Further Trigonometry – (Ext. 1 Worksheet)

Question 1

Find in surd form the exact values of i. $\sin 75°$ ii. $\tan 15°$

Question 2

If $\tan\frac{\theta}{2} = 2$, find the exact values of i. $\sin\theta$ ii. $\cos\theta$

Question 3

Solve the equations

i. $\sin 2\theta = \sin\theta,\ \ 0 \le \theta \le 2\pi$

ii. $\cos 2\theta = \cos\theta,\ \ 0 \le \theta \le 2\pi$

Question 4

i. Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. Hence solve $\sin 3\theta = \sin\theta$ for $0° \le \theta \le 360°$.

ii. Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. Hence solve $\cos 3\theta = \cos\theta$ for $0° \le \theta \le 360°$.

Question 5

Show that

i. $\dfrac{\cos 3\theta}{\sin\theta} + \dfrac{\sin 3\theta}{\cos\theta} = 2\cot 2\theta$.

ii. $\dfrac{\sin 3\theta}{\sin\theta} + \dfrac{\cos 3\theta}{\cos\theta} = 4\cos 2\theta$.

Question 6

i. Show that $\cos^4\theta - \sin^4\theta = \cos 2\theta$.

ii. If $\sec A = \sin B + \cos B$, show that $\tan^2 A = \sin 2B$.

Question 7

i. Show that $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$. Hence find the exact value of $\tan 15°$.

ii. Show that $\operatorname{cosec} 2\theta - \cot 2\theta = \tan\theta$. Hence find the exact value of $\tan\frac{\pi}{8}$.

Question 8

i. Show that $\dfrac{\sin 2\theta}{1-\cos 2\theta}=\cot\theta$. Hence find the exact value of $\cot 15°$.

ii. Show that $\operatorname{cosec} 2\theta+\cot 2\theta=\cot\theta$. Hence find the exact value of $\cot\frac{\pi}{8}$.

Question 9

i. Express $\tan\frac{3\pi}{4}$ in terms of $t=\tan\frac{3\pi}{8}$. Hence show that $t^2-2t-1=0$ and find the exact value of $\tan\frac{3\pi}{8}$.

ii. Express $\sin\frac{5\pi}{6}$ in terms of $t=\tan\frac{5\pi}{12}$. Hence show that $t^2-4t+1=0$ and find the exact of $\tan\frac{5\pi}{12}$.

Question 10

Solve the equation $\sin\theta + 3\cos\theta = 2$ for $0° \le \theta \le 360°$, giving answers correct to the nearest minute,

i. by first expressing $\sin\theta + 3\cos\theta$ in terms of $t = \tan\frac{\theta}{2}$.

ii. by first expressing $\sin\theta + 3\cos\theta$ in the form $R\sin(\theta + \alpha)$ where $R > 0$, $0° \le \alpha \le 360°$.

Question 11

i. Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$ where $R > 0$, $0° \le \alpha \le 360°$.

ii. Hence find the range of the functions $f(\theta) = 2\cos\theta - \sin\theta$ and $g(\theta) = (2\cos\theta - \sin\theta)^2$.

