SYDNEY GIRLS HIGH SCHOOL

2000

MATHEMATICS

4 UNIT - second paper

May Assessment

Time allowed - 90 minutes

DIRECTIONS TO CANDIDATES

NAME [redacted]

- Attempt ALL questions.
- Questions are not of equal value - part marks are shown
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

Question 1:

Find $\int \frac{d\theta}{1+\cos 2\theta}$ [3]

Question 2:

Find $\int \sqrt{\frac{2+x}{2-x}}.dx$ [4]

Question 3:

Find $\int \frac{dx}{\sqrt{4x^2 - 8x}}$ [5]

Question 4:

Find $\int \frac{x^3.dx}{(x+1)^2}$ [5]

Question 5:

Evaluate $\int_1^e \frac{\log_e x}{x^2}.dx$ [6]

Question 6:

Find $\int \frac{x^2.dx}{\sqrt{x^2+1}}$ [6]

Question 7:

Evaluate the following integral: $\int_3^4 \frac{6x.dx}{(x+1)^2(x-2)}$ [6]

Question 8:

Find $\int \frac{2dx}{x^3 - x^2 + x - 1}$ [6]

Question 9:

Find $\int \frac{5dx}{3 \sin x + 4 \cos x}$ [8]

Question 10:

a) Find a reduction formula for $\int \tan^n x.dx$ in terms of $\int \tan^{n-2} x.dx$ [5]

b) Evaluate $\int_0^{\frac{\pi}{4}} \tan^6 x.dx$ [3]

Question 11

Find the volume of a solid whose base is a circle of radius 3 cm and each cross section perpendicular to one of the diameter is a square. [6]

Question 12

The base of a solid is the area enclosed by $x^2 = 2y$ and $y = x$. If each cross section perpendicular to the Y axis is an equilateral triangle, find the volume of the solid. [7]

Question 13:

The area enclosed by the curve
$y = x^2 + 1$, the Y axis and the lines [illegible] and $y = 4x - 3$ is rotated about the Y axis.
Find the volume that is formed. $y = x$ [7]

Question 14

The area of the curve $y = \sin x$ enclosed by the X axis, $X = 0$ and $X = \pi$ is rotated about the line $x = 2\pi$. Find the volume that is formed. [7]

Question 15:

a) Show that the area enclosed by the parabola $x^2 = 4ay$ and its latus rectum is $\frac{8a^2}{3}$

(The latus rectum is the line passing through the focus, parallel to the X axis) [3]

b) A solid is a square with sides of 6 cm and has cross sectional areas perpendicular to the diagonal is that part of a parabola enclosed by its latus rectum, the latus rectum lying on the base of the solid. Find the volume of the solid. [5]

Question 16.

A circle centre (a,b) and radius r $(r<a)$ is rotated about the Y axis. Find the volume of the solid. [7]

4th, May 2000

1. $\int \frac{d\theta}{1+\cos 2\theta}$

$\cos 2\theta = 2\cos^2\theta - 1$

$\therefore 1+\cos 2\theta = 2\cos^2\theta$

$\therefore I = \int \frac{d\theta}{2\cos^2\theta} = \frac{1}{2}\int \sec^2\theta \, d\theta$

$= \frac{1}{2}\tan\theta + C$

2. $\int \sqrt{\frac{2+x}{2-x}} . dx \cdot \frac{\sqrt{2+x}}{\sqrt{2+x}}$

$= \int \frac{2}{\sqrt{4-x^2}} dx + \int \frac{x}{\sqrt{4-x^2}} dx$

$= 2\sin^{-1}\frac{x}{2} - \sqrt{4-x^2} + C$

3. $\int \frac{dx}{\sqrt{4x^2-8x}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2-2x+1-1}}$

$= \frac{1}{2}\int \frac{dx}{\sqrt{(x-1)^2-1}}$

$= \frac{1}{2}\ln\left(x-1+\sqrt{x^2-2x}\right) + C$

4. $\int \frac{x^3 . dx}{(x+1)^2}$

Now

$$\begin{array}{r|l} & x - 2 \\ \hline x^2+2x+1 & x^3+0x^2+0x+0 \\ & x^3+2x^2+x \\ \hline & -2x^2-x \\ & -2x^2-4x-2 \\ \hline & 3x+2 \end{array}$$

$\therefore I = \int (x-2)dx + \int \frac{3x+2}{x^2+2x+1} dx$

$= \int (x-2)dx + \frac{3}{2}\int \frac{2x+2}{x^2+2x+1} dx - \int \frac{1}{(x+1)^2} dx$

