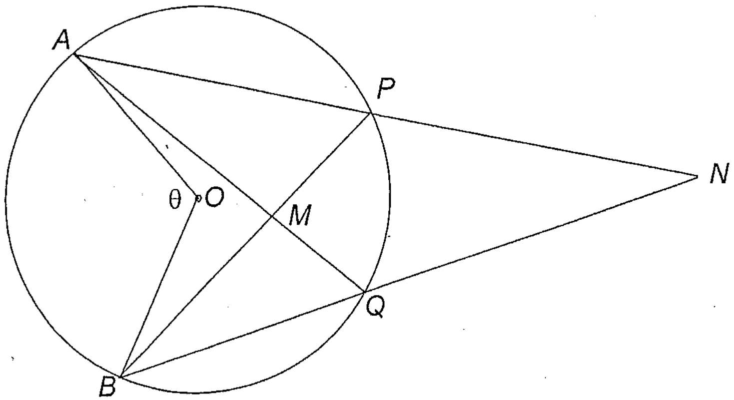
(vii) Find the Cartesian equation of the locus of R. 3

Question 5 [13 marks]

(a) (i) Sketch the curve $y=\cos^{-1}x$. **2**

(ii) Find the area bounded by the curve $y=\cos^{-1}x$, the x-axis, the y-axis and the line $x=\frac{1}{2}$. **3**

(b) Copy the diagram below onto your paper.



Given O is the centre of the circle above and $\angle AOB=\theta$, prove that :

(i) $\angle BPN=\angle AQN=180-\frac{\theta}{2}$ **3**

(ii) $\angle AMB+\angle ANB=\theta$ **2**

(c) State the domain and range for the function $y=\frac{3\sin^{-1}|4-x|}{2}$. **3**

SOLUTIONS : Question 1 [13 marks]

(a)

$$5\times x=4\times 12$$
$$x=\frac{48}{5}$$
i.e. $x=9\frac{3}{5}$

(b) (i)

$$\frac{d\left(\sin^{-1}(3x)\right)}{dx}=\frac{1}{\sqrt{1-(3x)^2}}\times 3=\frac{3}{\sqrt{1-9x^2}}$$

(b) (ii)

$$\frac{d\left(x^2\tan^{-1}x\right)}{dx}=x^2\times\frac{1}{1+x^2}+2x\times\tan^{-1}x=\frac{x^2}{1+x^2}+2x\tan^{-1}x$$

(c) (i)

$$x=4y-3$$
$$y=\frac{x+3}{4}$$
i.e. $g^{-1}(x)=\frac{x+3}{4}$

(c) (ii)

$$f\left(g^{-1}(-5)\right)=f\left(\frac{-5+3}{4}\right)=f\left(-\frac{1}{2}\right)=3-2\left(-\frac{1}{2}\right)^2=2\frac{1}{2}$$

(d)

$$u=1-\sin x$$
$$\frac{du}{dx}=-\cos x$$
$$\int(1-\sin x)^4\cos x\,dx=\int -u^4\,du=-\frac{u^5}{5}+C=-\frac{(1-\sin x)^5}{5}+C$$

SOLUTIONS : Question 2 [13 marks]

(a) ΔABC is isosceles ($AB = AC$, given)

$\angle ABC = \angle ACB$ (base $\angle$s of isosc. Δ)

$\angle ABC = \angle CAN$ ($\angle$ in alt. segment)

$\angle CAN = \angle ACB$ (both equal $\angle ABC$)

$\therefore MN \parallel BC$ (alt. angles only equal on parallel lines)

(b)

$$\int_1^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_1^{\sqrt{3}}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{6}$$

$$= \frac{\pi}{6}$$

(c)

$$u = 3 + x^2$$

$$\frac{du}{dx} = 2x$$

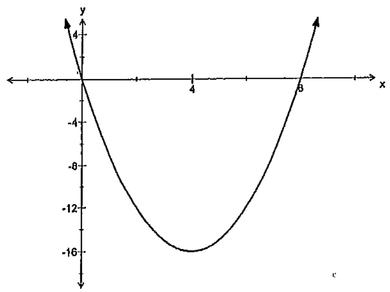
$$\int x\sqrt{3+x^2}\,dx = \int \frac{1}{2}\sqrt{u}\,du$$

$$= \frac{1}{2} \times \frac{2u^{\frac{3}{2}}}{3} + C$$

$$= \frac{(3+x^2)^{\frac{3}{2}}}{3} + C$$

$$= \frac{(3+x^2)\sqrt{3+x^2}}{3} + C$$

(d) (i) $H(x) = x^2 - 8x$



$\therefore x \geq 4$ is the largest positive domain for which an inverse function will exist.

