SYDNEY GIRLS HIGH SCHOOL

2007 HSC Assessment Task 2

March 7, 2007

MATHEMATICS Extension 1

Year 12

Time allowed: 75 minutes

Topics: Trigonometry (I & II) First part of Polynomials

DIRECTIONS TO CANDIDATES:

- Attempt all questions
- Questions are not of equal value
- There are 5 questions with part marks shown
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used
- Write on one side of the paper only

QUESTION 1 — Marks

a) Find $\frac{dy}{dx}$ if — 5

(i) $y = \sin 4x$

(ii) $y = \cos\left(\frac{1}{x}\right)$

(iii) $y = \tan^6 x$

b) Find — 6

(i) $\int 3\sin 3x \, dx$

(ii) $\int \sec^2 4x \, dx$

(iii) $\int (\cos 2x + 2\cos x) \, dx$

(iv) $\int \sin(4x - 3) \, dx$

c) Given that $\cos 36^0 = \frac{\sqrt{5}+1}{4}$ — 4

find in simplest form, the exact value of $\cos 72^0$

QUESTION 2

a) Given the lines $L_1: 3y = x+1$ *and* $L_2: 3x-4y=12$, Show that the acute angle formed by them is equal to the acute angle formed by L_1 and the x-axis. 5

b) If $\tan\frac{\theta}{2} = \frac{1}{2}$ find the exact values of 6

(i) $\tan\theta$

(ii) $\cos 2\theta$

c) (i) Factorise $3x^3+3x^2-x-1$ 4

(ii) hence solve the equation

$3\tan^3\theta + 3\tan^2\theta - \tan\theta - 1 = 0$ for $0 \le \theta \le \pi$

QUESTION 3

a) Sketch the polynomial $f(x) = x^2(x+1)(x+2)$ 2

(without using calculus)

b) For the polynomial $P(x) = 2 + x - 5x^2 + 8x^4$ 3

state the

(i) degree

(ii) leading term

(ii) constant term

c) (i) Find the remainder when ax^2+bx+c is divided by $x-1$ 10

(ii) Under what conditions is $x-1$ a factor of the quadratic ?

(iii) Show that $x-1$ is a factor of $ax^3+(b-a)x^2+(c-b)x-c$ and find the other factor

(iv) State necessary and sufficient conditions on a,b,c for the cubic to have three real and distinct roots

QUESTION 4

a) Find the volume of the solid of revolution obtained by revolving the area between $y = 2\sec x$ and the x-axis between $x = o$ and $x = \frac{\pi}{3}$ about the x-axis — 3

b) (i) Express $\sqrt{2}\sin x + \sqrt{2}\cos x$ in the form $R\sin(x+\alpha)$ *where* $R > 0$ *and* $0 \le \alpha \le \frac{\pi}{2}$ — 5

(ii) Hence, sketch $y = \sqrt{2}\sin x + \sqrt{2}\cos x$ *for* $0 \le x \le 2\pi$ (show intercepts and endpoints clearly)

(iii) Hence, find the value(s) of k for which $\sqrt{2}\sin x + \sqrt{2}\cos x = k$ has 3 solutions in the domain $0 \le x \le 2\pi$

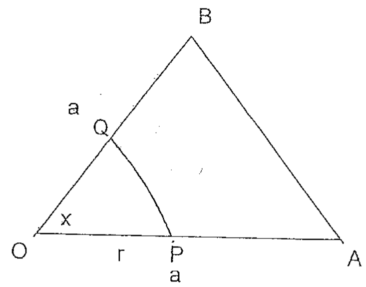
c) Express the solution of the equation $\sin 2\theta = \sin\theta$ in general form, if θ is in radians — 4

(d) Find $\int \cos^2\theta \, d\theta$ — 3

QUESTION 5

a) Evaluate $\lim_{x \to 0} \frac{\sin 2x}{3x}$ — 2

b) In $\triangle AOB$, $OA = OB = a$, which is constant. $< AOB = x$ radians, where x is variable. PQ is a circular arc, centre O and radius r. If the area of $\triangle AOB$ is twice that of the sector OPQ — 6



(i) Express r^2 in terms of a and x

(ii) Find r in terms of a when $< AOB$ is a right angle

(iii) Describe the behaviour of r as $x \to 0$

c) A monic polynomial $P(x)$ of degree 4 is known to have zeros 2 and -2 — 7

(i) Write down an equation for $P(x)$ to the extent specified so far.

