Sydney Girls High School

2012 HSC Assessment Task 2

MATHEMATICS

Extension 2

Time Allowed: 90 minutes

Topic: **Complex Numbers**

Total Marks: 90

Directions to Candidates:

- There are SIX (6) questions, of equal value.
- All questions must be attempted.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Write on one side of the paper only.
- Start each question on a new page.
- Diagrams are NOT to scale.

HSC STANDARD INTEGRAL SHEET

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \pm 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a \neq 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\ x > 0$

Question 1: (15 marks)

a) If $z_1 = 2+5i$ and $z_2 = 3-i$, find:

(i) $z_1 - 2z_2$ **1**

(ii) $(z_1)^2$ **2**

b) Solve $x^2 - 2x + 5 = 0$. **2**

c) Form a quadratic equation with roots $2 \pm 3i$. **2**

d) Express $\dfrac{3+4i}{2-3i}$ in the form $x+iy$. **2**

e) Simplify $(i)^{2012}$. **2**

f) Solve for a and b: $\dfrac{2}{a+ib} + \dfrac{3}{2+i} = \dfrac{9-4i}{5}$. **4**

START QUESTION 2 ON A NEW PAGE

Question 2: (15 marks)

a) Simplify $\dfrac{8cis(50°)}{2cis(-70°)}$, expressing your answer in the form $x+iy$. **2**

b) (i) Express $-1-\sqrt{3}i$ in mod-arg form. **2**

(ii) Hence find $\left(-1-\sqrt{3}i\right)^7$ in Cartesian form. **3**

c) Find $\sqrt{4-4i\sqrt{3}}$ in the form $a+ib$ where a and b are real. **4**

d) **Sketch** the following loci on an Argand diagram:

(i) $|z-2| = 2$ **2**

(ii) $\text{Im}(z) = |z|$ **2**

START QUESTION 3 ON A NEW PAGE

Question 3: (15 marks)

a) (i) Show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ 4

(ii) Hence, solve the equation $32x^5 - 40x^3 + 10x - 1 = 0$. 4

b) If ω is a complex cube root of unity, find the value of:

(i) $1 + \omega + \omega^2$ 1

(ii) $(1 + 2\omega + 3\omega^2)(1 + 2\omega^2 + 3\omega)$ 3

c) On an Argand diagram, shade in the region determined by the inequalities $2 \le \text{Im}(z) \le 4$ and $-\frac{\pi}{4} \le \arg z \le \frac{\pi}{4}$. 3

y

START QUESTION 4 ON A NEW PAGE

Question 4: (15 marks)

a) (i) Given that $z = \cos\theta + i\sin\theta$ show that $z^n - z^{-n} = 2i\sin n\theta$. 2

(ii) Express $\sin^4\theta$ in terms of multiples of θ. 3

(iii) Hence find $\int \sin^4\theta \, d\theta$. 2

b) (i) Find the five fifth roots of -1 in mod-arg form. 4

(ii) Hence, factorise $z^5 + 1$ into real linear and quadratic factors. 4

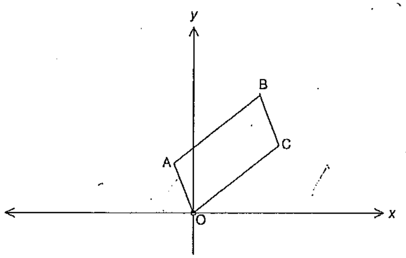
START QUESTION 5 ON A NEW PAGE

Question 5: (15 marks)

a) Form the quadratic equation with roots $1+\omega, 1+\omega^2$ where ω is the solution to $z^3-1=0$ with the smallest positive argument. **2**

b) Find z (where $z=a+ib$) such that $|z|^2-iz=16-2i$. **4**

c) In the diagram below, $OABC$ is a parallelogram with $OA=\frac{1}{2}OC$.



The point A represents the complex number $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$. If $\angle AOC=60°$, what complex number is represented by:

(i) C? **3**

(ii) B? **1**

QUESTION 5 CONTINUES ON THE NEXT PAGE

d) z_1 and z_2 are two complex numbers with $z_1=z_2 cis\theta$.

