SYDNEY GIRLS HIGH SCHOOL

YEAR 12
MATHEMATICS
EXTENTION 1

Assessment Task 2
MARCH -2004

TOPICS : Exponential and Logarithmic Functions
The Trigonometric Function
Trigonometric Functions II

Time allowed- 75 minutes

DIRECTIONS TO CANDIDATES NAME

- Attempt All questions
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

QUESTION 1 (35 marks)

(a) Simplify
(i) $\log_3 81$
(ii) $\log_e \sqrt{e}$
(iii) $\log_6 3 + \log_6 12$

(b) Differentiate the following with respect to x
(i) $y = e^{-3x}$
(ii) $y = 2e^{\sqrt{x}}$
(iii) $y = \ln x^3$

(c) Find the following indefinite integrals
(i) $\int e^{3x} + e^{-x}\, dx$
(ii) $\int 2xe^{x^2+1}\, dx$
(iii) $\int \frac{x}{x^2 - 3}\, dx$

(d) Solve $4^{y-3} = 9$ correct to 2 decimal places.

(e) Find $\frac{d}{dx}(xe^x - e^x)$ hence, evaluate $\int_1^e xe^x\, dx$

(f) If $y = e^{2x} + e^{4x}$ show that $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$

(g) Determine the equation of the tangent to the curve $y = \ln(x^2 + e)$ at $x = 0$

(h) Differentiate the following
(i) $\log_2 x$
(ii) 2^x

(i) Find the exact area bounded by the curve $y = \frac{1}{x-1}$, the x-axis and the lines $x = 4$ and $x = 7$.

(j) The curve $y = \sqrt{e^x + 1}$ is rotated about the x-axis from $x = 0$ to $x = 1$. Find the exact volume of the solid formed.

(k) Show that $\frac{x-6}{x-1} = 1 - \frac{5}{x-1}$ and hence, find $\int \frac{x-6}{x-1}\, dx$.

QUESTION 2 (30 marks)

(a) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin x}{4x}$

(b) Find $\dfrac{dy}{dx}$ of the following:

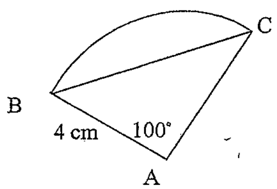
(i) $y = \cos \dfrac{x}{4}$

(ii) $y = \tan(3x + 1)$

(iii) $y = \dfrac{1 - \sin x}{1 + \sin x}$

(c) The area of a circle is 450 cm^2.
Find in degrees and minutes, the angle subtended at the centre of the circle by a 2.7 cm arc.

(d)



Arc BC subtends an angle of 100° at the centre A of a circle with radius 4cm.
Find:

(i) The exact perimeter of the sector ABC.

(ii) The approximate ratio of the area of the minor segment to the area of the sector.

(e) Differentiate $\log_e (\sin x)$

(f) Find $\int 3 \sec^2 \dfrac{x}{3} \, dx$

(g) For the graph $y = 3\cos 2x$
State its (i) Amplitude
(ii) Period
and (iii) Sketch the curve for $0 \le x \le 2\pi$

hence or otherwise, Evaluate

(iv) $\int_0^{\frac{\pi}{2}} 3\cos 2x \, dx$

(h) A curve has $\dfrac{d^2y}{dx^2} = 18\sin 3x$ and a stationary point at $(\dfrac{\pi}{6}, -2)$.
Find the equation of the curve.

QUESTION 3 (35 marks)

(a) Solve $2\sin^2\theta - \sin\theta - 1 = 0$ for $0 \le \theta \le 2\pi$

(b) If $\sin A = \dfrac{1}{2}$ and $\cos B = \dfrac{1}{\sqrt{2}}$ where $0 < A < \dfrac{\pi}{2}$ and $0 < B < \dfrac{\pi}{2}$.
Find the exact value for: $\cos(A + B)$

(c) Prove the following identity:
$$\frac{\sin A}{\cos A + \sin A} + \frac{\sin A}{\cos A - \sin A} = \tan 2A.$$

(d) Solve the equation $\sin 2x = 2\sin^2 x$ for $0 < x < \pi$

(e) Find the acute angle between the lines
$2x + y = 4$ and $x - y = 2$, to the nearest degree.

