SYDNEY GIRLS H.S. - EXT. 2 - TASK 2 - MAR 09

Question 1: (25 marks)

a) Solve the quadratic equation $x^2+4x+5=0$, giving your answers in the form $a+ib$. **2**

b) Form the quadratic equation with roots $1+3i$ and $1-3i$. **2**

c) If $z=3-2i$ and $w=2+i$, find, expressing in the form $a+ib$:

i) $3z-2w$ **2**

ii) iw **2**

iii) $\frac{w}{z}$ **2**

iv) w^2-z^2 **2**

d)

i) Express $\sqrt{8+6i}$ in the form $a+ib$. **4**

ii) Hence solve the equation: $z^2+2(1+2i)z-(11+2i)=0$ **3**

e)

i) Express $\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$ in mod-arg form. **2**

ii) Use De Moivre's theorem to simplify $\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)^{60}$, expressing your answer in the form $a+ib$ **2**

f) Evaluate i^{2009}. **2**

Question 2: (25 marks)

a)

i) Simplify $\left(cis\frac{\pi}{3}\right)\left(cis\frac{\pi}{4}\right)$. Express your answer in the form $a+ib$. **3**

ii) Hence evaluate $\cos\frac{7\pi}{12}$ in surd form. **2**

b)

i) Use De Moivre's theorem to express $\cos 4\theta$ in terms of $\sin\theta$ and $\cos\theta$. **3**

ii) Express $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$. **1**

iii) Hence show that $\tan 4\theta=\frac{4t-4t^3}{1-6t^2+t^4}$ where $t=\tan\theta$. **3**

iv) Using this result or otherwise, solve the equation $x^4+4x^3-6x^2-4x+1=0$. **3**

c)

i) If $z=\cos\theta+i\sin\theta$, use de Moivre's Theorem to show that $z^n+\frac{1}{z^n}=2\cos n\theta$ **1**

ii) Find an expression for $\cos^4\theta$ in terms of multiples of $\cos\theta$. **3**

d)

i) Solve the equation $z^6+1=0$, giving the roots in the form $a+ib$. **3**

ii) Hence factorise z^6+1 into real quadratic factors. **3**

Question 3: (25 marks)

a) One vertex of an equilateral triangle OAB is A $(1+i)$. If O is the origin, find the coordinates of B if it lies in the second quadrant. **3**

b) Given O is the origin and A is represented by the complex number $\sqrt{3}+i$:

i) find C if OA is rotated through $\left(-\frac{\pi}{3}\right)$ and doubled in length to form C; **2**

ii) find the point B which forms the parallelogram $OABC$ **2**

c) Prove that $|z_1-z_2|^2+|z_1+z_2|^2=2|z_1|^2+2|z_2|^2$ **4**

d) If w is the complex cube root of unity with the smallest positive argument:

i) show that the other complex cube root is w^2 **2**

ii) prove that $1+w+w^2=0$ **2**

iii) evaluate $(1+3w+w^2)(1+w+3w^2)$ **1**

iv) find the quadratic equation which has the roots $(1+3w+w^2)$ and $(1+w+3w^2)$ **2**

e) If $z=\frac{1+i}{1-i}$ and $w=\frac{2}{1-\sqrt{3}i}$:

i) express z and w in modulus argument form; **4**

ii) plot $z, w, z+w$ on an Argand diagram; **1**

iii) show that $\tan\frac{5\pi}{12}=\sqrt{3}+2$ **2**

END OF TEST

EXTENSION 2 MATHEMATICS TASK 2 SOLUTIONS

Question 1:

a)
$$x^2+4x+5=0$$
$$x=\frac{-4\pm\sqrt{16-20}}{2}=\frac{-4\pm\sqrt{-4}}{2}=\frac{-4\pm 2i}{2}=-2\pm i$$

b)
$$\alpha+\beta=(1+3i)+(1-3i)=2$$
$$\alpha\beta=(1+3i)(1-3i)=1-9i^2=10$$
$$x^2-2x+10=0$$

c) i)
$$3z-2w=3(3-2i)-2(2+i)=9-6i-4-2i=5-8i$$

ii)
$$iw=i(2+i)=2i+i^2=-1+2i$$

iii)
$$\frac{w}{z}=\frac{2+i}{3-2i}\times\frac{3+2i}{3+2i}=\frac{(2+i)(3+2i)}{(3-2i)(3+2i)}=\frac{6+7i+2i^2}{9-4i^2}=\frac{4}{13}+i\frac{7}{13}$$

iv)
$$w^2-z^2=(2+i)^2-(3-2i)^2=(4+4i+i^2)-(9-12i+4i^2)=(3+4i)-(5-12i)=-2+16i$$

d) i) $\sqrt{8+6i}=a+ib$

$8+6i=(a+ib)^2$

$=a^2-b^2+i2ab$

$a^2-b^2=8$ (1)