$O(0,0)$, $A(z_1)$, $B(z_1+z_2)$ and $C(z_2)$ are points on the Argand diagram.

Show that:

(i) $\arg\left(\dfrac{z_1-z_2}{z_1+z_2}\right)=\dfrac{\pi}{2}$ **2**

(ii) $\arg\left(1+\dfrac{z_1}{z_2}\right)=\dfrac{\theta}{2}$ **3**

START QUESTION 6 ON A NEW PAGE

Question 6: (15 marks)

a) Find the Cartesian equation of $|z-1|=|z+2|$. **2**

b) Find the least positive integer k such that $\cos\left(\frac{4\pi}{7}\right)+i\sin\left(\frac{4\pi}{7}\right)$ is a solution of $z^k=1$. **2**

c) Prove that if $3|z+1|^2=|z-1|^2$, then $|z+2|^2=3$. **3**

d) By considering the roots of $z^6-1=0$, solve $z^4+z^2+1=0$ writing the answer in the form $r\text{cis}\theta$. **4**

e) If z lies on the unit circle and $\arg z = 2\theta$:

(i) show that $|z^2-z|=|z-1|$ **2**

(ii) Find $\arg(z^2-z)$ in terms of π and θ. **2**

END OF TEST

Question 1:

a) $z_1 = 2 + 5i \qquad z_2 = 3 - i$

i) $z_1 - 2z_2 = 2 + 5i - 2(3 - i)$
$= 2 + 5i - 6 + 2i$
$= -4 + 7i$ (1)

ii) $(z_1)^2 = (2+5i)^2$
$= 4 + 20i + 25i^2$ (1)
$= 4 + 20i - 25$
$= -21 + 20i$ (2)

b) $x^2 - 2x + 5 = 0$

$x = \dfrac{2 \pm \sqrt{4 - 20}}{2}$

$= \dfrac{2 \pm \sqrt{-16}}{2}$

$= \dfrac{2 \pm 4i}{2}$ (1)

$= 1 \pm 2i$ (2)

c) $x^2 - (2 + 3i + 2 - 3i)x + (2+3i)(2-3i) = 0$

$x^2 - \underset{(1)}{4x} + \underset{(2)}{13} = 0$

d) $\dfrac{3 + 4i}{2 - 3i} \times \dfrac{2 + 3i}{2 + 3i} = \dfrac{6 + 9i + 8i + 12i^2}{4 + 9}$ (1)

$= \dfrac{-6 + 17i}{13}$ (2)

$= \dfrac{-6}{13} + \dfrac{17i}{13}$

e) $i^{2012} = (i^2)^{1006}$
$= (-1)^{1006}$ (1)
$= 1$ (2)

f) $\dfrac{2}{a+ib} + \dfrac{3}{2+i} = \dfrac{9}{5} - \dfrac{4i}{5}$

$\dfrac{2}{a+ib} = \dfrac{9-4i}{5} - \dfrac{3}{2+i}$

$\dfrac{2}{a+ib} = \dfrac{9-4i}{5} - \dfrac{3(2-i)}{(2+i)(2-i)}$ (1)

$\dfrac{2}{a+ib} = \dfrac{9-4i}{5} - \dfrac{3(2-i)}{5}$

$\dfrac{2}{a+ib} = \dfrac{3-i}{5}$ (2)

$\dfrac{a+ib}{2} = \dfrac{5}{3-i}$

$a + ib = \dfrac{10}{3-i} \times \dfrac{3+i}{3+i}$ (3)

$a + ib = \dfrac{30 + 10i}{10}$

$a + ib = 3 + i$ (4)