(f) If $\sin A = \dfrac{\sqrt{3}}{2}$ and $0 < A < \dfrac{\pi}{2}$, find the exact values for

(i) $\cos 2A$

(ii) $\tan 2A$.

(g) Find the indefinite integral of: $\int (\sin^2 x) \, dx$

(h) Write down the expansion for $\sin 2x$.
Hence find : $\int \sin x . \cos x \, dx$

(i) Find the exact volume of the solid formed if the curve $y = \cos x + 1$ is rotated about the x –axis from $x = 0$ to $x = \dfrac{\pi}{2}$

(j) Using 't' method solve, where $t = \tan \dfrac{\theta}{2}$ (Give answer to 3 decimal places)
$2\cos\theta + \sin\theta = 1$ for $0^\circ \le \theta \le 360^\circ$.

(k) (i) Express $\cos\theta + \sqrt{3}\sin\theta$ in the form $R\sin(\theta + a)$, where a is in radians.
(ii) Hence or otherwise find all the values of θ in the range $0 \le \theta \le 2\pi$ for which $\cos\theta + \sqrt{3}\sin\theta = 1$.

THE END

Marking Scale.

YEAR 12
MATHEMATICS
EXTENTION 1 - 2004
MARCH

$\frac{}{100}$

Question 1: (35 marks)

(a) (i) $\log_3 81 = \log_3 3^4$
$= 4$ ①

(ii) $\log_e \sqrt{e} = \log_e e^{1/2}$
$= \frac{1}{2}$ ①

(iii) $\log_6 3 + \log_6 12 = \log_6 (3 \times 12)$
$= \log_6 36$
$= \log_6 6^2$
$= 2$ ①

(b) (i) $y = e^{-3x}$
$\frac{dy}{dx} = -3e^{-3x}$ ①

(ii) $y = 2e^{\sqrt{x}}$, $y = 2e^{x^{1/2}}$
$\frac{dy}{dx} = 2 \times \frac{1}{2} x^{-1/2} e^{\sqrt{x}}$
$= \frac{e^{\sqrt{x}}}{\sqrt{x}}$ ②

(iii) $y = \ln x^3$
$\frac{dy}{dx} = \frac{3x^2}{x^3}$
$= \frac{3}{x}$ ①

c) (i) $\int e^{3x} + e^{-x}\,dx$
$= \frac{1}{3}e^{3x} - e^{-x} + C$ ②

(ii) $\int 2x\, e^{x^2+1}\,dx$
$= e^{x^2+1} + C$ ②

(iii) $\int \frac{x}{x^2-3}\,dx$
$= \frac{1}{2}\ln(x^2-3) + C$ ②

d) $4^{y-3} = 9$
$\log_e 4^{y-3} = \log_e 9$
$y - 3 = \frac{\log_e 9}{\log_e 4}$
$\therefore y = \frac{\ln 9}{\ln 4} + 3$ ②
$y \doteqdot 4.58$

e) $\frac{d}{dx}(x e^x - e^x)$
$= x.e^x + 1.e^x - e^x$
$= xe^x$

$\int_1^e x e^x dx = \left[xe^x - e^x\right]_1^e$
$= \left[e.e^e - e^e\right] - \left[e - e\right]$
$= e^e\left[e-1\right]$ ③