$= \frac{x^2}{2} - 2x + \frac{3}{2}\ln(x+1)^2 + \frac{1}{x+1} + C$

OR $= \frac{x^2}{2} - 2x + 3\ln(x+1) + \frac{1}{x+1} + C$

5. $\int_1^e \frac{\ln x}{x^2} dx$ let $u = \ln x$ $dV = \frac{dx}{x^2}$

$du = \frac{dx}{x}$ $V = -\frac{1}{x}$

$\therefore I = \left[-\frac{1}{x}\ln x\right]_1^e + \int_1^e \frac{1}{x^2} dx$

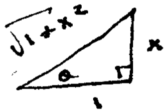
$= \left[-\frac{1}{x}\ln x - \frac{1}{x}\right]_1^e$

$= \left[\frac{1}{x}\ln x + \frac{1}{x}\right]_e^1$

$= 1 - \frac{1}{e}\ln e - \frac{1}{e} = 1 - \frac{2}{e}$

6. $\int \frac{x^2 dx}{\sqrt{x^2+1}}$ let $x = \tan\theta$

$\therefore dx = \sec^2\theta \, d\theta$



$I = \int \frac{\tan^2\theta . \sec^2\theta . d\theta}{\sqrt{\tan^2\theta+1}}$

$= \int \tan^2\theta \sec\theta . d\theta$

$= \int (\sec^2\theta - 1)\sec\theta . d\theta$

$= \int (\sec^3\theta - \sec\theta) d\theta$

For $\int \sec^3\theta . d\theta$ let $u = \sec\theta$ $dV = \sec^2\theta d\theta$

$du = \sec\theta\tan\theta . d\theta$ $V = \tan\theta$

$I = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta . d\theta$

$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1) d\theta$

$= \sec\theta\tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta$

$\therefore 2\int \sec^3\theta \, d\theta = \sec\theta\tan\theta + \ln(\sec\theta+\tan\theta) + c$

$\therefore \int \sec^3\theta . d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln(\sec\theta+\tan\theta) + C$

$\therefore \int (\sec^3\theta - \sec\theta) d\theta$

$= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln(\sec\theta+\tan\theta) + C$

$= \frac{x}{2}\sqrt{1+x^2} - \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right) + C$

7. $\int_3^4 \frac{6x\,dx}{(x+1)^2(x-2)}$

let $\frac{6x}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$

$\therefore 6x = A(x+1)(x-2) + B(x-2) + C(x+1)^2$

at $x=2$, $12 = 9C$, $C = 4/3$

at $x=-1$, $-6 = -3B$, $B = 2$

Equate x^2, $0 = A+C$, $A = -4/3$

$\therefore I = \int_3^4 -\frac{4}{3} \cdot \frac{1}{x+1} dx + \frac{4}{3}\int_3^4 \frac{1}{x-2} dx + \int_3^4 \frac{2}{(x+1)^2} dx$

$= \left[\frac{4}{3}\ln\left(\frac{x-2}{x+1}\right) - \frac{2}{(x+1)}\right]_3^4$

$= \frac{4}{3}\ln\left(\frac{2}{5}\right) - \frac{2}{5} - \frac{4}{3}\ln\left(\frac{1}{4}\right) + \frac{1}{2}$

$= \frac{4}{3}\ln 8/5 + \frac{1}{2} - 2/5$

$= 4/3 \ln 8/5 + \frac{1}{10}$

8. $\int \frac{2dx}{x^3-x^2+x-1}$

$x^3-x^2+x-1 = x^2(x-1)+1(x-1)$

$= (x^2+1)(x-1)$

let $\frac{2}{x^3-x^2+x-1} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$

$\therefore 2 = (Ax+B)(x-1) + C(x^2+1)$

at $x=1$, $2 = 2C$ $\therefore C = 1$

Equate x^2: $0 = A+C$, $A = -1$

at $x=0$, $2 = -B+C$, $\therefore B = -1$

$\therefore I = \int \left(\frac{1}{x-1} - \frac{x}{x^2+1} - \frac{1}{x^2+1}\right) dx$

$= \ln(x-1) - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x + C$

9. $\int \frac{5dx}{3\sin x + 4\cos x}$ let $t = \tan\frac{x}{2}$

$\therefore dx = \frac{2dt}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

$\therefore I = \int \frac{\frac{10dt}{1+t^2}}{\frac{6t}{1+t^2} + \frac{4-4t}{1+t^2}}$