$a = 3, \quad b = 1$

V. poorly answered question.

ii) $4 \text{ cis } 120°$

$= 4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$

$= -2 + 2\sqrt{3}\, i$

b)(i) $|-1-\sqrt{3}\, i|$

$= \sqrt{1^2 + (\sqrt{3})^2}$

$= 2$

$\tan\theta = \frac{-\sqrt{3}}{-1}$

$= \sqrt{3}$

$\theta = -120°$

(ii) $2^7 \text{ cis}(7 \times -120°)$

$= 128 \text{ cis}(-840°)$

$= 128 \text{ cis}(-120°)$

$= 128\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)$

$= -64 - 64\sqrt{3}\, i$

c) $(a+ib)^2 = 4 - 4i\sqrt{3}$

$a^2 - b^2 = 4 \qquad 2ab = -4\sqrt{3}$

$b = \frac{-2\sqrt{3}}{a}$

$a^2 - \frac{12}{a^2} = 4$

$a^4 - 4a^2 - 12 = 0$

$(a^2-6)(a^2+2) = 0$

$a = \pm\sqrt{6}$

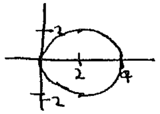
when $a = \sqrt{6}$, $b = \frac{-2\sqrt{3}}{\sqrt{6}}$

$= -\frac{2}{\sqrt{2}}$

" $a = -\sqrt{6}$, $b = \frac{-2\sqrt{3}}{-\sqrt{6}}$

$= \frac{2}{\sqrt{2}}$

d) (i)

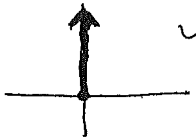


(ii) $y = \sqrt{x^2+y^2}$

$y^2 = x^2 + y^2$

$x^2 = 0$

$x = 0, \quad y \geq 0$



Question 3

(a)(i)

$$(cis\theta)^5 = c^5+5c^4is+10c^3i^2s^2+10c^2i^3s^3+5ci^4s^4+i^5s^5$$
$$cis5\theta = c^5-10c^3s^2+5cs^4+i\left(5c^4s-10c^2s^3+s^5\right)$$

Equating real parts

$$\cos5\theta = c^5-10c^3s^2+5cs^4$$
$$=c^5-10c^3\left(1-c^2\right)+5c\left(1-c^2\right)^2$$
$$=c^5-10c^3+10c^5+5c-10c^3+5c^5$$
$$\cos5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta$$

(a)(ii)

$$16x^5-20x^3+5x=\frac{1}{2}$$

let $x=\cos\theta$

$$\cos5\theta=\frac{1}{2}$$
$$5\theta=\frac{\pi}{3},\frac{5\pi}{3},\frac{7\pi}{3},\frac{11\pi}{3},\frac{13\pi}{3}....$$
$$\theta=\frac{\pi}{15},\frac{\pi}{3},\frac{7\pi}{15},\frac{11\pi}{15},\frac{13\pi}{15}....$$
$$x=\cos\frac{\pi}{15},\cos\frac{\pi}{3},\cos\frac{7\pi}{15},\cos\frac{11\pi}{15},\cos\frac{13\pi}{15}$$

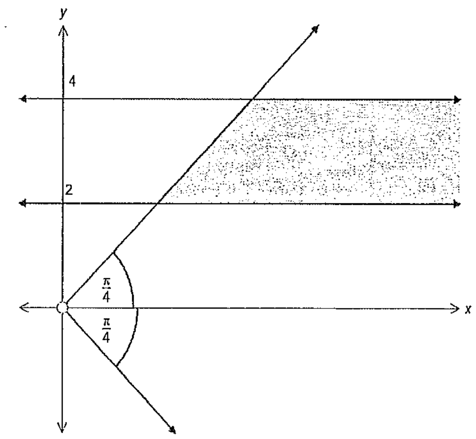
(b)(i)

$$1+\omega+\omega^2=0$$

(b)(ii)

$$\left(1+2\omega+3\omega^2\right)\left(1+2\omega^2+3\omega\right)$$
$$=\left(1+2\omega+3(-1-\omega)\right)\left(1+2(-1-\omega)+3\omega\right)$$
$$=(-2-\omega)(-1+\omega)$$
$$=2+\omega-2\omega-\omega^2$$
$$=2-\omega-\omega^2$$
$$=2-\left(\omega+\omega^2\right)$$
$$=2-(-1)$$
$$=3$$

(c)



