SYDNEY GIRLS HIGH SCHOOL

2001

MATHEMATICS

3 UNIT

ASSESSMENT TASK 2

Topics : Exponential & Logarithmic Functions, Trigonometric Functions and Polynomials

Time allowed - 75 minutes

DIRECTIONS TO CANDIDATES

NAME [redacted]

- Attempt ALL questions.
- Questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

QUESTION ONE

a) A circle has a radius of 12 cm. Find ,in exact form the length of an arc that subtends an angle of 120° at the centre. [2]

b) Find the derivative of the following functions

i) $e^{4x}+3x$ [1]

ii) $\sin 3x$ [1]

iii) $x^2\log_e(2-5x)$ [2]

iv) $\tan^2 5x$ [1]

v) $\log_e\sqrt{x+1}$ [2]

c) Integrate the following

i) $\dfrac{4}{1+3x}$ [1]

ii) $\tan^2\left(\dfrac{x}{3}\right)$ [2]

iii) $5e^{2x+3}$ [1]

d) Solve $\log_3 4x = 2$ [2]

QUESTION TWO

a) Find the following definite integrals

i) $\int_1^e \frac{4}{x}\,dx$ [2]

ii) $\int_0^1 \frac{e^{2x}+1}{e^x}\,dx$ e^{-x} [2]

b) Consider the polynomial $p(x) = 6x^3 - 5x^2 - 2x + 1$

i) Show that one is the zero of $p(x)$ [1]
ii) Express $p(x)$ as a product of 3 linear factors [3]
iii) Solve the inequality $p(x) \le 0$ [3]

c) If $\tan\alpha = \frac{3}{4}$ and α is acute , and $\cos\beta = \frac{-2}{5}$ and β is obtuse , find the exact values of [4]

i) $\sin 2\alpha$
ii) $\cos(\alpha + \beta)$

QUESTION THREE

a) Prove that $\frac{\sin 2x}{1+\cos 2x} = \tan x$ [3]

b) $(x+1)(x-3)$ is a factor of $ax^3 - bx^2 - 5x + 2a$. Find a , b and the third factor of the polynomial. [4]

c) If $\cos\theta = \frac{8}{9}$, and θ is acute, find the exact value of $\tan\frac{\theta}{2}$ [2]

d) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ [2]

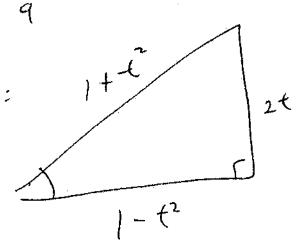
e) The gradient of a curve is given by $y' = \frac{2}{x+1}$ and the curve passes through the point $(0,1)$. What is the equation of this curve? [2]

f) Find the value of a if

$$\int_a^e \frac{1}{x}\,dx = 5$$ [2]

$\cos\theta = \frac{8}{9}$

θ :



QUESTION FIVE

a) The roots of $x^3 - 4x^2 - 9 = 0$ are α, β and γ. Find the value of

i) $\alpha^2 + \beta^2 + \gamma^2$ [2]

ii) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ [2]

iii) $(\alpha + 2)(\beta + 2)(\gamma + 2)$ [3]

b) Show that the cubic $x^3 - 12x + 11$ has one real root in the domain $0 \le x \le 2$. [2]

c) Find the acute angle between $y = x^4 + 2x + 2$ and $y = \frac{1}{x}$ at the point where $x = 1$. [3]

d) If $3\cot x = 2$, find the value of $\frac{5\sin x - 7\cos x}{\operatorname{cosec} x + \sec x}$ [3]

$\cot x = \frac{2}{3}$

$\tan x = \frac{3}{2}$

$3x^2 - 12 = 0$

$x^2 =$

$x^{-1} = -1x^{-2}$

$-\frac{1}{x^2}$

$\frac{4}{0}$

QUESTION FOUR

a) Sketch the curve $y = 4\cos x + 2$ in the domain $0 \le x \le 2\pi$. [2]

b) Solve for x

$$\log_a(x+2) - \log_a(2) = \log_a(x) + \log_a(2)$$ [2]

c) Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x + \alpha), R > 0$. Hence or otherwise find the solution for $\sqrt{3}\cos x - \sin x = 1$ in the domain $0 \le x \le 2\pi$. [4]

d) Solve $\cot 2x = \tan 2x$ for $0 \le x \le 2\pi$. [3]

e) Prove that $\tan(\frac{\pi}{4} + \frac{\theta}{2}) = \sec\theta + \tan\theta$ [4]