$2ab=6$ (2)

Using $(a^2+b^2)^2=(a^2-b^2)^2+(2ab)^2$

$=8^2+6^2$

$=100$

$a^2+b^2=10$ $(\text{as } a^2+b^2>0)$ (3)

(1)+(3)

$2a^2=18$

$a^2=9$

$a=\pm 3$

When $a=3, b=1$

When $a=-3, b=-1$

$\therefore \sqrt{8+6i}=\pm(3+i)$

ii) $z^2+2(1+2i)z-(11+2i)=0$

$z^2+2(1+2i)z=11+2i$

$z^2+2(1+2i)z+(1+2i)^2=11+2i+(1+2i)^2$

$(z+[1+2i])^2=11+2i+1+4i+4i^2$

$=12+6i-4$

$=8+6i$

$z+1+2i=\pm\sqrt{8+6i}$

$=\pm(3+i)$

$z+1+2i=3+i$ $\qquad$ $z+1+2i=-3-i$

$z=2-i$ $\qquad$ $z=-4-3i$

e) i) $r=\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{-\sqrt{2}}{2}\right)^2}$

$=1$

$\theta=\tan^{-1}(1)$ in 4^{th} quadrant

$=-\frac{\pi}{4}$

$\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}=\text{cis}\left(-\frac{\pi}{4}\right)$

ii) $\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)^{60}=\left(\text{cis}\left[-\frac{\pi}{4}\right]\right)^{60}$

$=\left(\cos\left[-\frac{\pi}{4}\right]+i\sin\left[-\frac{\pi}{4}\right]\right)^{60}$

$=\cos(-15\pi)+i\sin(-15\pi)$

$=\cos 15\pi-i\sin 15\pi$

$=-1$

f) $i^{2009}=(i^4)^{502}\times i$

$=1\times i$

$=i$

Question 2:

a) i) $\left(\text{cis}\frac{\pi}{3}\right)\times\left(\text{cis}\frac{\pi}{4}\right)=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$$=\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)$$
$$=\frac{1}{2}\left(1+i\sqrt{3}\right)\frac{1}{\sqrt{2}}(1+i)$$
$$=\frac{1}{2\sqrt{2}}\left(1+i+i\sqrt{3}+i^2\sqrt{3}\right)$$
$$=\frac{\sqrt{2}}{4}\left(1-\sqrt{3}\right)+i\frac{\sqrt{2}}{4}\left(1+\sqrt{3}\right)$$

ii) $\left(\text{cis}\frac{\pi}{3}\right)\left(\text{cis}\frac{\pi}{4}\right)=\text{cis}\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$

$$=\text{cis}\frac{7\pi}{12}$$

Equating real parts:

$$\cos\frac{7\pi}{12}=\frac{\sqrt{2}\left(1-\sqrt{3}\right)}{4}$$

b) i) $(\cos\theta+i\sin\theta)^4=\cos4\theta+i\sin4\theta$

$$(\cos\theta+i\sin\theta)^4=\cos^4\theta+4\cos^3\theta(i\sin\theta)+6\cos^2\theta(i\sin\theta)^2+4\cos\theta(i\sin\theta)^3+(i\sin\theta)^4$$
$$=\cos^4\theta+i4\cos^3\theta\sin\theta+i^2 6\cos^2\theta\sin^2\theta+i^3 4\cos\theta\sin^3\theta+i^4\sin^4\theta$$
$$=\left(\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta\right)+i\left(4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta\right)$$
$$\cos4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$$

ii) $\sin4\theta=4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta$

iii) $\tan4\theta=\frac{\sin4\theta}{\cos4\theta}$

$$=\frac{4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta}{\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta}$$
$$=\frac{\frac{4\cos^3\theta\sin\theta}{\cos^4\theta}-\frac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\frac{\cos^4\theta}{\cos^4\theta}-\frac{6\cos^2\theta\sin^2\theta}{\cos^4\theta}+\frac{\sin^4\theta}{\cos^4\theta}}$$
$$=\frac{4t-4t^3}{1-6t^2+t^4}\quad\text{where } t=\tan\theta$$

iv) $x^4+4x^3-6x^2-4x+1=0$

$$4x-4x^3=1-6x^2+x^4$$
$$\frac{4x-4x^3}{1-6x^2+x^4}=1$$

Let $x=\tan\theta=t$:

$$\frac{4t-4t^3}{1-6t^2+t^4}=1$$
$$\tan4\theta=1$$
$$4\theta=\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4},\frac{13\pi}{4}$$
$$\theta=\frac{\pi}{16},\frac{5\pi}{16},\frac{9\pi}{16},\frac{13\pi}{16}$$
$$x=\tan\frac{\pi}{16},\tan\frac{5\pi}{16},\tan\frac{9\pi}{16},\tan\frac{13\pi}{16}$$

