Sydney Girls High School

MATHEMATICS- Extension 1
HSC ASSESSMENT TASK 2

March 2005

Topics: Exponential and Logarithmic Functions, Trigonometric Functions, Trigonometric Functions 2.

Time Allowed: 75 minutes + 5 minutes reading time.

Instructions:

- There are four (4) questions of equal value.
- Attempt all questions.
- Start each question on a new page.
- Show all necessary working.
- Marks may be deducted for careless or poor setting out.
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.

Total = 80 marks

QUESTION 1 (20 marks) **Marks**

(a) Given $\log_2 3 \doteqdot 1.6$ and $\log_2 5 \doteqdot 2.3$ evaluate:

(i) $\log_2 15$ 2

(ii) $\log_2 30$ 2

(b) Solve $e^{2x-3} = 17$ for x, correct to 1 decimal place. 2

(c) Draw a neat sketch of $y = \log_e(x-4)$, showing all relevant features. 2

(d) Differentiate each of the following with respect to x:

(i) e^{3x} 1

(ii) 7^x 1

(iii) $6xe^{4x^2}$ 2

(iv) $\log_e(5x^2+2x)$ 2

(e) Find the equation of the tangent to the curve $y = 4e^{3x+1}$ at the y-intercept. 3

(f) Find:

(i) $\int e^{6x+1}\,dx$ 1

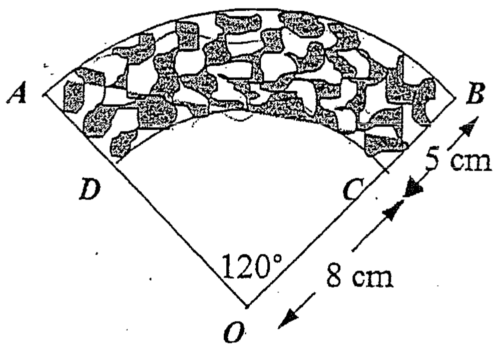
(ii) $\int \frac{x^2}{x^3-10}\,dx$ 2

QUESTION 2 (20 marks) — **Marks**

(a) A fan in the shape of a sector of a circle has its frame made of cane and the shaded area is material. The frame consists of two circular arcs, AB and DC, and the sides AO and OB.

(i) How much cane is used in the frame, to the nearest cm? — 2

(ii) What area of material is used in the shaded part of the fan, to the nearest cm^2? — 2



(b) Differentiate the following with respect to x:

(i) $y = \cos(2x+5)$ — 1

(ii) $y = \log_e(\sin x)$ — 2

(ii) $y = \dfrac{x}{\tan 4x}$ — 2

(c) Evaluate:

(i) $\lim\limits_{x \to 0} \dfrac{\sin \frac{x}{5}}{x}$ — 2

(ii) $\int_0^{\frac{\pi}{6}} \tan^2 x \, dx$ — 2

(d) Find the equation of the normal to the curve $y = \sin x$ at the point where $x = \dfrac{\pi}{4}$. — 3

(e) Sketch the following curves for $0 \le x \le 2\pi$. State the amplitude and period of each curve.

(i) $y = 3\sin 2x$ — 2

(ii) $y = \cos\left(2x + \dfrac{\pi}{2}\right)$ — 2

QUESTION 3 (20 marks) **Marks**

(a) If $\sin a = \frac{\sqrt{3}}{2}$, $\frac{\pi}{2} < a < \pi$, and $\cos b = \frac{1}{\sqrt{2}}$, $0 < b < \frac{\pi}{2}$, evaluate $\cos(a-b)$. 2

(b) Prove the following identities

(i) $\cos\left(\frac{\pi}{6}+x\right)+\sin\left(\frac{\pi}{3}+x\right)=\sqrt{3}\cos x$ 3

(ii) $\sin 4x = 4\sin x\cos^3 x - 4\cos x\sin^3 x$ 3

(c)

(i) Show that $\sin^4 x = \left[\frac{1}{2}(1-\cos 2x)\right]^2$ 2

(ii) Hence or otherwise, show that $\int \sin^4 x \, dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$ 3

(d)