(d) (ii)

$$x = y^2 - 8y$$

$$y^2 - 8y + 16 = x + 16$$

$$(y-4)^2 = x + 16$$

$$y = 4 \pm \sqrt{x+16}$$

Since $y \geq 4$, the inverse function is

$$H^{-1}(x) = 4 + \sqrt{x+16}$$

SOLUTIONS : Question 3 [13 marks]

(a) (i) $xx_1 = 2a(y + y_1)$

(a) (ii)

$$x^2 = y$$

$$4a = 1$$

$$a = \frac{1}{4}$$

$$x \times -2 = 2 \times \frac{1}{4} \times (y - 7)$$

$$-2x = \frac{1}{2}(y-7)$$

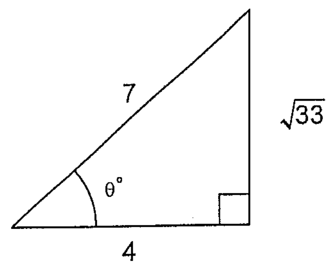
$$-4x = y - 7$$

$$4x + y - 7 = 0$$

(b)

let $y = \cos^{-1}\left(\frac{4}{7}\right)$

$$\cos y = \frac{4}{7}$$



$$\sin\left(2\cos^{-1}\left(\frac{4}{7}\right)\right) = \sin 2y$$

$$= 2\sin y \cos y$$

$$= 2 \times \frac{\sqrt{33}}{7} \times \frac{4}{7}$$

$$= \frac{8\sqrt{33}}{49}$$

(c) (i)

$$\frac{1}{x^2+1} - \frac{1}{x^2+3} = \frac{x^2+3}{(x^2+1)(x^2+3)} - \frac{x^2+1}{(x^2+3)(x^2+1)}$$

$$= \frac{x^2+3-x^2-1}{(x^2+1)(x^2+3)}$$

$$= \frac{2}{(x^2+1)(x^2+3)}$$

SOLUTIONS : Question 3 [continued]

(c) (ii)

$$\int_{-1}^{\sqrt{3}} \frac{dx}{(x^2+1)(x^2+3)} = \int_{-1}^{\sqrt{3}} \frac{1}{2}\left(\frac{1}{x^2+1} - \frac{1}{x^2+3}\right)dx$$

$$= \frac{1}{2}\left[\tan^{-1} x - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_{-1}^{\sqrt{3}}$$

$$= \frac{1}{2}\left(\tan^{-1}(\sqrt{3}) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) - \left(\tan^{-1}(-1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{3} - \frac{1}{\sqrt{3}} \times \frac{\pi}{4} - \left(-\frac{\pi}{4} - -\frac{1}{\sqrt{3}} \times \frac{\pi}{6}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{4\sqrt{3}} + \frac{\pi}{4} - \frac{\pi}{6\sqrt{3}}\right)$$

$$= \frac{21 - 5\pi\sqrt{3}}{72}$$

(d) Step 1 : Prove true for $n = 1$.

$3^{2\times1} + 7 = 3^2 + 7$

$= 9 + 7$

$= 16$

$= 8 \times 2$

$\therefore$ Divisible by 8 for n = 1

Step 2 : Assume true for $n = k$.

i.e. Assume $3^{2k} + 7 = 8A$ where A is some integer

$\therefore 3^{2k} = 8A - 7$

Step 3 : If true for $n = k$, prove true for $n = k+1$.

$3^{2(k+1)} + 7 = 3^{2k+2} + 7$

$= 3^2(3^{2k}) + 7$

$= 9(8A - 7) + 7$

$= 72A - 63 + 7$

$= 72A - 56$

$= 8(9A - 7)$

$= 8B$ where $B = 9A - 7$

$\therefore$ Divisible by 8 for $k+1$.

Step 4 : If true for $n = k$, proven true for $n = k+1$. Since proven true for $n = 1$, then true for $n = 2$. Since true for $n = 2$, then true for $n = 3$. And so on.