Question 4

(a)(i)

$$z=cis\theta$$
$$z^n=\left(cis\theta\right)^n$$
$$=cis(n\theta) \quad \text{by De Moivre's Theorem}$$
$$z^{-n}=\left(cis\theta\right)^{-n}$$
$$=cis(-n\theta) \quad \text{by De Moivre's Theorem}$$
$$z^n-z^{-n}=\cos(n\theta)+i\sin(n\theta)-\left(\cos(-n\theta)+i\sin(-n\theta)\right)$$
$$=\cos(n\theta)+i\sin(n\theta)-\left(\cos(n\theta)-i\sin(n\theta)\right)$$

since cos is even and sin is odd

$$=\cos(n\theta)+i\sin(n\theta)-\cos(n\theta)+i\sin(n\theta)$$
$$z^n-z^{-n}=2i\sin(n\theta)$$

(a)(ii)

$$(2i\sin\theta)^4=\left(z-\frac{1}{z}\right)^4$$
$$=z^4-4z^2+6-\frac{4}{z^2}+\frac{1}{z^4}$$
$$=z^4+\frac{1}{z^4}-4\left(z^2+\frac{1}{z^2}\right)+6$$
$$2^4\sin^4\theta=(2\cos4\theta)-4(2\cos2\theta)+6$$
$$\sin^4\theta=\frac{\cos4\theta}{8}-\frac{\cos2\theta}{2}+\frac{3}{8}$$

(a)(iii)

$$\int\sin^4\theta d\theta=\int\left(\frac{\cos4\theta}{8}-\frac{\cos2\theta}{2}+\frac{3}{8}\right)d\theta$$
$$=\frac{\sin4\theta}{32}-\frac{\sin2\theta}{4}+\frac{3\theta}{8}+C$$

(b)(i)

$$z^5=-1$$

let $z=rcis\theta$

$$r^5cis5\theta=cis(\pi+2k\pi)$$
$$r=1,\theta=\frac{\pi+2k\pi}{5}$$
$$\therefore z=cis\frac{\pi}{5},cis\frac{3\pi}{5},-1,cis\frac{7\pi}{5},cis\frac{9\pi}{5}$$

(b)(ii)

$$z^5+1=(z+1)\left(z-cis\frac{\pi}{5}\right)\left(z-cis\frac{9\pi}{5}\right)\left(z-cis\frac{3\pi}{5}\right)\left(z-cis\frac{7\pi}{5}\right)$$

Conjugate roots $\left(cis\frac{\pi}{5},cis\frac{9\pi}{5}\right)$:

$$z+\bar{z}=2\operatorname{Re}(z)=2\cos\frac{\pi}{5}$$
$$z\bar{z}=\cos^2\frac{\pi}{5}+\sin^2\frac{\pi}{5}=1$$

Similarly for conjugate pair $\left(cis\frac{3\pi}{5},cis\frac{7\pi}{5}\right)$

$$z+\bar{z}=2\operatorname{Re}(z)=2\cos\frac{3\pi}{5}$$
$$z\bar{z}=x^2+y^2=\cos^2\frac{3\pi}{5}+\sin^2\frac{3\pi}{5}=1$$
$$\therefore z^5+1=(z+1)\left(z^2-2\cos\frac{\pi}{5}z+1\right)\left(z^2-2\cos\frac{3\pi}{5}z+1\right)$$

Question Five

a) Now $1+\omega+\omega^2=0$, $\omega^3=1$

let $\alpha=1+\omega$, $\beta=1+\omega^2$

eqn of form $(x^2-(\alpha+\beta)x+\alpha\beta=0$

$\alpha+\beta=1+\omega+1+\omega^2$

$=1$

$\alpha\beta=(1+\omega)(1+\omega^2)$

$=1+\omega+\omega^2+\omega^3$

$=0+\omega^3$

$=1$ (2)

ie $x^2-x+1=0$

b) $|z|^2-iz=16-2i$

$z=a+ib$

ie $(a^2+b^2)-i(a+ib)=16-2i$

$a^2+b^2-ia+b=16-2i$

$a=2$

$a^2+b^2+b=16$

$4+b^2+b=16$

$b^2+b=12$

$b^2+b-12=0$

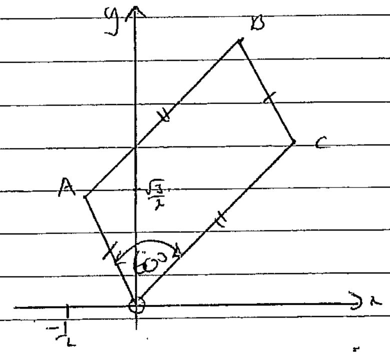
$(b+4)(b-3)=0$

$b=-4$ or 3

$z=2-4i$ or $z=2+3i$ (4)

