COMPLEX NUMBERS 10 February 2005

The Complex Roots of Unity

Question One

If $z^5 = 1$,

a) Use de Moivre's theorem to find the five complex solutions in mod/arg form.
b) Plot these solutions on an Argand diagram as $1, z_1, z_2, z_3, z_4$
c) If $z_1 = \alpha$, show that $z_2 = \alpha^2, z_3 = \alpha^3$ etc
d) Show $\alpha^5 = 1$
e) Show that $1+\alpha+\alpha^2+\alpha^3+\alpha^4 = 0$
f) If $z_2 = \beta$, express all the solutions in terms of β
g) The roots plotted in part b) form a pentagon. Find the length of each side and the area of the pentagon.
h) Form the quadratic equation with roots $\alpha+\alpha^4$ and $\alpha^2+\alpha^3$

We **note** the following about the complex roots of unity (i.e. one);

a) The roots of $z^n = 1$ are; $z = cis\left(\frac{2k\pi}{n}\right), k = 0, 1, 2, \dots (n-1)$
b) On an Argand diagram, the *n*th roots of 1 are the vertices of a regular polygon of *n* sides, centered at the origin. The distance from the origin to each vertex is 1 unit.
c) If *n* is even, two roots of $z^n - 1 = 0$ are real (± 1).
d) If *n* is odd, only one root of $z^n - 1 = 0$ is real (1)
e) If *w* is the complex root with the smallest positive argument, then the complete set of roots is $1, w, w^2 \dots, w^{n-1}$
g) $1 + w + w^2 + \dots + w^{n-1} = 0$ (Either sum roots or sum GP)

The Complex Roots of -1

Question Two

If $z^5 = -1$;

a) Use de Moivre's theorem to find the five complex solutions in mod/arg form
b) Plot these solutions on an Argand diagram as $-1, z_1, z_2, z_3, z_4$
c) If α is a complex root with the smallest positive argument show the other roots are $\alpha^3, \alpha^5, \alpha^7, \alpha^9$

Set Work

1. Patel Exercise 4I, questions 7-9 plus
2. Given the following;
   a) <u>If *n* odd</u>

   $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \dots + z + 1)$

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E.g. $(z^5-1)=(z-1)(z^4+z^3+z^2+z+1)$

b) <u>If n even</u>

$z^n-1=(z-1)(z+1)(z^{n-2}+z^{n-4}+...+z^2+1)$

E.g. $z^8-1=(z-1)(z+1)(z^6+z^4+z^2+1)$

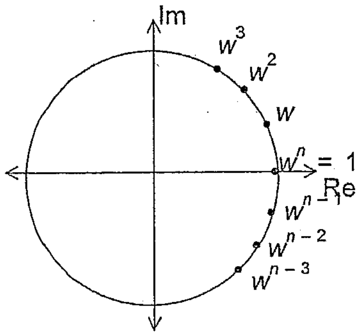
Find; a) z^4-1 b) z^3-1 c) z^6-1 d) z^7-1 e) $z^{10}-1$

Hints on Exercise 4I questions 7 – 9

7 a, b Similar to solving $z^5=1$ above. Note that $w^3=-1$.

7c use you answers from 7b

8a If w is a complex root of $z^n-1=0$ then $w^n=1$. The roots are shown on an Argand Diagram below.



Thus $(z^n-1)=(z-w^n)(z-w)..........(z-w^{n-2})(z-w^{n-1})$

8b Assume n odd then from the supplementary notes for sheets 9-10 use

$z^n-1=(z-1)(z^{n-1}+z^{n-2}+....+z+1)$

8c Substitute $z = 1$ into the expression deduced in 8b

9 Refer to Complex Numbers 5, 6.

9a Use you answers from Q7

9b 1.First factorise (z^6-1) using the difference of two squares.

2.Now complete the factorisation using the sum and difference of two cubes.

3. Divide both sides by $(x-1)(x+1)$ to obtain z^4+z^2+1.

4.Substitute answers from 9a to complete the question.

Answers to 2 above

a) $(z^4-1)=(z-1)(z+1)(z^2+1)$ b) $(z^3-1)=(z-1)(z^2+z+1)$

c) $(z^6-1)=(z-1)(z+1)(z^4+z^2+1)$

d) $(z^7-1)=(z-1)(z+1)(z^6+z^5+z^4+z^3+z^2+z+1)$

e) $(z^{10}-1)=(z+1)(z-1)(z^8+z^6+z^4+z^2+1)$