(i) Find $\frac{d}{dx}\left(\sin^3 3x\right)$ 2

(ii) Hence or otherwise find $\int \sin^2 3x \cos 3x \, dx$ 2

(e) Find the acute angle, to the nearest minute, between the following lines: 3

$3x + y - 5 = 0 \quad \text{and} \quad x - 2y + 2 = 0$

QUESTION 4 (20 marks) Marks

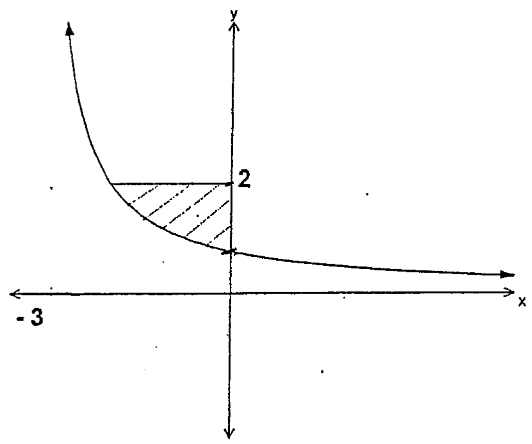
(a) Show that $y = xe^{-x}$ has a maximum point at $\left(1, \frac{1}{e}\right)$ 3

(b) Evaluate $\displaystyle\int_1^{\log_e 2} \frac{e^x + 1}{e^x}\, dx$ 2

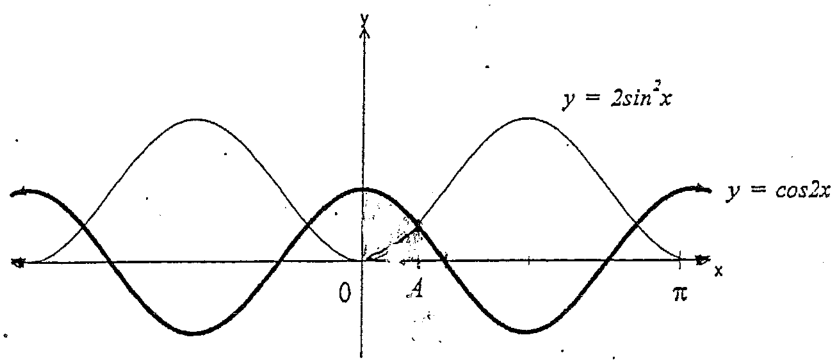
(c) (i) Show that $\dfrac{x+2}{x+5} = 1 - \dfrac{3}{x+5}$ 1

(ii) Hence or otherwise evaluate $\displaystyle\int_2^3 \frac{x+2}{x+5}\, dx$ 2

(d) Find the exact area of the region bounded by the curve $y = \dfrac{2}{x+3}$, the y-axis, and the line $y = 2$. 3



(e) The diagram below shows the curves $y = \cos 2x$ and $y = 2\sin^2 x$ for $-\pi \le x \le \pi$.



(i) Find the coordinates of A, a point of intersection of the two curves. 3

(ii) Find the exact area enclosed between the curves $y = \cos 2x$ and $y = 2\sin^2 x$ from the origin to the point A. 3

(iii) Find the volume of the solid formed if the area in part (ii) is rotated about the x-axis. [Hint: Question 3(c) (ii) may be of some use] 3

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QUESTION 1 (20 marks)

(a) $\log_2 3 \doteqdot 1.6$ and $\log_2 5 \doteqdot 2.3$

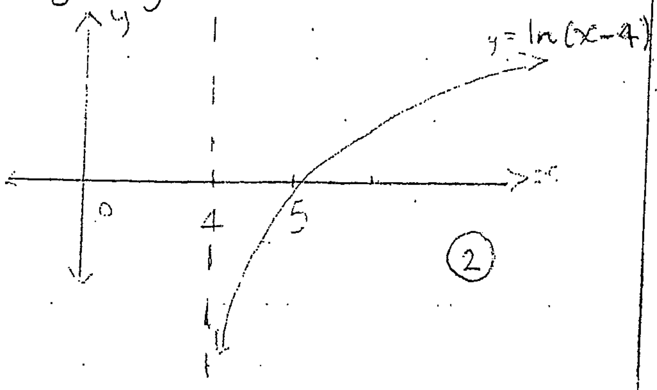
(i) $\log_2 15 = \log_2 (3\times 5)$
$= \log_2 3 + \log_2 5$
$= 3.9$ (2)