Question 12

Find in exact radian measure the general solutions of

i. $\cos 2\theta = \frac{1}{2}$ ii. $\tan 2\theta = -1$

1. i. $\sin(30°+45°)=\frac{1+\sqrt{3}}{2\sqrt{2}}$

ii. $\tan(60°-45°)=\frac{\sqrt{3}-1}{1+\sqrt{3}}=2-\sqrt{3}$

2. i. $\sin\theta=\dfrac{2\times 2}{1+2^2}=\frac{4}{5}$

ii. $\cos\theta=\dfrac{1-2^2}{1+2^2}=-\frac{3}{5}$

3. i. $2\sin\theta\cos\theta=\sin\theta,\ 0\le\theta\le 2\pi$

$\sin\theta\left(2\cos\theta-1\right)=0$

$\sin\theta=0\ \text{or}\ \cos\theta=\frac{1}{2}$

$\theta=0,\frac{\pi}{3},\pi,\frac{5\pi}{3},2\pi$

ii. $2\cos^2\theta-1=\cos\theta,\ 0\le\theta\le 2\pi$

$\left(2\cos\theta+1\right)\left(\cos\theta-1\right)=0$

$\cos\theta=-\frac{1}{2}\ \text{or}\ \cos\theta=1$

$\theta=0,\frac{2\pi}{3},\frac{4\pi}{3},2\pi$

4. i. $\sin 3\theta=\sin 2\theta\cos\theta+\cos 2\theta\sin\theta$

$=\sin\theta\left(2\cos^2\theta+\cos^2\theta-\sin^2\theta\right)$

$=\sin\theta\left(3(1-\sin^2\theta)-\sin^2\theta\right)$

$=3\sin\theta-4\sin^3\theta$

$3\sin\theta-4\sin^3\theta=\sin\theta$

$2\sin\theta\left(1-2\sin^2\theta\right)=0$

$\sin\theta=0,\ \sin\theta=\pm\frac{1}{\sqrt{2}}$

$\theta=0°,45°,135°,180°,225°,315°,360°$

ii. $\cos 3\theta=\cos 2\theta\cos\theta-\sin 2\theta\sin\theta$

$=\cos\theta\left(\cos^2\theta-\sin^2\theta-2\sin^2\theta\right)$

$=\cos\theta\left(\cos^2\theta-3(1-\cos^2\theta)\right)$

$=4\cos^3\theta-3\cos\theta$

$4\cos^3\theta-3\cos\theta=\cos\theta$

$4\cos\theta\left(\cos^2\theta-1\right)=0$

$\cos\theta=0,\ \text{or}\ \cos\theta=\pm1$

$\theta=0°,90°,180°,270°,360°$

5. i. $LHS=\dfrac{\cos\theta\cos 3\theta+\sin\theta\sin 3\theta}{\sin\theta\cos\theta}$

$=\dfrac{\cos(3\theta-\theta)}{\frac{1}{2}\sin 2\theta}$

$=2\dfrac{\cos 2\theta}{\sin 2\theta}$

$=2\cot 2\theta$

$=RHS$

ii. $LHS=\dfrac{\cos\theta\sin 3\theta+\sin\theta\cos 3\theta}{\sin\theta\cos\theta}$

$=\dfrac{\sin 4\theta}{\frac{1}{2}\sin 2\theta}$

$=\dfrac{2\sin 2\theta\cos 2\theta}{\frac{1}{2}\sin 2\theta}$

$=4\cos 2\theta$

$=RHS$

6. i.

$\cos^4\theta-\sin^4\theta$

$=\left(\cos^2\theta+\sin^2\theta\right)\left(\cos^2\theta-\sin^2\theta\right)$

$=\cos^2\theta-\sin^2\theta$

$=\cos 2\theta$

ii.

$\sec^2 A=\left(\sin B+\cos B\right)^2$

$1+\tan^2 A=\sin^2 B+\cos^2 B+2\sin B\cos B$

$\tan^2 A=\sin 2B$

7. i. $\dfrac{\sin 2\theta}{1+\cos 2\theta}=\dfrac{2\sin\theta\cos\theta}{2\cos^2\theta}=\tan\theta$

$\tan 15°=\dfrac{\left(\frac{1}{2}\right)}{1+\frac{\sqrt{3}}{2}}=\dfrac{1}{2+\sqrt{3}}=2-\sqrt{3}$

ii.