f) $y = e^{2x} + e^{4x}$
$\frac{dy}{dx} = 2e^{2x} + 4e^{4x}$

$\frac{d^2y}{dx^2} = 4e^{2x} + 16e^{4x}$

$\therefore \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$

$= 4e^{2x} + 16e^{4x} - 6(2e^{2x} + 4e^{4x}) + 8(e^{2x} + e^{4x})$

$= 4e^{2x} + 16e^{4x} - 12e^{2x} - 24e^{4x} + 8e^{2x} + 8e^{4x}$ ③

$= 0$

g) $y = \ln(x^2 + e)$
$\frac{dy}{dx} = \frac{2x}{x^2+e}$ at $x = 0$

$m = 0$
at $x = 0$, $y = \ln(0 + e)$
$= \ln e$
$y = 1$
$\therefore$ Equation of tangent
$y - y_1 = m(x - x_1)$
$y - 1 = 0(x - 0)$
$\therefore y - 1 = 0$ or
$y = 1$ ③

h) (i) $y = \log_2 x$
$= \frac{\log_e x}{\log_e 2}$
$= \frac{1}{\log_e 2} \times \log_e x$
$\frac{dy}{dx} = \frac{1}{\log_e 2} \times \frac{1}{x}$
$= \frac{1}{x.\log_e 2}$ ①

(ii) $y = 2^x$
$2 = e^{\ln 2}$
$\therefore 2^x = (e^{\ln 2})^x$
$= e^{x\ln 2}$

$\therefore \frac{dy}{dx} = \ln 2 . e^{x \ln 2}$
$= \ln 2 . 2^x$ ①
or $= 2^x . \ln 2$.

(i) $A = \int_4^7 \frac{1}{x-1}\,dx$
$= \left[\ln(x-1)\right]_4^7$
$= \ln(7-1) - \ln(4-1)$
$= \ln 6 - \ln 3$
$= \ln \frac{6}{3}$
$= \log_e 2$ units^2 ③

dx

(j) $y = \sqrt{e^x + 1}$

$$\text{Volume} = \pi \int_a^b y^2\,dx$$

$$= \pi \int_0^1 e^x + 1\,dx$$

$$= \pi \left[e^x + x \right]_0^1$$

$$= \pi \left[(e + 1) - (1 - 0) \right] \quad 3$$

$$= e\pi \text{ units}^3$$

k) $\frac{x-6}{x-1} = \frac{x-1}{x-1} - \frac{5}{x-1}$

$$= 1 - \frac{5}{x-1}$$

$$\int \frac{x-6}{x-1}\,dx$$

$$= \int 1 - \frac{5}{x-1}\,dx \quad 3$$

$$= x - 5\log(x-1) + C$$

Question 2 (30 marks)

a) $\lim_{x \to 0} \frac{\sin x}{4x} = \frac{1}{4} \lim_{x \to 0} \frac{\sin x}{x}$

$$= \frac{1}{4} \quad 2$$

b) (i) $y = \cos \frac{x}{4}$

$$\frac{dy}{dx} = -\frac{1}{4} \sin \frac{x}{4} \quad 2$$

(ii) $y = \tan(3x+1)$

$$\frac{dy}{dx} = 3\sec^2(3x+1) \quad 2$$

(iii) $y = \frac{1 - \sin x}{1 + \sin x}$ $\frac{u}{v}$

let $u = 1 - \sin x$ $v = 1 + \sin x$

$\frac{du}{dx} = -\cos x$ $\frac{dv}{dx} = \cos x$

$$\therefore \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(1+\sin x) - \cos x - \cos x(1 - \sin x)}{(1+\sin x)^2}$$

$$= \frac{-\cos x(1+\sin x) - \cos x(1-\sin x)}{(1+\sin x)^2}$$

$$= \frac{-\cos x - \cancel{\sin x \cos x} - \cos x + \cancel{\sin x \cos x}}{(1+\sin x)^2}$$