(ii) $\log_2 30 = \log_2 (2\times 3\times 5)$
$= \log_2 2 + \log_2 3 + \log_2 5$
$= 4.9$ (2)

or $\log_2 30 = \log_2 (2\times 15) = 1 + 3.9$
$= 4.9$

(b) $e^{2x-3} = 17$
$\log_e e^{2x-3} = \log_e 17$
$2x - 3 = \log_e 17$
$x = \frac{1}{2}(\log_e 17 + 3)$
$\doteqdot 2.9$ (2)

(c) $y = \log_e (x-4)$



(2)

(d) (i) $y = e^{3x}$

$\frac{dy}{dx} = 3e^{3x}$ (1)

(ii) $y = 7^x$
$= \log_e 7 . 7^x$ (1)

(iii) $y = 6xe^{4x^2}$

$\frac{dy}{dx} = uv' + vu'$

$u = 6x$
$u' = 6$
$v = e^{4x^2}$
$v' = 8xe^{4x^2}$

$= 6x(8xe^{4x^2}) + e^{4x^2}.6$
$= 6e^{4x^2}(8x^2+1)$ (2)

(iv) $y = \log_e (5x^2 + 2x)$

$\frac{dy}{dx} = \frac{10x+2}{5x^2+2x}$ (2)

(e) $y = 4e^{3x+1}$
$\frac{dy}{dx} = 12e^{3x+1}$ (3)
at $x = 0$
$m = 12e$

Note: at $x=0$
$y = 4e$

$y - y_1 = m(x - x_1)$
$y - 4e = 12e(x-0)$
$0 = 12ex - y + 4e$

(f) (i) $\int e^{6x+1}\,dx$
$= \frac{1}{6}e^{6x+1} + C$ (1)

(ii) $\int \frac{x^2}{x^3-10}\,dx$

$f(x) = x^3 - 10$
$f'(x) = 3x^2$

$= \frac{1}{3}\int \frac{3x^2}{x^3-10}\,dx$

$= \frac{1}{3}\log_e (x^3-10) + C$ (2)

QUESTION 2 (20 marks)

(a) (i)

Amount of cane used :

$= 2(8+5) + \left(\frac{120}{180} \times \pi \times 8\right) + \left(\frac{120\pi}{180} \times 13\right)$

$= 26 + \frac{16\pi}{3} + \frac{26\pi}{3}$

$= 26 + 14\pi$

$\doteqdot 70$ cm (2)

(ii) $A = \frac{1}{2} r^2 \theta$

Area of material $= \left(\frac{1}{2} \times 13^2 \times \frac{2\pi}{3}\right) - \left(\frac{1}{2} \times 8^2 \times \frac{2\pi}{3}\right)$

$\doteqdot 110$ cm^2 (2)

(b) (i) $y = \cos(2x+5)$

$\frac{dy}{dx} = -2\sin(2x+5)$ (1)

(ii) $y = \log_e(\sin x)$

$\frac{dy}{dx} = \frac{\cos x}{\sin x}$

$= \cot x$ (2)

(iii) $y = \frac{x}{\tan 4x}$

Let $u = x$ $\quad v = \tan 4x$

$u' = 1$ $\quad v' = 4\sec^2 4x$ (2)

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{\tan 4x(1) - x(4\sec^2 4x)}{\tan^2 4x}$

$= \frac{\tan 4x - 4x\sec^2 4x}{\tan^2 4x}$

(c) (i) $\lim_{x \to 0} \frac{\sin \frac{x}{5}}{x}$

$= \lim_{x \to 0} \frac{\sin \frac{x}{5}}{\frac{x}{5}} \times \frac{1}{5}$

$= \frac{1}{5} \lim_{x \to 0} \frac{\sin \frac{x}{5}}{\frac{x}{5}}$ (2)

$= \frac{1}{5}(1)$

$= \frac{1}{5}$

(ii) $\int_0^{\frac{\pi}{6}} \tan^2 x \, dx$

$= \int_0^{\frac{\pi}{6}} (\sec^2 x - 1) \, dx$

$= \left[\tan x - x\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{\sqrt{3}} - \frac{\pi}{6}$

$= \frac{6 - \sqrt{3}\pi}{6\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$ (2)