(ii) given further that $P(0) = 4$ and $P(1) = -3$, find P(x)

(iii) Solve $P(x) = 0$ for real roots

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Q(1) (a) (i) $y' = 4\cos 4x$ (1)

(ii) $y = \cos\left(\frac{1}{x}\right)$ let $u = \frac{1}{x}$

$\frac{dy}{du} = -\sin u \quad \frac{du}{dx} = -\frac{1}{x^2}$

$\therefore \frac{dy}{dx} = \frac{1}{x^2}\sin\left(\frac{1}{x}\right)$ (2)

(iii) $y = (\tan x)^6$

$y' = 6(\tan x)^5 \times \sec^2 x$

$= 6\tan^5 x \sec^2 x$ (2)

(b) (i) $\int 3\sin 3x \, dx$

$= 3\int \sin 3x \, dx$

$= 3\left(-\frac{1}{3}\cos 3x\right) + c$

$= -\cos 3x + c$ (2)

(ii) $\int \sec^2 4x \, dx = \frac{1}{4}\tan 4x + C$ (1)

(iii) $\int (\cos 2x + 2\cos x) \, dx$

$= \frac{1}{2}\sin 2x + 2\sin x + c$ (2)

(iv) $\int \sin(4x-3) \, dx$

$= -\frac{1}{4}\cos(4x-3) + c$ (1)

(c) $\cos 72^\circ = \cos(2 \times 36^\circ)$

$= 2\cos^2 36 - 1$

$= 2\left(\frac{6+2\sqrt{5}}{16}\right) - 1$

$= \frac{6+2\sqrt{5}}{8} - \frac{8}{8}$

$= \frac{2\sqrt{5}-2}{8}$

$= \frac{\sqrt{5}-1}{4}$ (4)

Q(2) (a) $L_1: y = \frac{1}{3}x + \frac{1}{3}$

$L_2: y = \frac{3}{4}x - 3$

$\therefore M_1 = \frac{1}{3}, \quad M_2 = \frac{3}{4}$

$\tan(\theta_2 - \theta_1) = \frac{M_2 - M_1}{1 + M_2 \times M_1}$

$= \frac{\frac{3}{4} - \frac{1}{3}}{1 + \frac{3}{4} \times \frac{1}{3}}$

$= \frac{5}{12} \div \frac{5}{4}$

$= \frac{1}{3}$

$= M_1$

$= \tan\theta_1$ (5)

(b) (i) $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}$

$= \frac{2 \times \frac{1}{2}}{1 - \frac{1}{4}}$

$= \frac{4}{3}$ (3)

Q(2) (b) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(\frac{1-t^2}{1+t^2}\right)^2 - \left(\frac{2t}{1+t^2}\right)^2$

$= \left(\frac{1-\frac{1}{4}}{1+\frac{1}{4}}\right)^2 - \left(\frac{2 \times \frac{1}{2}}{1+\frac{1}{4}}\right)^2$

$= \frac{9}{25} - \frac{16}{25}$

$= -\frac{7}{25}$ (3)

(c) (i) $3x^2(x+1) - 1(x+1)$

$= (x+1)(3x^2-1)$ (1)

(ii) let $x = \tan\theta \quad 0 \le \theta \le \pi$

$\therefore (\tan\theta + 1)(3\tan^2\theta - 1) = 0$

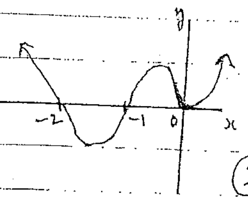
$\tan\theta = -1$ or $\tan^2\theta = \frac{1}{3}$

$\theta = \frac{3\pi}{4} \qquad \tan\theta = \pm\frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore \theta = \frac{\pi}{6}, \frac{3\pi}{4}, \frac{5\pi}{6}$ (3)

(3) (a)



(2)

(b) (i) 4 (1)

(ii) $8x^4$ (1)

(iii) 2 (1)

(c) (i) $f(1) = a + b + c$ (1)

(ii) $a + b + c = 0$ (1)