c) i) $z^n+\frac{1}{z^n}=z^n+z^{-n}$

$$=(\cos\theta+i\sin\theta)^n+(\cos\theta+i\sin\theta)^{-n}$$
$$=\cos n\theta+i\sin n\theta+\cos(-n\theta)+i\sin(-n\theta)$$
$$=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$$
$$=2\cos n\theta$$

ii) Let $n=1$:

$$z+\frac{1}{z}=2\cos\theta$$
$$\left(z+\frac{1}{z}\right)^4=16\cos^4\theta$$
$$16\cos^4\theta=z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4}$$
$$=\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6$$
$$=2\cos4\theta+8\cos2\theta+6$$
$$\cos^4\theta=\frac{1}{8}\cos4\theta+\frac{1}{2}\cos2\theta+\frac{3}{8}$$

d) i) $z^6 = -1$

$= cis\pi$

$z = (cis\pi)^{\frac{1}{6}}$

$= cis\left(\dfrac{\pi + 2k\pi}{6}\right)$ where $k = 0,1,2,3,4,5$

$z_0 = cis\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}$

$z_1 = cis\dfrac{\pi}{2} = i$

$z_2 = cis\dfrac{5\pi}{6} = -\dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}$

$z_3 = cis\dfrac{7\pi}{6} = -\dfrac{\sqrt{3}}{2} - i\dfrac{1}{2}$

$z_4 = cis\dfrac{3\pi}{2} = -i$

$z_5 = cis\dfrac{11\pi}{6} = \dfrac{\sqrt{3}}{2} - i\dfrac{1}{2}$

ii) Note:

$$z_3 = cis\frac{7\pi}{6} = cis\left(-\frac{5\pi}{6}\right) = \overline{z_2}$$

$$z_5 = cis\frac{11\pi}{6} = cis\left(-\frac{\pi}{6}\right) = \overline{z_0}$$

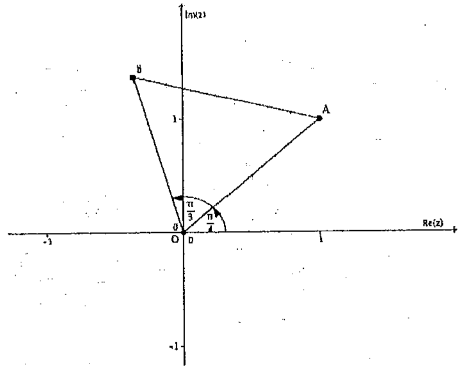
$$z + \bar{z} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta = 2\mathrm{Re}(z)$$

$$z\bar{z} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta - i^2\sin^2\theta = \cos^2\theta + \sin^2\theta = 1$$

$$z^6 + 1 = (z - i)(z + i)(z - z_0)(z - \overline{z_0})(z - z_2)(z - \overline{z_2})$$
$$= (z^2 - i^2)(z^2 - [z_0 + \overline{z_0}]z + z_0\overline{z_0})(z^2 - [z_2 + \overline{z_2}]z + z_2\overline{z_2})$$
$$= (z^2 + 1)(z^2 - \sqrt{3}z + 1)(z^2 + \sqrt{3}z + 1)$$

Question 3:

a)



$$\left(cis\frac{\pi}{3}\right)(1+i) = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)(1+i)$$
$$= \frac{1}{2} + \frac{1}{2}i + i\frac{\sqrt{3}}{2} + i^2\frac{\sqrt{3}}{2}$$
$$= \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) + i\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$$

Coordinates of B: $\left(\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}, \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\right)$

b) i) $|OA| = \sqrt{(\sqrt{3})^2 + 1^2}$

$= 2$

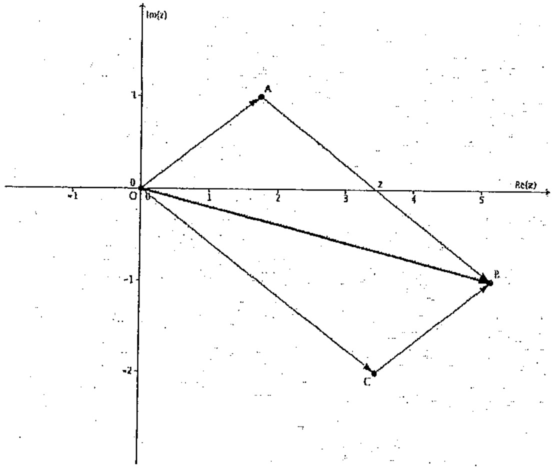
$|OC| = 2 \times |OA|$

$= 4$

$$2cis\left(-\frac{\pi}{3}\right)(\sqrt{3} + i) = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)(\sqrt{3} + i)$$
$$= (1 - i\sqrt{3})(\sqrt{3} + i)$$
$$= \sqrt{3} + i - 3i - i^2\sqrt{3}$$
$$= 2\sqrt{3} - 2i$$