$= \int \frac{5dt}{3t+2-2t^2}$

$= \int \frac{-5dt}{2t^2-3t-2} = -\int \frac{5dt}{(2t+1)(t-2)}$

let $\frac{-5}{(2t+1)(t-2)} = \frac{A}{2t+1} + \frac{B}{t-2}$

$\therefore -5 = A(t-2) + B(2t+1)$

at $t=2$, $-5 = 5B$, $\therefore B = -1$

at $t = -1/2$, $-5 = -2\frac{1}{2}A$, $\therefore A = 2$

$\therefore I = \int\left(\frac{2}{2t+1} - \frac{1}{t-2}\right)dt$

$= \ln\left(\frac{2t+1}{t-2}\right) + c$

$= \ln\left(\frac{2\tan x/2 + 1}{\tan x/2 - 2}\right) + c$

10. $\int \tan^n x\,dx = \int \tan^{n-2}x . \tan^2 x\,dx$

$= \int \tan^{n-2}x . (\sec^2 x - 1)\,dx$

$= \int \tan^{n-2}x . \sec^2 x\,dx - \int \tan^{n-2}x . dx$

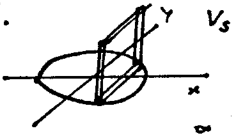
$= \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x . dx$

$\therefore \int_0^{\pi/4} \tan^6 x\,dx = \left[\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x\right]_0^{\pi/4}$

$= \frac{1}{5} - \frac{1}{3} + 1 - \pi/4$

$= \frac{13}{15} - \frac{\pi}{4}$

11.



V_{slice}, $\delta V = \ell^2 \delta x$

$= 4y^2 \delta x$

$= 4(9-x^2)\delta x$

$V_{solid}, V = \sum_{i=0}^{\infty} 4(9-x_i^2)\delta x$

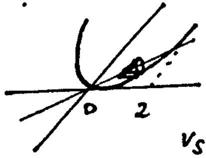
$= \int_{-3}^{3}(36-4x^2)\,dx$

$= 2\int_0^3 (36-4x^2)\,dx$

$= 2\left[36x - \frac{4}{3}x^3\right]_0^3$

$= 2(108-36)$, $\therefore Vol = 144\,cm^3$

12.



$Y = X$

$X^2 = 2Y$

$\therefore$ meet at $x=0, x=2$

$V_{slice} = \frac{1}{2}ab\sin c\,\delta y$

$a=b = x_1 - x_2$

$= \sqrt{2y} - y$

$\delta V = \frac{1}{2}(\sqrt{2y}-y)^2 . \sqrt{3}/2\,\delta y$

$= \frac{\sqrt{3}}{4}(2y - 2\sqrt{2}y^{3/2} + y^2)\delta y$

$V_{solid}\ V = \sum_{i=0}^{n} \frac{\sqrt{3}}{4}(2y_i - 2\sqrt{2}y_i^{3/2} + y_i^2)\delta y$

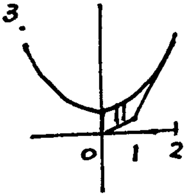
$= \frac{\sqrt{3}}{4}\int_0^2 (2y - 2\sqrt{2}y^{3/2} + y^2)\,dy$

$= \frac{\sqrt{3}}{4}.\left[y^2 - 2\sqrt{2}\frac{2}{5}y^{5/2} + y^3/3\right]_0^2$

$= \frac{\sqrt{3}}{4}\left(4 - \frac{32}{5} + \frac{8}{3}\right) = \frac{\sqrt{3}}{4}\left(\frac{60-96+40}{15}\right) = \sqrt{3}/15$

$\therefore$ Volume $= \sqrt{3}/15\ u^3$

13.



$y = x^2+1$, $y = x$, $y = 4x-3$

$y = x$ & $y = 4x-3$ meet at $(1,1)$

$y = x^2+1$ & $y = 4x-3$ " " $(2,5)$

$\delta V_1 = \pi R^2 H - \pi r^2 h$

$= \pi(y_1-y_2)\{(x+\delta x)^2 - x^2\}$

$= \pi(x^2+1-x)(2x\delta x$, $\delta x^2 \doteqdot 0)$

$= \pi(2x^3-2x^2+2x)\delta x$

$\delta V_2 = \pi R^2 H - \pi r^2 h$

$= \pi(y_1-y_2)\{(x+\delta x)^2 - x^2\}$

$= \pi(x^2+1-4x+3).2x\,dx$

$= \pi(x^2-4x+4)2x\,\delta x$

$= \pi(x^3-4x^2+4x)\delta x$

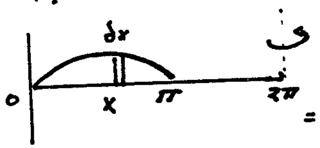
$\therefore V_1 = \pi\left(\frac{x^4}{2} - \frac{2}{3}x^3 + x^2\right)\Big]_0^1$ $\quad V_2 = 2\pi\left[\frac{x^4}{4} - \frac{4}{3}x^3 + \right.$

$= \pi\left(\frac{1}{2} - \frac{2}{3} + 1\right) + 2\pi\left\{4 - \frac{32}{3} + 8 - \frac{1}{4} + \frac{4}{3} - 2\right\}$

$= \pi\left(\frac{5}{6}\right) + 2\pi\left(5/12\right)$

$\therefore$ Volume $= \frac{5\pi}{6}\ u^3$.