(iii) $f(1) = a + (b-a) + (c-b) - c$

$= 0$

$\therefore x - 1$ is a factor (2)

$$\begin{array}{r} ax^2 + bx + c \\ x-1 \overline{\smash{)}\, ax^3 + (b-a)x^2 + (c-b)x - c} \\ \underline{ax^3 - ax^2} \qquad\qquad\qquad \\ bx^2 + (c-b)x \qquad \\ \underline{bx^2 - bx} \qquad\quad \\ cx - c \\ \underline{cx - c} \\ 0 \end{array}$$ (4)

$\therefore$ other factor $ax^2 + bx + c$

(iv) $ax^2 + bx + c$ has 2 real distinct roots if $\Delta > 0$

ie; $b^2 - 4ac > 0$

and also $a + b + c \ne 0$

ie; $x - 1$ is not a factor of $ax^2 + bx + c$ (2)

4a) $V = \pi \int y^2 \, dx$

$= \pi \int_0^{\pi/3} 4\sec^2 x \, dx$

$= 4\pi \left[\tan x\right]_0^{\pi/3}$ 3

$= 4\pi . \sqrt{3}$ $\therefore$ Volume $= 4\sqrt{3}\pi \; u^3$.

b) i) $\sqrt{2}\sin x + \sqrt{2}\cos x = R\sin(x+\alpha)$

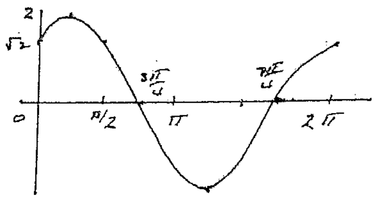
$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

$\therefore R\cos\alpha = \sqrt{2}$, $R\sin\alpha = \sqrt{2}$ 2

$\therefore \tan\alpha = 1$, $\alpha = \pi/4$

$\therefore R . \frac{1}{\sqrt{2}} = \sqrt{2}$, $R = 2$

$\therefore y = 2\sin(x + \frac{\pi}{4})$



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iii) 3 solutions if $k = \sqrt{2}$. 1

c) $\sin 2\theta = \sin\theta$

$\therefore 2\sin\theta\cos\theta = \sin\theta$

$\therefore \sin\theta(2\cos\theta - 1) = 0$

$\therefore \sin\theta = 0$ $\cos\theta = 1/2$ 3

$\therefore \theta = n\pi$, $2n\pi \pm \pi/3$ 1

d) $\int \cos^2\theta . d\theta = \frac{1}{2}\int (1 + \cos 2\theta) d\theta$ 1

$= \frac{1}{2}(\theta + \frac{1}{2}\sin 2\theta) + C$ 1

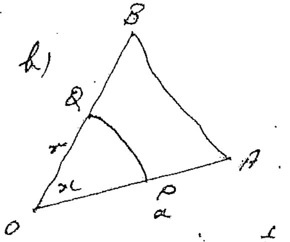
$= \theta/2 + \frac{1}{4}\sin 2\theta + C$. 1

Q5 a) $\lim_{x \to 0} \frac{\sin 2x}{3x}$

as $x \to 0$, $\sin 2x \to 2x$. 2

$\therefore \lim_{x \to 0} \frac{\sin 2x}{3x} \Rightarrow \frac{2x}{3x} = 2/3$.

b)



i) Area $\triangle AOB = \frac{1}{2}a^2\sin x$ 2

Area $POQ = \frac{1}{2}r^2 x$

$\therefore \frac{1}{2}a^2\sin x = r^2 x$.

$\therefore r^2 = \frac{a^2\sin x}{2x}$.

ii) If $\angle AOB = \pi/2$, $r^2 = \frac{a^2 . 2}{2 . \pi} = \frac{a^2}{\pi}$

$\therefore r = \frac{a}{\sqrt{\pi}}$ 2

iii) as $x \to 0$, $\frac{\sin x}{x} \to 1$ 2

$\therefore r^2 \to a^2/2$

$r \to a/\sqrt{2}$

c) i) $P(x) = (x+2)(x-2)(x^2 + bx + c)$ 2

$P(0) = 4$ $\therefore 4 = -4 . c$ $c = -1$

$P(1) = -3$ $\therefore -3 = (3)(-1)(1 + b + c)$

$\therefore b + c + 1 = +1$ 3

$\therefore b - 1 + 1 = +1$, $b = 1$

$\therefore P(x) = (x+2)(x-2)(x^2 + x - 1)$ 1

$\therefore$ Roots are $x = \pm 2$, $\frac{-1 \pm \sqrt{5}}{2}$. 1