$= \frac{2\sqrt{3} - \pi}{6}$

(d) $y = \sin x$

$\frac{dy}{dx} = \cos x$

at $x = \frac{\pi}{4}$ $\quad \frac{dy}{dx} = \cos \frac{\pi}{4}$

$m_1 = \frac{1}{\sqrt{2}}$

$m_2 = -\sqrt{2}$

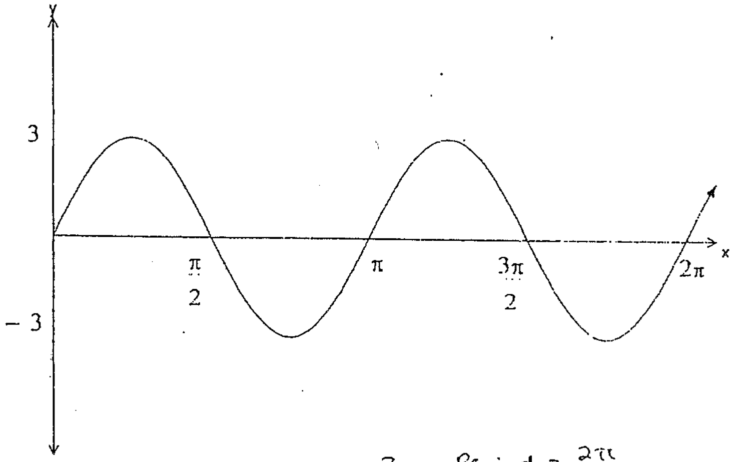
at $x = \frac{\pi}{4}$ $\quad y = \frac{1}{\sqrt{2}}$ (3)

$y - y_1 = m_2(x - x_1)$

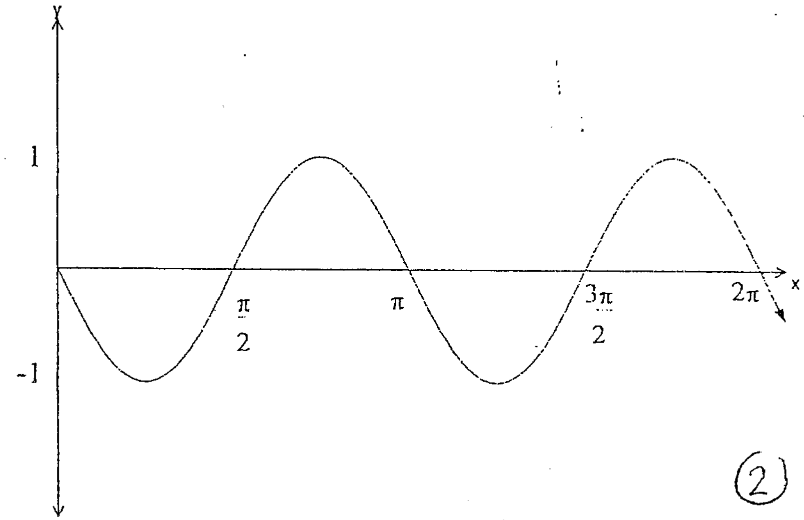
$y - \frac{1}{\sqrt{2}} = -\sqrt{2}\left(x - \frac{\pi}{4}\right)$

$y = -\sqrt{2}x + \frac{\pi\sqrt{2}}{4} + \frac{\sqrt{2}}{2}$

$\sqrt{2}x + y + \frac{\pi\sqrt{2}}{4} - \frac{\sqrt{2}}{2} = 0$



y = 3sin2x $\quad$ Amplitude = 3 $\quad$ Period $= \frac{2\pi}{2} = \pi$ (2)



(2)

$y = \cos\left(2x + \frac{\pi}{2}\right)$

$= \cos 2\left(x + \frac{\pi}{4}\right)$

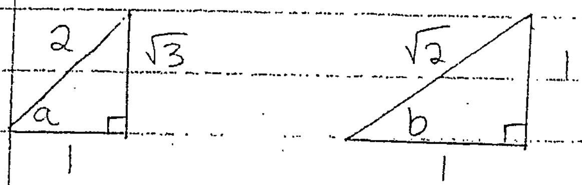
Amplitude = 1

Period = π

QUESTION 3 (20 marks)

(a) $\sin a = \frac{\sqrt{3}}{2}$, $\frac{\pi}{2} < a < \pi$

$\cos b = \frac{1}{\sqrt{2}}$, $0 < b < \frac{\pi}{2}$



$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$
$$= \left(-\frac{1}{2} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right)$$
$$= \frac{-1+\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4} \quad (2)$$