QUESTION SIX

a) Differentiate $x\log_e x$ and hence find $\int_{\sqrt{e}}^{e} \log_e x \, dx$ [4]

b)

i) Find the point of intersection of $y = 2\cos x$ and $y = \frac{1}{2}\sec x, 0 \le x \le \frac{\pi}{2}$. [2]

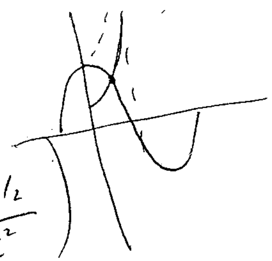
ii) The area between the two curves above is rotated about the x-axis. Find the volume of the solid of the revolution. [3]

c) Solve $4x^3 - 12x^2 + 11x - 3 = 0$ if the roots are in arithmetic progression. [3]

d) Show that $1 - \cos x = 2\sin^2 \frac{x}{2}$, hence or otherwise evaluate

$$\lim_{x \to 0} \frac{1 - \cos x}{3x^2}$$ [3]

THE END



$$\frac{2\sin^2 x/2}{3x^2} = \frac{2}{3}\left(\frac{\sin^2 x/2}{x^2}\right)$$

$$\frac{\sin x/2}{x} \cdot \frac{\sin x/2}{x} \quad 2\sin\frac{x/2}{x/2} \quad \frac{1}{2}$$

QUESTION ONE 86/90 1187 4924

EXCELLENT

(a) $l = r\theta$
$= 12 \cdot 2\pi/3$
$= 8\pi$ cm ✓

(b) i $= 4e^{4x} + 3$ ✓ 1
ii $= 3\cos 3x$ ✓ 1
iii $= 2x \log_e(2-5x) + x^2 (-5)(\log_e(2-5x))$
$= 2x \log_e(2-5x) - \cancel{5x^2(\log_e(2-5x))}$
$= \cancel{\log_e(2-5x)\ (2x - 5x^2)}$ X
$= \frac{-5x^2}{2-5x} + 2x \ln(2-5x)$ ✓ 1
iv $= 2\tan 5x \cdot 5\sec^2 5x = 10\tan 5x\sec^2 5x$ 1
v $= \frac{d}{dx}\frac{1}{2}\log_e(x+1) = \frac{1}{2(x+1)}$ ✓ 2

(c) i $= \frac{4}{3}\int \frac{3}{1+3x}\, dx = \frac{4}{3}\ln(1+3x) + c$ ✓ 1

ii $= 1 + \tan^2\frac{x}{3} = \sec^2\frac{x}{3}$
$\therefore \tan^2 x/3 = \sec^2 x/3 - 1$