Question Five (cont)

c)



A $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$

i) To find C multiply A by $2\operatorname{cis}-\frac{\pi}{3}$

ie $2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$ ie $(1-i\sqrt{3})$

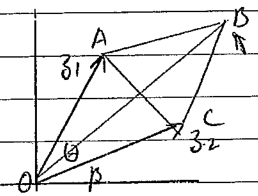
$(1-i\sqrt{3})\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-\frac{1}{2}+\frac{3}{2}+\frac{\sqrt{3}i}{2}+\frac{\sqrt{3}i}{2}$

$=1+\sqrt{3}i$ (3)

ii) B at $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)+(1+\sqrt{3}i)$

$=\frac{1}{2}+\frac{3\sqrt{3}i}{2}$ (1)

d) i)



z_1 is found by rotating z_2 through θ deg.

Note $|z_1|=|z_2|$

i) $\arg\left(\frac{z_1-z_2}{z_1+z_2}\right)=\arg(z_1-z_2)-\arg(z_1+z_2)$

OABC is a rhombus and the angle between the diagonals is $\frac{\pi}{2}$ which is given by $\arg\left(\frac{z_1-z_2}{z_1+z_2}\right)$ (2)

ii) $\arg\left(1+\frac{z_1}{z_2}\right)=\arg\frac{z_1+z_2}{z_2}$

$=\arg(z_1+z_2)-\arg(z_2)$

let $\arg z_2=\beta$.

then $\arg(z_1+z_2)=\beta+\frac{\theta}{2}$ (Diagonals bisect $\angle$.)

then $\arg(z_1+z_2)-\arg(z_2)=\beta+\frac{\theta}{2}-\beta$ (3)

1.

Question 5 d) (continued)

OR

ii) $\angle BOC = \frac{\theta}{2}$ diagonals of a parallelogram bisect $\angle$'s through which they pass

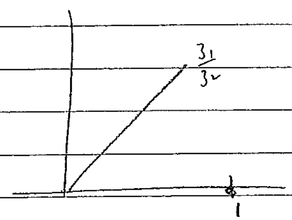
then $\arg z_2 + \frac{\theta}{2} = \arg(z_1 + z_2)$

$\arg z_2 - \arg(z_1 + z_2) = -\frac{\theta}{2}$

$\arg(z_1 + z_2) - z_1 = \frac{\theta}{2}$

$\arg\left(\frac{z_1 + z_2}{z_1}\right) = \frac{\theta}{2}$

$\arg\left(1 + \frac{z_2}{z_1}\right) = \frac{\theta}{2}$



Question Six

a) $|z-1| = |z+2|$ ie the points equidistant from $(1,0)$ and $(-2,0)$

ie $x = \frac{1-2}{2}$

$= -\frac{1}{2}$

(2)

b) $z^k = 1$ $1 = \text{cis } 0,\ \text{cis } 2\pi,\ \text{cis } 4\pi$ etc

$\left(\text{cis } \frac{4\pi}{7}\right)^k = \text{cis}(n\pi)$ k, n, integers

$\text{cis } \frac{4k\pi}{7} = \text{cis } n\pi$

$k = 7$

(2)

c) $|z+2|^2 = 3$ is a circle centre $(-2,0)$ r:

Now $3|z+1|^2 = |z-1|^2$

$3((x+1)+iy)((x+1)-iy)) = ((x-1)+iy)((x-1)-iy)$

$3x^2 + 6x + 3y^2 + 3 = x^2 - 2x + 1 + y^2$

$2x^2 + 8x + 2y^2 + 2 = 0$

$x^2 + 4x + 1 + y^2 = 0$

$x^2 + 4x + 4 + y^2 = -1 + 4$

$(x+2)^2 + y^2 = 3$ ie circle centre $(-2,0)$, $r = \sqrt{3}$

(3)

d) $z^6 = 1$

$= \text{cis}(0 + 2\pi k)$ $k = 0 \to 5$

$z_0 = \text{cis } 0 = 1$, $z_1 = \text{cis } \frac{\pi}{3}$, $z_2 = \text{cis } \frac{2\pi}{3}$, $z_3 = \text{cis } \pi = -1$