14.



$V_{shell} = \pi R^2 H - \pi r^2 H$

$= \pi y\{(2\pi - x)^2 - (2\pi - x - \delta x)^2\}$

$= \pi y\{4\pi - 2x\}\delta x$, $\delta x^2 \doteqdot 0$

$= \pi\sin x(4\pi - 2x)\delta x$

$\therefore V_{solid}\ V = \sum \pi\sin x(4\pi - 2x)\delta x$

$= \pi\int_0^{\pi}(4\pi\sin x - 2x\sin x)\,dx$

For $\int 2x\sin x\,dx$. Let $u = 2x$ $\quad dv = \sin x\,dx$

$du = 2dx$ $\quad v = -\cos x$

$I = -2x\cos x + \int 2\cos x\,dx$

$= -2x\cos x + 2\sin x + c$

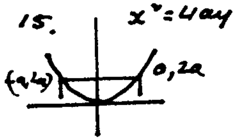
$\therefore V_{solid} = \pi\left[-4\pi\cos x + 2x\cos x - 2\sin x\right]_0^{\pi}$

$= \pi\{(4\pi - 2\pi - 0) - (-4\pi + 0)]$

$= \pi[2\pi + 4\pi]$

$\therefore$ Volume $= 6\pi^2\ u^3$

15.



$x^2 = 4ay$ $\quad A = LB - 2\int_0^{2a}\frac{x^2}{4a}\,dx$

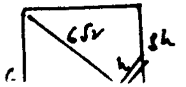
$= 4a^2 - 2\left[\frac{x^3}{12a}\right]_0^{2a}$

$= 4a^2 - \frac{1}{6a}.8a^3 = \frac{8a^2}{3}$

V_{slice}, $\delta V = \frac{8a^2}{3}\delta h$, $4a = 2h$

$= \frac{8}{3}.\frac{h^2}{4}\delta h = \frac{2}{3}h^3\delta h$

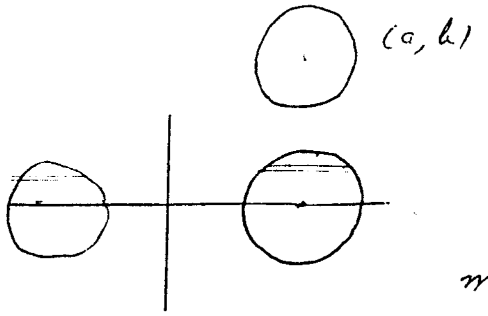
$V_{solid} = 2\int_0^{3\sqrt{2}}\frac{2}{3}h^2\,dh$



$= 2\frac{2}{3}.\left[\frac{h^3}{3}\right]_0^{3\sqrt{2}}$

$2\ 2\ 3^3.2\sqrt{2} = 24\sqrt{2}$

16.



$(x-a)^2 + (y-b)^2 = r^2$

Move circle to $(a, 0)$

$\therefore$ Circle is $(x-a)^2 + y^2 = r^2$

$V_{slice}, \quad \delta V = (\pi R^2 - \pi r^2)\,\delta y.$

$R = x_1, \quad r = x_2.$

So $(x-a)^2 = r^2 - y^2$

$x - a = \pm\sqrt{r^2 - y^2}$

$x = a \pm \sqrt{r^2 - y^2}$

$\therefore x_1 = a + \sqrt{r^2 - y^2}, \qquad x_2 = a - \sqrt{r^2 - y^2}$

$\therefore \delta V = \pi\{R + r\}\{R - r\}\,\delta y$

$= \pi\{2a\}\, 2\sqrt{r^2 - y^2}\,\delta y$

$= 4\pi a\sqrt{r^2 - y^2}\,\delta y$

$\therefore V_{solid}, \quad V = \sum_{i=0}^{\infty} 4\pi a\sqrt{r^2 - y^2}\,\delta y$

$= 4\pi a \int_{-r}^{r} \sqrt{r^2 - y^2}\,dy$

$= 4\pi a \cdot \frac{1}{2}\pi r^2$ (semi-circle)

$= 2\pi^2 a r^2$