$\therefore \int \sec^2 x/3 - 1\, dx = 3\tan x/3 - x + c$ ✓ 2

iii $= \frac{5}{2}e^{2x+3} + c$ ✓ 1

(d) $\log_3 4x = 2$ $\quad 3^2 = 4x$
$\therefore x = 9/4$ ✓ 2

1- 14
2- 15
3- 15
4- 15
5- 14
6- 13
7-
8-
9-
10-

QUESTION TWO

(a) i: $[4\ln x]_1^e = 4\log_e e - 4\log_e 1 = 4$ ✓ 2

ii = $\int_0^1 e^x + \frac{1}{e^x}\,dx = \int_0^1 e^x\,dx + \int_0^1 e^{-x}\,dx$

$= [e^x]_0^1 + [-e^{-x}]_0^1$

~~$e^1 + (-e^{-1}) - (-e^0) = e - \frac{1}{2} + 1$~~

$(e^1 - e^0) + (-e^{-1} + e^0) = e - \frac{1}{e}$ 2

(b) i = $P(1) = 6 - 5 - 2 + 1 = 0$ ✓ 1

∴ 1 is a zero of $p(x)$

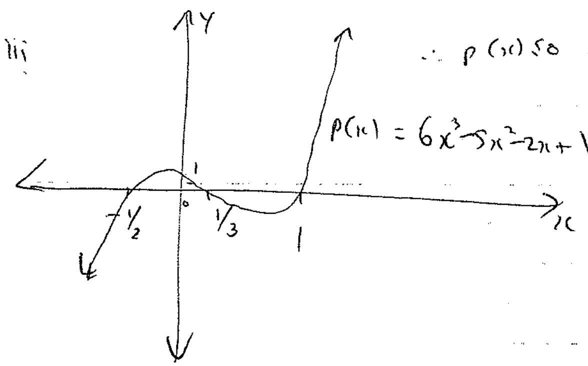
ii =

$$
\begin{array}{r}
6x^2 + x - 1 \\
x - 1 \overline{)\, 6x^3 - 5x^2 - 2x + 1} \\
-\;\underline{6x^3 - 6x^2} \\
x^2 - 2x \\
\underline{x^2 - x} \\
-x + 1 \\
\underline{-x + 1}
\end{array}
$$

$6x^2 + x - 1 = (3x - 1)(2x + 1)$ ✓ 3

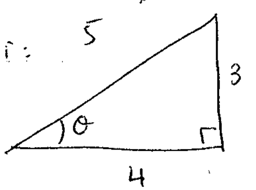
∴ $p(x) = (3x-1)(2x+1)(x-1)$

iii

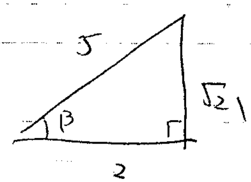


∴ $p(x) \le 0$ when $x \le -\frac{1}{2}$ or $\frac{1}{3} \le x \le 1$ ✓ 3

(c) i:



$\tan a = \frac{3}{4}$
$\sin a = \frac{3}{5}$
$\cos a = \frac{4}{5}$



$\cos\beta = -\frac{2}{5}$

$\sin\beta = \frac{\sqrt{21}}{5}$ (2nd Quadrant sin is positive)

$\tan\beta = -\frac{\sqrt{21}}{2}$ (2nd Quad tan is odd)

i = $\sin 2a = 2\sin a\cos a = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$ ✓ 2

ii = $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{4}{5} \times \left(-\frac{2}{5}\right) - \frac{3}{5}\left(\frac{\sqrt{21}}{5}\right)$

$= -\frac{8}{25} - \frac{3\sqrt{21}}{25}$

$= \frac{-8 - 3\sqrt{21}}{25}$ ✓ 2

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QUESTION THREE

(a) LHS $= \frac{\sin 2x}{1+\cos 2x}$

$= \frac{2\sin x\cos x}{1+(2\cos^2 x-1)}$

$= \frac{2\sin x\cos x}{2\cos^2 x}$

$= \tan x$

$=$ RHS ✓ 3

(b) $p(x) = ax^3 - bx^2 - 5x + 2a$

$p(-1) = -a - b + 5 + 2a = 0$

$\Rightarrow a - b + 5 = 0$

$p(3) = 27a - 9b - 15 + 2a = 0$

$\Rightarrow 29a - 9b = 15$

$- \quad 9a - 9b = -45$

$20a = 60$ ✓

$a = 3$ ✓

$\therefore b = 8$

$\therefore 3x^3 - 8x^2 - 5x + 6$ is $p(x)$ ✓ 4

$\therefore$ other factor is $(3x-2)$ ie $x = 2/3$

(c) $\cos\theta = \frac{1-t^2}{1+t^2}$

$\frac{8}{9} = \frac{1-t^2}{1+t^2}$

$8 + 8t^2 = 9 - 9t^2$

$17t^2 = 1$

$\therefore t^2 = \frac{1}{17}$

$\therefore t = \pm\frac{1}{\sqrt{17}}$

Now $\tan\theta/2 = t$

and since θ is acute

$\therefore t = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17}$ 2

(d) $\cos 2x = 2\cos^2 x - 1$

$\therefore \cos^2 x = \frac{\cos 2x + 1}{2}$

$\frac{1}{2}\int_0^{\pi/2} \cos 2x + 1 \; dx$

$\frac{1}{2}\left[\frac{1}{2}\sin 2x + x\right]_0^{\pi/2}$

$\frac{1}{2}\left[\left(0 + \frac{\pi}{2}\right) - (0 + 0)\right] = \frac{\pi}{4}$ ✓ 2