$z_4 = \text{cis } \frac{4\pi}{3} = \text{cis } -\frac{2\pi}{3}$ $z_5 = \text{cis } \frac{5\pi}{3} = \text{cis } -\frac{\pi}{3}$

(4)

$(z^2 - 1)(z^4 + z^2 + 1) = z^6 - 1$

If $(z^2-1)(z^4+z^2+1) = 0$

$z = \pm 1$ which are the roots of $z^2 - 1 = 0$

$\therefore$ roots of $z^4 + z^2 + 1$ are $\text{cis } \pm\frac{\pi}{3}$, $\text{cis } \pm\frac{2\pi}{3}$

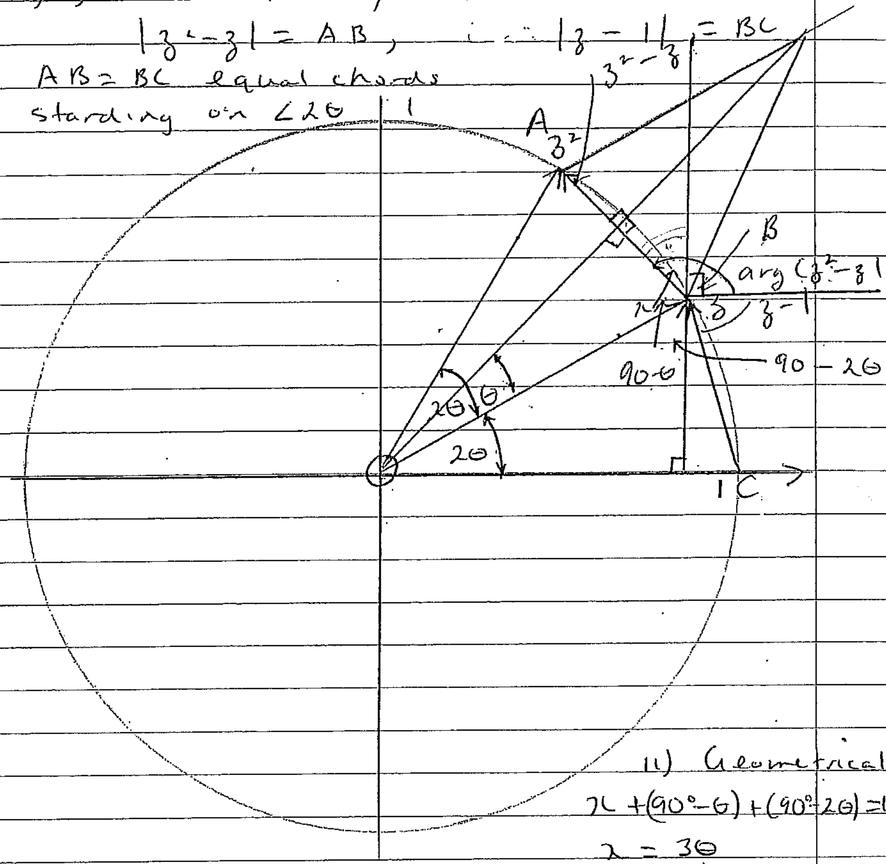
Question Six (continued)

e) i) Geometrically

$|z^2 - z| = AB$, $\therefore$ $|z-1| = BC$

AB = BC equal chords

standing on $\angle 2\theta$



ii) Geometrically

$x + (90° - \theta) + (90° - 2\theta) = 180°$

$x = 3\theta$

$\arg(z^2 - z) = 90° + 3\theta$

$= \frac{\pi}{2} + 3\theta$

1. Algebraically

z lies on a unit circle $\therefore$ $|z| = 1$

Show $|z^2 - z| = |z - 1|$

$LHS = |z^2 - z|$

$= |z(z-1)|$

$= |z|\,|z-1|$ $\quad (\,|z_1 z_2| = |z_1|\,|z_2|$

$= 1|(z-1)|$

$= RHS$