$\operatorname{cosec} 2\theta-\cot 2\theta=\dfrac{1-\cos 2\theta}{\sin 2\theta}$

$=\dfrac{2\sin^2\theta}{2\sin\theta\cos\theta}$

$=\tan\theta$

$\tan\frac{\pi}{8}=\operatorname{cosec}\frac{\pi}{4}-\cot\frac{\pi}{4}=\sqrt{2}-1$

8. i. $\dfrac{\sin 2\theta}{1-\cos 2\theta}=\dfrac{2\sin\theta\cos\theta}{2\sin^2\theta}=\cot\theta$

$\cot 15°=\dfrac{\left(\frac{1}{2}\right)}{1-\frac{\sqrt{3}}{2}}=\dfrac{1}{2-\sqrt{3}}=2+\sqrt{3}$

ii.

$$\operatorname{cosec}2\theta+\cot 2\theta=\frac{1+\cos 2\theta}{\sin 2\theta}=\frac{2\cos^2\theta}{2\sin\theta\cos\theta}=\cot\theta$$

$\cot\frac{\pi}{8}=\operatorname{cosec}\frac{\pi}{4}+\cot\frac{\pi}{4}=\sqrt{2}+1$

9. i. $\tan\frac{3\pi}{4}=\dfrac{2t}{1-t^2}\Rightarrow\dfrac{2t}{1-t^2}=-1$

$$\left.\begin{array}{l}t^2-2t-1=0\\(t-1)^2=2\end{array}\right\}\Rightarrow t=1\pm\sqrt{2}$$

But $\tan\frac{3\pi}{8}>0$. $\therefore\tan\frac{3\pi}{8}=1+\sqrt{2}$

ii. $\sin\frac{5\pi}{6}=\dfrac{2t}{1+t^2}\Rightarrow\dfrac{2t}{1+t^2}=\dfrac{1}{2}$

$$\left.\begin{array}{l}t^2-4t+1=0\\(t-2)^2=3\end{array}\right\}\Rightarrow t=2\pm\sqrt{3}$$

But $\frac{\pi}{4}<\frac{5\pi}{12}<\frac{\pi}{2}\Rightarrow\tan\frac{5\pi}{12}>1$.

$\therefore\tan\frac{5\pi}{12}=2+\sqrt{3}$.

10. i. $\dfrac{2t+3\left(1-t^2\right)}{1+t^2}=2$

$5t^2-2t-1=0\qquad\therefore t=\dfrac{1\pm\sqrt{6}}{5}$

$t\approx 0\cdot 6899,\ -0\cdot 2899$

$\frac{\theta}{2}\approx 34°36',\ 163°50'$

$\theta\approx 69°12',\ 327°40'$

ii.

$\sin\theta+3\cos\theta$

$=\sqrt{10}\left(\frac{1}{\sqrt{10}}\sin\theta+\frac{3}{\sqrt{10}}\cos\theta\right)$

$=\sqrt{10}\sin(\theta+\alpha)$

where $\tan\alpha=3,\ \alpha\approx 71°34'$.

$\sin(\theta+\alpha)=\frac{2}{\sqrt{10}}$

$\theta+\alpha\approx 180°-39°14',\ 360°+39°14'$

$\theta\approx 69°12',\ 327°40'$

11. i. $2\cos\theta-\sin\theta$

$=\sqrt{5}\left(\frac{2}{\sqrt{5}}\cos\theta-\frac{1}{\sqrt{5}}\sin\theta\right)$

$=\sqrt{5}\cos(\theta+\alpha)$

where $\tan\alpha=\frac{1}{2},\ \alpha=26°34'$.

ii. $y=f(\theta)$ has range

$\left\{y:-\sqrt{5}\le y\le\sqrt{5}\right\}$

$y=g(\theta)$ has range

$\left\{y:0\le y\le 5\right\}$

12. i. $2\theta=\pm\frac{\pi}{3}+2n\pi,\ n=0,\pm1,\pm2,\ldots$

$\theta=(6n\pm1)\frac{\pi}{6},\ n=0,\pm1,\pm2,\ldots$

ii. $2\theta=-\frac{\pi}{4}+m\pi,\ m=0,\pm1,\pm2,\ldots$

$\theta=(4m-1)\frac{\pi}{8},\ m=0,\pm1,\pm2,\ldots$