(b)(i)

$$LHS = \cos\left(\frac{\pi}{6}+x\right) + \sin\left(\frac{\pi}{3}+x\right)$$
$$= \cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}\sin x + \sin\frac{\pi}{3}\cos x + \cos\frac{\pi}{3}\sin x$$
$$= \frac{\sqrt{3}}{2}\cos x - \frac{\sin x}{2} + \frac{\sqrt{3}}{2}\cos x + \frac{\sin x}{2}$$
$$= \sqrt{3}\cos x$$
$$= RHS \quad (3)$$

(ii) $LHS = \sin 4x$
$$= 2\sin 2x \cos 2x$$
$$= 2(2\sin x\cos x)(\cos^2 x - \sin^2 x)$$
$$= 4\sin\cos^3 x - 4\cos x\sin^3 x$$
$$= RHS \quad (3)$$

(c)(i)

$$LHS = (\sin x)^4$$
$$= (\sin^2 x)^2$$
$$= \left[\frac{1}{2}(1-\cos 2x)\right]^2 \quad (2)$$

since $\cos 2x = 1 - 2\sin^2 x$
$$2\sin^2 x = 1 - \cos 2x$$
$$\sin^2 x = \frac{1}{2}\left[1 - \cos 2x\right]$$

(ii) $\int \sin^4 dx = \frac{1}{4}\int (1-\cos 2x)^2 dx$
$$= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x \, dx \quad (3)$$
$$= \frac{1}{4}\left[x - \frac{2}{2}\sin 2x + \frac{x}{2} + \frac{\sin 4x}{8}\right] + C$$
$$= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{x}{8} + \frac{1}{32}\sin 4x + C$$
$$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Note: $\int \cos^2 ax \, dx$
$$= \frac{1}{2}\int (1+\cos 2ax)\,dx$$
$$= \frac{x}{2} + \frac{1}{4a}\sin 2ax + C$$

(d)(i) $\frac{d}{dx}(\sin^3 3x)$
$$= 3(\sin 3x)^2 . 3\cos 3x \quad (2)$$
$$= 9\sin^2 3x \cos 3x$$

(ii) $\int 9\sin^2 3x\cos 3x \, dx = \sin^3 3x + C$

$\int \sin^2 3x \cos 3x \, dx = \frac{1}{9}\sin^3 3x + C($

(e) $m_1 = -\frac{3}{1}$ $\quad m_2 = \frac{1}{2}$

$= -3 \quad (3)$

Let θ be the acute angle

$$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right| = \left|\frac{-3-\frac{1}{2}}{1-\frac{3}{2}}\right| = 7$$

$\theta \doteqdot 81°52'$

QUESTION 4 (20 marks)

(a) $y = xe^{-x}$ $u = x \quad u' = 1$, $v = e^{-x} \quad v' = -e^{-x}$

$\frac{dy}{dx} = uv' + vu'$

$= x(-e^{-x}) + e^{-x}(1)$

$= e^{-x} - xe^{-x}$

For stat pts $\frac{dy}{dx} = 0$

$0 = e^{-x} - xe^{-x}$

$0 = e^{-x}(1 - x)$

$1 = x$ At $x = 1$ $y = \frac{1}{e}$

$\frac{d^2y}{dx^2} = -e^{-x} - [e^{-x} - xe^{-x}]$

$= -2e^{-x} - xe^{-x}$

at $x = 1$

$\frac{d^2y}{dx^2} = -2e^{-1} - e^{-1}$

$= -\frac{3}{e}$

< 0 ∴ max pt at $(1, \frac{1}{e})$

(3)

(b) $\int_1^{\log_e 2} \frac{e^x + 1}{e^x}\, dx$

$= \int_1^{\ln 2} 1 + e^{-x}\, dx$

$= [x - e^{-x}]_1^{\ln 2}$

$= [\ln 2 - e^{-\ln 2}] - [1 - e^{-1}]$

$= \ln 2 - e^{\ln(\frac{1}{2})} - 1 + \frac{1}{e}$

$= \ln 2 - \frac{1}{2} - 1 + \frac{1}{e}$

$= \ln 2 + \frac{1}{e} - \frac{3}{2}$

(2)