Coordinates of C: $(2\sqrt{3}, -2)$

ii)



$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$$
$$= (\sqrt{3}+i) + (2\sqrt{3}-2i)$$
$$= 3\sqrt{3} - i$$

Coordinates of C: $(3\sqrt{3}, -1)$

c) $$|z_1 - z_2|^2 + |z_1 + z_2|^2 = (z_1 - z_2)(\overline{z_1 - z_2}) + (z_1 + z_2)(\overline{z_1 + z_2})$$
$$= (z_1 - z_2)(\overline{z_1} - \overline{z_2}) + (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$
$$= z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}$$
$$= 2z_1\overline{z_1} + 2z_2\overline{z_2}$$
$$= 2|z_1|^2 + 2|z_2|^2$$

d) i) $z^3 = 1$
$$= cis0$$
$$z = (cis0)^{\frac{1}{3}}$$
$$= cis\left(\frac{0+2k\pi}{3}\right) \quad \text{where } k = 0,1,2$$

$z_0 = cis0$
$= 1$

$z_1 = cis\frac{2\pi}{3}$
$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$z_2 = cis\frac{4\pi}{3}$
$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Let $w = cis\frac{2\pi}{3}$:

$$w^2 = \left(cis\frac{2\pi}{3}\right)^2$$
$$= cis\frac{4\pi}{3}$$
$$= z_2$$

ii) $$1 + w + w^2 = \frac{1(w^3 - 1)}{w - 1}$$
$$= \frac{1-1}{w-1} \quad (\text{as } w^3 = 1)$$
$$= 0$$

iii) $$(1+3w+w^2)(1+w+3w^2) = ([1+w^2]+3w)([1+w]+3w^2)$$
$$= (-w+3w)(-w^2+3w^2)$$
$$= 2w \times 2w^2$$
$$= 4w^3$$
$$= 4$$

iii) $$(1+3w+w^2)+(1+w+3w^2) = 2w + 2w^2$$
$$= 2w(1+w)$$
$$= 2w(-w^2)$$
$$= -2w^3$$
$$= -2$$
$$(1+3w+w^2)(1+w+3w^2) = 4$$

$$x^2 + 2x + 4 = 0$$

e)

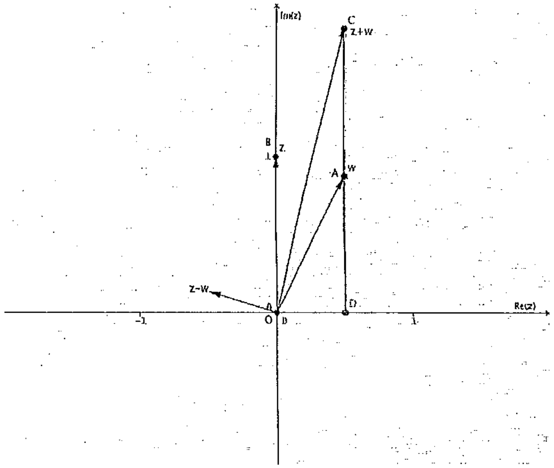
i) $z=\dfrac{\sqrt{2}cis\frac{\pi}{4}}{\sqrt{2}cis\left(-\frac{\pi}{4}\right)}$

$=cis\dfrac{\pi}{2}$

$w=\dfrac{2cis0}{2cis\left(-\frac{\pi}{3}\right)}$

$=cis\dfrac{\pi}{3}$

ii)



$\angle AOD=\arg\left(w\right)$

$=\dfrac{\pi}{3}$

$\angle BOA=\dfrac{\pi}{6}$ $\left(\text{adjacent complementary }\angle s\right)$

As $\left|z\right|=\left|w\right|$, $OBCA$ is a rhombus.

So diagonals bisect the angles through which they pass.

$\angle COA=\dfrac{\pi}{6}\div 2$

$=\dfrac{\pi}{12}$

$\angle COD=\dfrac{\pi}{3}+\dfrac{\pi}{12}$

$=\dfrac{5\pi}{12}$

In ΔAOD:

$\sin\angle AOD=\dfrac{DA}{1}$ $\qquad$ $\cos\angle AOD=\dfrac{DO}{1}$

$\sin\dfrac{\pi}{3}=DA$ $\qquad$ $\cos\dfrac{\pi}{3}=DO$

$DA=\dfrac{\sqrt{3}}{2}$ $\qquad$ $DO=\dfrac{1}{2}$

In ΔCOD:

$\tan\dfrac{5\pi}{12}=\dfrac{\frac{\sqrt{3}}{2}+1}{\frac{1}{2}}$

$=\dfrac{\frac{\sqrt{3}+2}{2}}{\frac{1}{2}}$

$=\sqrt{3}+2$