(e) $\int \frac{2}{x+1}\,dx = 2\ln(x+1) + c$

Now when $x = 0$ $y = 1$

$\therefore 1 = 2\ln(1) + c$

$\therefore c = 1$

$\therefore$ eqn of curve is $2\ln(x+1) + 1$ ✓ ✓

(f) $(\ln x)_a^e = 5$

$\log_e e - \log_e a = 5$

$1 - \log_e a = 5$

$\therefore 1 - 5 = \log_e a$

$-4 = \log_e a$ ✓ ✓

$e^{-4} = a$

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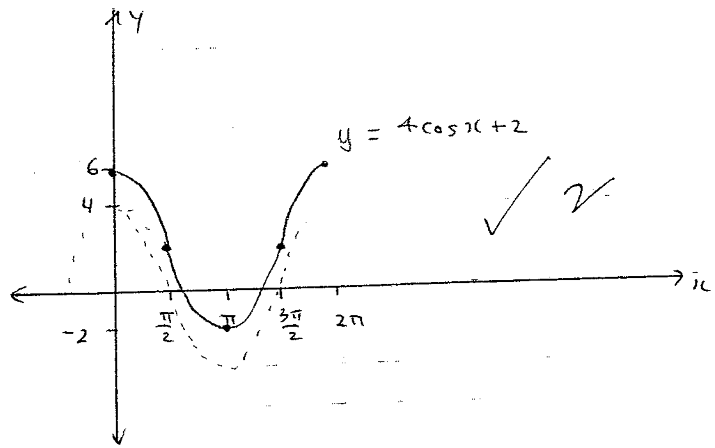
(a)



2

period 2π, amplitude 4, shift up 2

(b) $\log_a \frac{x+2}{2} = \log_a 2x$

$\frac{x+2}{2} = 2x$

$x + 2 = 4x$

$2 = 3x$

$\therefore x = 2/3$

2

(c) $\sqrt{3}\cos x - \sin x = 2\cos(x+\alpha)$

$2(\cos x\cos\alpha - \sin x \sin\alpha) = 2\cos x\cos\alpha - 2\sin x\sin\alpha$

$\therefore 2\cos\alpha = \sqrt{3}$ and $2\sin\alpha = 1$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \pi/6$

4

$\therefore 2\cos(x + \pi/6) = 1$

$\cos(x+\pi/6) = \frac{1}{2}$

$x + \pi/6 = \pi/3, \frac{5\pi}{3} \ldots \quad \therefore x = \pi/6$ or $3\pi/2$

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(d) $\frac{1}{\tan 2x} = \tan 2x$

$1 = \tan^2 2x$

$\therefore \tan 2x = \pm 1$

$2x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4, 9\pi/4, 11\pi/4, 13\pi/4,$

3

$x = \pi/8, 3\pi/8, 5\pi/8, 7\pi/8, 9\pi/8, 11\pi/8, 13\pi/8, 15\pi/8$

(e) $\tan(\pi/4 + \theta/2) = \frac{\tan\pi/4 + \tan\theta/2}{1 - \tan\pi/4\tan\theta/2}$

$= \frac{1+\tan\theta/2}{1-\tan\theta/2}$

$= \frac{1 + \frac{\sin\theta/2}{\cos\theta/2}}{1 - \frac{\sin\theta/2}{\cos\theta/2}}$

$= \frac{\frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2}}{\frac{\cos\theta/2 - \sin\theta/2}{\cos\theta/2}}$

$= \frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2 - \sin\theta/2}$

PTO

~~$= \frac{(\cos\theta/2 + \sin\theta/2)(\cos\theta/2 - \sin\theta/2)}{\cos^2\theta/2 + \sin^2\theta/2 - 2\sin\theta/2\cos\theta/2}$~~

~~$= \frac{\cos^2\theta/2 - \sin^2\theta/2}{1 - \sin\theta}$~~

~~$= \frac{\cos\theta}{1 - \sin\theta}$~~

$$= \frac{(\cos\theta/2 + \sin\theta/2)(\cos\theta/2 + \sin\theta/2)}{(\cos\theta/2 - \sin\theta/2)(\cos\theta/2 + \sin\theta/2)}$$

$$= \frac{\cos^2\theta/2 + \sin^2\theta/2 + 2\sin\theta/2\cos\theta/2}{\cos^2\theta/2 - \sin^2\theta/2}$$

$$= \frac{1 + \sin\theta}{\cos\theta}$$ ✓ 4

$$= \sec\theta + \tan\theta$$

13

QUESTION 5

(a) $\therefore \alpha+\beta+\gamma = 4$

$\Sigma\alpha\beta = 0$

$\alpha\beta\gamma = 9$

(i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2\Sigma\alpha\beta$

$= 16 - 2(0)$

$= 16$ ✓ ✓

(ii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}$

$= \frac{0}{9}$ ✓ ✓

$= 0$

(iii) $(\alpha+2)(\beta+2) = (\alpha\beta+2\beta+2\alpha+4)(\gamma+2)$

$= \alpha\beta\gamma + 2\beta\gamma + 2\alpha\gamma + 4\gamma + 2\alpha\beta + 4\beta + 4\alpha + 8$