$$= \frac{-2\cos x}{(1+\sin x)^2} \quad 3$$

c) $A = \pi r^2$

$450 = \pi r^2$

$\frac{450}{\pi} = r^2$

$r = \sqrt{\frac{450}{\pi}}$

$r = 11.97$

c) Now, $l = r\theta$

$2.7 = 11.97\theta$

$\frac{2.7}{11.97} = \theta$

$\therefore \theta = 0.226$

π radians $= 180^\circ$

1 radian $= \frac{180^\circ}{\pi}$

$\therefore 0.226 \text{ radians} \times \frac{180^\circ}{\pi}$

$= 12.93^\circ$ 2

$\theta = 12^\circ 56'$

d) (i) $l = r\theta$ $1^\circ = \frac{\pi}{180}$

$l = 4 \times 100^\circ$

$= 4 \times \frac{100}{180}\pi$

$= 2\frac{2}{9}\pi$

$l = \frac{20}{9}\pi$ cm 2

$\therefore$ Exact Perimeter

$= \left(8 + \frac{20}{9}\right)\pi$ cm

(ii) Area of minor segment

$A = \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2}(4)^2\left(\frac{5}{9}\pi - \sin 100^\circ\right)$

$= 6.08$

Area of sector

$= \frac{1}{2}r^2\theta$

$= \frac{1}{2}(4)^2 \frac{5\pi}{9}$

$= 13.96$

$\therefore$ Area of minor segment : Area of sector

is approximately

6 : 14

3 : 7 4

e)

let $y = \log_e(\sin x)$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x}$$

$$= \cot x \quad 2$$

f) Find $\int 3\sec^2 \frac{x}{3}\,dx$

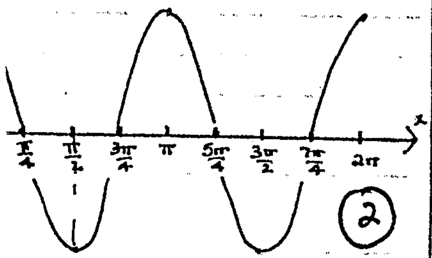
$$= 9\tan\frac{x}{3} + C \quad 2$$

$y = 3\cos 2x$ 1

i) Amplitude = 3

i) Period = $\frac{2\pi}{2} = \pi$ 1

ii) Each interval = $\frac{\pi}{4}$



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) $\int_0^{\pi/2} 3\cos 2x \, dx$

$\int_0^{\pi/2} 3\cos 2x \, dx$

$\left[\frac{3}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$

$\frac{3}{2}\left[\sin 2\frac{\pi}{2} - \sin 0\right]$

$= \frac{3}{2}[\,0\,]$

$= 0$ 2

h) $\frac{d^2y}{dx^2} = 18\sin 3x$

$\int 18\sin 3x \, dx$

$\frac{dy}{dx} = {}^-6\cos 3x + C$

$\frac{dy}{dx} = 0$ at $x = \frac{\pi}{6}$

$-6\cos 3\left(\frac{\pi}{6}\right) + C = 0$

$\therefore C = 0$

$\frac{dy}{dx} = -6\cos 3x$

$\int -6\cos 3x \, dx$

$y = {}^-2\sin 3x + C$

at $x = \frac{\pi}{6}$, $y = {}^-2$

$-2 = -2\sin 3\left(\frac{\pi}{6}\right) + C$

$-2 = -2 + C$

$\therefore C = 0$

Eqn of curve

$y = -2\sin 3x$ 3

Question 3 (35 marks)