$\therefore$ True for all integers, $n \ge 1$

SOLUTIONS : Question 4 [13 marks]

(i)

$$m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq}$$

$$= \frac{a(p^2 - q^2)}{2a(p-q)}$$

$$= \frac{(p+q)(p-q)}{2(p-q)}$$

$$m_{PQ} = \frac{p+q}{2}$$

(ii) Since PQ is parallel to $y = x$, $m_{PQ} = 1$

i.e. $\frac{p+q}{2} = 1$

Hence $p + q = 2$

(iii)

$$y = \frac{x^2}{4a}$$

$$\frac{dy}{dx} = \frac{2x}{4a}$$

$$= \frac{x}{2a}$$

at P, $m_T = \frac{2ap}{2a}$

$= p$

(iv) Since $m_T = p$, $m_N = -\frac{1}{p}$

$$y - ap^2 = -\frac{1}{p}(x - 2ap)$$

$$py - ap^3 = -x + 2ap$$

$$x + py = 2ap + ap^3$$

(v) Equation of normal at Q is $x + qy = 2aq + aq^3$.

(vi)

$$\left.\begin{aligned} x + py &= 2ap + ap^3 \\ x + qy &= 2aq + aq^3 \end{aligned}\right\}$$

$$y(p-q) = 2a(p-q) + a(p^3 - q^3)$$

$$y(p-q) = 2a(p-q) + a(p-q)(p^2 + pq + q^2)$$

$$y = 2a + a(p^2 + pq + q^2)$$

$$= a(p^2 + pq + q^2 + 2)$$

$$x + pa(p^2 + pq + q^2 + 2) = 2ap + ap^3$$

$$x + ap^3 + ap^2q + apq^2 + 2ap = 2ap + ap^3$$

$$x = -ap^2q - apq^2$$

$$= -apq(p+q)$$

$\therefore R$ is $\left(-apq(p+q),\ a(p^2 + pq + q^2 + 2)\right)$

(vii) $x = -apq(p+q)$

$x = -2apq$

$pq = -\dfrac{x}{2a}$

$y = a\left(p^2 + pq + q^2 + 2\right)$

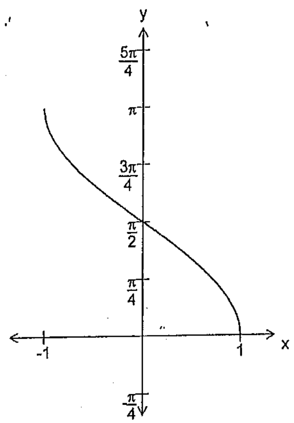
$= a\left((p+q)^2 - pq + 2\right)$

$= a\left(4 + \dfrac{x}{2a} + 2\right)$

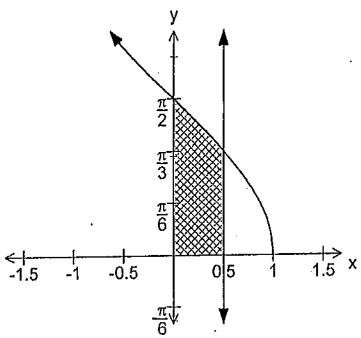
$y = \dfrac{x}{2} + 6a$

SOLUTIONS : Question 5 [13 marks]

(a) (i)



(a) (ii)



$A = \dfrac{\pi}{3} \times \dfrac{1}{2} + \displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos y \, dy$

$= \dfrac{\pi}{6} + \left[\sin y\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \dfrac{\pi}{6} + \left(\sin\dfrac{\pi}{2} - \sin\dfrac{\pi}{3}\right)$

$= \dfrac{\pi}{6} + 1 - \dfrac{\sqrt{3}}{2}$

(b) (i) $\angle BPA = \dfrac{1}{2}\angle AOB$ ($\angle$ at circumf. is half $\angle$ at centre)

$= \dfrac{\theta}{2}$

$\angle BPN = 180 - \dfrac{\theta}{2}$ (adj. supp. $\angle$s)

$\angle BQA = \dfrac{1}{2}\angle AOB$ ($\angle$ at circumf. is half $\angle$ at centre)

$= \dfrac{\theta}{2}$

$\angle AQN = 180 - \dfrac{\theta}{2}$ (adj. supp. $\angle$s)

$= \angle BPN$

(b) (ii) $\angle AMB = \angle PMQ$ (vert. opp. $\angle$s)

$\angle PMQ + \angle MQN + \angle QNP + \angle NPM = 360$ ($\angle$ sum of quad.)

$\angle AMB + 180 - \dfrac{\theta}{2} + \angle ANB + 180 - \dfrac{\theta}{2} = 360$

$\angle AMB + \angle ANB + 360 - \theta = 360$

$\angle AMB + \angle ANB = \theta$

(c) Domain :

$-1 \le |4 - x| \le 1$

$-1 \le 4 - x \le 1$

$-5 \le -x \le -3$

$5 \ge x \ge 3$

i.e. $3 \le x \le 5$

Range : $0 \le y \le \dfrac{3\pi}{4}$