$= \alpha\beta\gamma + 2\Sigma\alpha\beta + 4\Sigma\alpha + 8$

$= 9 + 2(0) + 4(4) + 8$

$= 33$ ✓ 3

(b)

$P(2) = 8 - 12(2) + 11$

$= -5$

$P(0) = 11$

$\therefore$ since sign changes there is a real root

$f(1) = 0$

$\therefore$ root is $x = 1$

✓ $\therefore$ there is one real root

$0 \le x \le 2$

(c) At $x=1$:

$\frac{dy}{dx}$ $x^4+2x+2 = 4x^3+2$

$\therefore$ m at $x=1$ is 6 when $x=1$ $y=5$

$\therefore 6 = \frac{y-5}{x-1}$

$6x-6 = y-5$

$\therefore 6x-1 = y$

$\therefore m_1 = 6$

$\frac{dy}{dx}\ \frac{1}{x} = \frac{-1}{x^2}$ at $x=1$ $m=-1$

at $x=1$ $y=1$

$-1 = \frac{y-1}{x-1}$

$-x+1 = y-1$

$-x+2 = y$

$\therefore m_2 = -1$

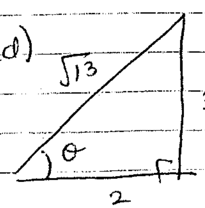
$\tan\theta = \left|\frac{6+(-1)}{1-6(-1)}\right|$

2

$= \left|\frac{5}{7}\right|$

$\therefore \theta = 35^\circ 32'$ (nearest whole minute)

(d)



$$\frac{\frac{5\cdot 3}{\sqrt{13}} - \frac{7\cdot 2}{\sqrt{13}}}{\frac{\sqrt{13}}{3} + \frac{\sqrt{13}}{2}} = \frac{\frac{1}{\sqrt{13}}}{\frac{2\sqrt{13}+3\sqrt{13}}{6}}$$

$$\frac{\frac{1}{\sqrt{13}}}{\frac{5\sqrt{13}}{6}} = \frac{1}{\sqrt{13}} \times \frac{6}{5\sqrt{13}} = \frac{6}{65}$$

✓3

QUESTION SIX

(a) $\frac{d}{dx}\, x\log_e x = x \cdot \frac{1}{x} + \log_e x$

$= 1 + \log_e x$

$\int \frac{d}{dx} x\log_e x \, dx = \int 1 \, dx + \int \log_e x \, dx$

$\int \frac{d}{dx} x\log_e x \, dx = \int \log_e x \, dx$

$\therefore x\log_e x - \int 1\, dx = \int \log_e x \, dx$

$x\log_e x - x = \int \log_e x \, dx$

$\int_{\sqrt{e}}^{e} \log_e x \, dx = \int_{\sqrt{e}}^{e} x\log_e x - x \, dx$

$= (e\log_e e - e) - (\sqrt{e}\log_e \sqrt{e} - \sqrt{e})$

$= (e - e) - \left(\sqrt{e} \cdot \frac{1}{2} - \sqrt{e}\right)$

$= \sqrt{e} - \frac{\sqrt{e}}{2}$

$= \frac{\sqrt{e}}{2}$

(b) ~~$r = 2\cos x = \frac{1}{2}\sec x$~~

~~$2\cos x = \frac{1}{2\cos x}$~~

~~$4\cos^2 x = 1$~~

~~$\cos x = \pm\frac{1}{2}$~~

~~$x = \pi/3$ in the restricted domain~~

(i) $2\cos x = \frac{1}{2}\sec x$

$2\cos x = \frac{1}{2\cos x}$

$4\cos^2 x = 1$

$\cos x = \pm\frac{1}{2}$

$\therefore x = \pi/3$ in the res domc

~~ii = Volume = $\pi \int_0^{\pi/3} \left(2\cos x - \frac{1}{2}\sec x\right) dx$~~