(a) $2\sin^2\theta - \sin\theta - 1 = 0$

let $m = \sin\theta$

$2m^2 - m - 1 = 0$

$(2m+1)(m-1) = 0$

$2m+1 = 0$ or $m - 1 = 0$

$2m = -1$ $\quad m = 1$

$m = -\frac{1}{2}$ $\quad \sin\theta = 1$

$\therefore \sin\theta = -\frac{1}{2}$ $\quad \theta = 90^\circ$

S | A
T | C

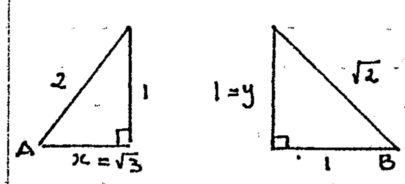
$\therefore \theta = 210^\circ$ or 330°

3

$\therefore \theta = 90^\circ, 210^\circ$ or 330°

$= \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

(b) $\sin A = \frac{1}{2}$ $\quad \cos B = \frac{1}{\sqrt{2}}$



$x^2 = 2^2 - 1^2$ $\quad y^2 = (\sqrt{2})^2 - 1^2$

$x^2 = 4 - 1$ $\quad y^2 = 2 - 1$

$x = \sqrt{3}$ $\quad y = 1$

$\cos(A+B) \equiv \cos A\cos B - \sin A \sin B$

$\equiv \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$

3

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

OR $= \frac{\sqrt{6}-\sqrt{2}}{4}$

c) $\frac{\sin A}{\cos A + \sin A} + \frac{\sin A}{\cos A - \sin A} = \tan 2A$

L.H.S

$\frac{\sin A(\cos A - \sin A) + \sin A(\cos A + \sin A)}{(\cos A + \sin A)(\cos A - \sin A)}$

$\frac{\sin A\cos A - \cancel{\sin^2 A} + \sin A\cos A + \cancel{\sin^2 A}}{(\cos A + \sin A)(\cos A - \sin A)}$

$= \frac{2\sin A\cos A}{\cos^2 A - \sin^2 A}$

$= \frac{\sin 2A}{\cos 2A}$

3

$= \tan 2A$

$=$ R.H.S.

d) $\sin 2x = 2\sin^2 x$

$2\sin x\cos x = 2\sin^2 x$

$2\sin^2 x - 2\sin x\cos x = 0$

$2\sin x(\sin x - \cos x) = 0$

$2\sin x = 0$ or $\sin x = \cos x$

$\sin x = 0$ $\quad 1 = \tan x$

$\therefore x = 0^\circ$ or 360°, 180° $\quad \therefore x = 45^\circ, 225^\circ$

OR/

$x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$

3

$\therefore$ only solution

$x = \frac{\pi}{4}$ as

e) $2x+y=4 \qquad x-y=2$

$y=-2x+4 \qquad y=x-2$

$m_1=-2 \qquad \therefore m_2=1$

The angle between two lines ϕ, is given by

$$\tan\phi = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$= \left|\frac{1+2}{1+-2.1}\right|$$

$$\tan\phi = \left|\frac{3}{1-2}\right|$$

3

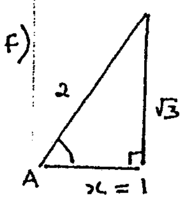
$$= \left|\frac{3}{-1}\right|$$

$$\tan\phi = 3$$

$$\phi = 71.565\ldots^\circ$$

$= 72^\circ$ (to the nearest degree)

f)