(c) (i) $\frac{x+2}{x+5} = \frac{x+5-3}{x+5}$ (1)

$= 1 - \frac{3}{x+5}$

(ii) $\int_2^3 \frac{x+2}{x+5}\, dx$

$= \int_2^3 \left(1 - \frac{3}{x+5}\right) dx$ (2)

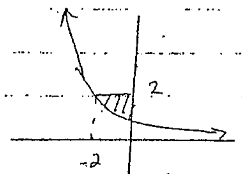
$= \left[x - 3\log_e(x+5)\right]_2^3$

$= (3 - 3\log_e 8) - (2 - 3\log_e 7)$

$= 1 - 3(\log_e 8 - \log_e 7)$

$= 1 - 3\log_e(\frac{8}{7})$

(d) $y = \frac{2}{x+3}$



Area = Area of square $- \int_{-2}^{0} \frac{2}{x+3}\, dx$

$= 4 - 2\int_{-2}^{0} \frac{dx}{x+3}$

$= 4 - 2[\ln(x+3)]_{-2}^{0}$ (3)

$= 4 - 2[\ln 3 - \ln 1]$

$\doteqdot 1.8\, u^2$

or, $A = \left|\int_{\frac{2}{3}}^{2} \frac{2}{y} - 3\, dy\right|$

$= |2\ln 3 - 4|$

$= 4 - 2\ln 3 \doteqdot 1.8\, u^2$

(e) (i) Point of intersection, $= A(x, y)$

$2\sin^2 x = \cos 2x$

$2\sin^2 x - \cos 2x = 0$

$2\sin^2 x - (1 - 2\sin^2 x) = 0$

$2\sin^2 x - 1 + 2\sin^2 x = 0$

$4\sin^2 x = 1$

$\sin^2 x = \frac{1}{4}$

$\sin x = \pm\frac{1}{2}$

But A lies between 0 & $\frac{\pi}{2}$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$ (3)

if $x = \frac{\pi}{6}$ $y = \cos 2(\frac{\pi}{6}) = \frac{1}{2}$

∴ $A(x, y) = (\frac{\pi}{6}, \frac{1}{2})$

(ii) $A = \int_0^{\frac{\pi}{6}} \cos 2x\, dx - 2\int_0^{\frac{\pi}{6}} \sin^2 x\, dx$

$= \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}} - 2 \cdot \frac{1}{2}\int_0^{\frac{\pi}{6}} 1 + \cos 2x\, dx$

$= \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}} - \left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}}$

$= \left[\frac{1}{2}\sin\frac{\pi}{3} - 0\right] - \left[\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3} - 0\right]$

$= \frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\sqrt{3}}{4}$ (3)

$= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$

$= \frac{3\sqrt{3} - \pi}{6}\, u^2$

(iii) $V = \pi\left[\int_0^{\frac{\pi}{6}} \cos^2 2x\, dx - \int_0^{\frac{\pi}{6}} 4\sin^4 x\, dx\right]$

Now for

$\int_0^{\frac{\pi}{6}} \cos^2 2x\, dx = \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right]$

$= \frac{1}{2}\left[\frac{\pi}{6} + \frac{1}{4}\sin\frac{2\pi}{3} - 0\right]$

$= \frac{\pi}{12} + \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{16}$ (3)

Now for

$4\int_0^{\frac{\pi}{6}} \sin^4 x\, dx = 4\left[\frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x\right]$

$= 4\left[\frac{3\pi}{48} - \frac{1}{4}\sin\frac{\pi}{3} + \frac{1}{32}\sin\frac{2\pi}{3} - 0\right]$

$= \frac{\pi}{4} - \sin\frac{\pi}{3} + \frac{1}{8}\sin\frac{2\pi}{3}$

$= \frac{\pi}{4} - \frac{\sqrt{3}}{2} + \frac{1}{8}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{4} - \frac{7\sqrt{3}}{16}$

∴ Vol. $= \pi\left[\frac{\pi}{12} + \frac{\sqrt{3}}{16} - \frac{\pi}{4} + \frac{7\sqrt{3}}{16}\right]$

$= \pi\left[\frac{8\sqrt{3}}{16} - \frac{\pi}{6}\right] u^3$

$\doteqdot \left(\frac{\sqrt{3}\pi}{2} - \frac{\pi^2}{6}\right) u^3$

(3)