~~$= \pi \int_0^{\pi/3} \left(4\cos^2 x + \frac{1}{4\cos^2 x} - 2\right) dx$~~

~~$= \pi \int_0^{\pi/3} 2(\cos 2x + 1) + 4\sec^2 x - 2 \; d$~~

$0 \le x \le \frac{\pi}{2}$

$\text{Volume} = \frac{\pi}{2}\int_{-\pi/3}^{\pi/3} (2\cos x)^2 - \left(\frac{1}{2}\sec x\right)^2 dx$

$= \frac{\pi}{2}\int_{-\pi/3}^{\pi/3} \left(4\cos^2 x - \frac{1}{4}\sec^2 x - 2\right) dx$

$= \frac{\pi}{2}\int_{-\pi/3}^{\pi/3} \left(2\cos 2x + 2 - \frac{1}{4}\sec^2 x - 2\right) dx$

$= \frac{\pi}{2}\int_{-\pi/3}^{\pi/3} \left(2\cos 2x - \frac{1}{4}\sec^2 x\right) dx$

$= \frac{\pi}{2}\left[\sin 2x - \frac{1}{4}\tan x\right]_{-\pi/3}^{\pi/3}$

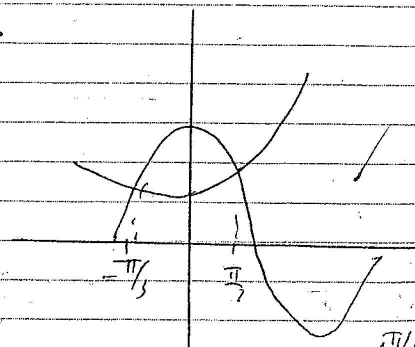
$= \frac{\pi}{2}\left[\left(\frac{\sqrt{3}}{2} - \frac{1}{4}\cdot\sqrt{3}\right) - \left(-\frac{\sqrt{3}}{2} - \frac{1}{4}(-\sqrt{3})\right)\right]$

$= \frac{\pi}{2}\left[\sqrt{3} - \frac{\sqrt{3}}{2}\right]$

$= \left(\frac{2\sqrt{3} - \sqrt{3}}{2}\right)\frac{\pi}{2}$

$= \frac{\sqrt{3}}{2} \cdot \frac{\pi}{2}$

$= \frac{\sqrt{3}}{4}\pi \text{ units}^3$



$\therefore \text{Volume} = \frac{\pi}{2}\int_{-\pi/3}^{\pi/3}$

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$$\pi \int_0^{\pi/3} 2\cos 2x + 4\sec^2 x \, dx$$

$$\pi \left[\sin 2x + 4\tan x \right]_0^{\pi/3}$$

$$\pi \left\{ \left[\frac{\sqrt{3}}{2} + 4\sqrt{3} \right] - (0 + 0) \right\}$$

$$\pi \left(\frac{\sqrt{3} + 8\sqrt{3}}{2} \right) = \pi \left(\frac{9\sqrt{3}}{2} \right) \text{ units}^3$$

(c) let roots be $a-d$, a and $a+d$

$$3a = -\frac{(-12)}{4} = 3$$
$$\therefore a = 1$$

$$a(a-d)(a+d) = 3/4$$
$$1 - d^2 = 3/4$$
$$\therefore d^2 = 1/4$$
$$d = \pm \tfrac{1}{2}$$

✓ 3

$\therefore$ roots are $1 - \frac{1}{2}$, 1, $1 + \frac{1}{2}$ $= \frac{1}{2}, 1, 3/2$

(d) $1 - \cos x = 1 - (1 - 2\sin^2 x/2)$
$= 2\sin^2 x/2$

$\frac{1/2}{1}$

$$\lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \lim_{x \to 0} \frac{2\sin^2 x/2}{3x^2}$$

$$= \frac{2}{3} \lim_{x \to 0} \frac{\sin x/2}{x} \cdot \frac{\sin x/2}{x}$$

$$= \frac{2}{3} \lim_{x \to 0} \frac{\sin x/2}{x} \cdot \frac{1/2}{\frac{1}{2}} \cdot \frac{\sin x/2}{x} \cdot \frac{1/2}{1/2}$$

$$= \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{2}$$
$$= \frac{1}{6}$$

✓ 2