$x^2 = 2^2-(\sqrt{3})^2$

$x^2 = 4-3$

$x^2=1$

$\therefore x = 1$

$\sin A = \frac{\sqrt{3}}{2} \qquad \tan A = \frac{\sqrt{3}}{1}$

$\cos A = \frac{1}{2} \qquad = \sqrt{3}$

(i) $\cos 2A = 2\cos^2 A - 1$

$= 2\left(\frac{1}{2}\right)^2 - 1$

$= -\frac{1}{2}$

2

(ii) $\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$

$$= \frac{2\times\sqrt{3}}{1-(\sqrt{3})^2}$$

$$= \frac{2\sqrt{3}}{1-3} = \frac{2\sqrt{3}}{-2}$$

2

$\tan 2A = -\sqrt{3}$

g) $\int(\sin^2 x)\,dx$

$= \frac{1}{2}\int(1-\cos 2x)\,dx$

2

$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$

h) $\sin 2x = 2\sin x\cos x$

$\therefore \int \sin x \cos x\,dx$

$= \frac{1}{2}\int \sin 2x\,dx$

$= \frac{1}{2}\times\frac{-1}{2}\cos 2x + C$

3

$= -\frac{1}{4}\cos 2x + C$

i) $V = \pi\int_a^b y^2\,dx$

$= \pi\int_0^{\pi/2} (\cos x + 1)^2\,dx$

$= \pi\int_0^{\pi/2} \cos^2 x + 2\cos x + 1\,dx$

(i) $\cos 2x \equiv 2\cos^2 x - 1$

$\cos 2x + 1 \equiv 2\cos^2 x$

$\therefore \frac{1}{2}\cos 2x + \frac{1}{2} \equiv \cos^2 x$

$= \pi\int_0^{\frac{\pi}{2}} \frac{1}{2}\cos 2x + \frac{1}{2} + 2\cos x + 1\,dx$

$= \pi\int_0^{\frac{\pi}{2}} \frac{1}{2}\cos 2x + 2\cos x + \frac{3}{2}\,dx$

$= \pi\left[\frac{1}{4}\sin 2x + 2\sin x + \frac{3}{2}x\right]_0^{\frac{\pi}{2}}$

$= \pi\left[\frac{1}{4}\sin 2\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{\pi}{2}\right) + \frac{3}{2}\left(\frac{\pi}{2}\right) - 0\right]$

$= \pi\left[0 + 2 + \frac{3\pi}{4}\right]$

$= \left(2\pi + \frac{3\pi^2}{4}\right)$ units^3

OR

$\frac{\pi}{4}(8+3\pi)\,u^3$

3

(j) $2\cos\theta + \sin\theta = 1$

$2\left(\frac{1-t^2}{1+t^2}\right) + \frac{2t}{1+t^2} = 1$

$\frac{2-2t^2+2t}{1+t^2} = 1$

$-2t^2+2t+2 = 1+t^2$

$-3t^2+2t+1=0$

$3t^2-2t-1=0$

$(3t+1)(t-1)=0$

$3t+1=0 \qquad t-1=0$

$3t=-1 \qquad t=1$

$t=\frac{-1}{3} \qquad \tan\frac{\theta}{2} = 1$

S A / T C

$\tan\frac{\theta}{2} = -\frac{1}{3}$

S A / T C

$\frac{\theta}{2} = 161^\circ.565$ or $341^\circ.565$

$\frac{\theta}{2} = 45^\circ$ or 225°

$\therefore \theta = 323.130^\circ$ or 90°

(only solutions in $0^\circ \le \theta \le 360^\circ$)

4

k) (i) $\cos\theta + \sqrt{3}\sin\theta \equiv R\sin(\theta+\alpha)$

$R = \sqrt{1^2+(\sqrt{3})^2}$

$= \sqrt{4}$

$R = 2$

$\cos\theta + \sqrt{3}\sin\theta \equiv \sin A\cos\theta + \cos A\sin\theta$

where $\sin A = 1$

$\cos A = \sqrt{3}$

$\therefore \tan A = \frac{1}{\sqrt{3}}$

$A = \frac{\pi}{6} = \alpha$

$\therefore \cos\theta + \sqrt{3}\sin\theta \equiv 2\sin\left(\theta + \frac{\pi}{6}\right)$

2

(ii) Solve $\cos\theta + 3\sin\theta = 1$

$\cos\theta + \sqrt{3}\sin\theta \equiv 2\sin\left(\theta + \frac{\pi}{6}\right)$

$(\div 2)$

$\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta \equiv \frac{1}{2} \equiv \sin\left(\theta+\frac{\pi}{6}\right)$

$\sin\left(\theta+\frac{\pi}{6}\right) = \frac{1}{2}$

$\therefore \left(\theta+\frac{\pi}{6}\right) = \frac{\pi}{6}, \pi-\frac{\pi}{6}, 2\pi+\frac{\pi}{6}\ldots$

$\therefore \theta = 0, \frac{2\pi}{3}, 2\pi \quad (0\le\theta\le 2\pi)$

2