# Sydney Girls High School

LABOR OMNIA VINCIT

# MATHEMATICS
# HSC ASSESSMENT TASK 2

## March 2006

Topics: Applications of the Second Derivative,
Probability, Sequences and Series.

Time Allowed: 90 minutes

Instructions:

- There are five (5) questions of equal value.
- Attempt all questions.
- Start each question on a new page.
- Show all necessary working.
- Marks may be deducted for careless or poor setting out.
- Board approved calculators may be used.

Total = 100 marks

Marks

QUESTION 1 (20 marks)

(a) In a production line, a batch of 200 finished items are tested and 8 are found to be faulty.

(i) What is the probability that an item selected at random is faulty? 1

(ii) How many faulty items could be expected if a batch of 600 were tested? 1

(b) The first three terms of an arithmetic series are 50, 43, 36.

(i) Find the common difference. 1

(ii) Write down a formula for the nth term. 1

(iii) If the last term of the series is –27, how many terms are there in the series? 2

(iv) Find the sum of the series. 2

(c) At Harbord High School the probability that Melissa is chosen as a prefect is $\frac{3}{5}$. The probability that Sarah is chosen as a prefect is $\frac{4}{5}$, whilst the probability that Tara is chosen as a prefect is $\frac{2}{5}$. Find the probability that out of the three girls:

(i) all three are chosen. 1

(ii) only Sarah and Tara are chosen . 1

(iii) at least one of the three is chosen. 1

(d) The sum of the first $n$ terms of a sequence is given by $S_n = 132n - 4n^2$.

(i) Find the sum of the first 7 terms 1

(ii) Find the sum of the first 6 terms 1

(iii) Hence find the 7th term. 1

(e) If $y = x^3 - 3x^2 - 7x - 5$, find:

(i) $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ 2

(ii) the value of $x$ for which $\frac{d^2y}{dx^2} = 0$ 1

(iii) the gradient of the tangent at the point where $\frac{d^2y}{dx^2} = 0$ 1

(iv) the equation of the tangent at the point where $\frac{d^2y}{dx^2} = 0$ 2

Marks

## QUESTION 2 (20 marks)

(a) In a group of 50 students there are 35 who watch the television show Desperate Housewives and 27 who watch Lost, while 6 watch neither .

(i) Draw a Venn diagram using the information above. 2

Find the probability that a student watches:

(ii) both television shows. 1

(iii) only Desperate Housewives. 1

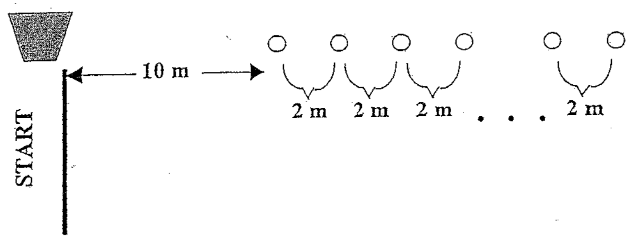
(iv) at least one of the television shows. 1

(b) For what value(s) of $x$ is the function $f(x)=2x^3-3x^2-12x+8$ : 4

(i) concave upwards

(ii) concave downwards

(c) In a game, competitors must run 10 metres from the starting point, pick up a ball, then run back to the start and place the ball in a bucket. The competitor then runs to the next ball which is 2 metres further than the first ball, picks that ball up, runs back to the start and places the ball in the bucket. This process is repeated with each successive ball 2 metres further than the previous one.




(i) How far does Bridget run if nine balls are placed in the bucket? 2

(ii) If Mark runs 644 metres in total to complete the race, how many balls were there? 3

(d) If $f(x)=x^3+ax^2+bx+3$, find the value of $a$ and $b$ given $f'(2)=9$ and $f''(4)=40$. 4

(e) Evaluate $\sum_{1}^{8} 3^r$ 2

## QUESTION 3 (20 marks)

Marks

(a) Justine is a talented soccer goal keeper. The probability that she can stop a penalty shot at goal is 70%. During a match the opposition had 3 penalty shots at goal. What is the probability that Justine stopped:

(i) all 3 penalty shots ? 1

(ii) exactly 1 penalty shot ? 1

(iii) at most 2 penalty shots ? 1

(b) By considering the recurring decimal $0.\dot{4}\dot{6}$ as the sum of an infinite geometric series, express the recurring decimal as a rational number. 3

(c) If $y=(x^2-1)(1+x)$ show that $x\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+2x-2=0$. 4

(d) Forty tickets are sold in a raffle that has 1st, 2nd and 3rd prizes given. Ryan purchases 8 tickets.

(i) Draw a probability tree using the information above. 2

What is the probability that Ryan wins:

(ii) the 1st prize only 1

(iii) at least two prizes 2

(e) The third term of a geometric sequence is 32 and the sixth term is 4.

(i) Find the first term and the common ratio. 2

(ii) Hence find an expression for the $n$th term. 1

(iii) Which term of the sequence is equal to $\frac{1}{2}$? 2

**QUESTION 4 (20 marks)** Marks

(a) On the day of her birth, $2^{nd}$ of January 1984, Sabrina's father deposited $2000 into a savings account which earned interest at 6% per annum, compounded annually.

(i) How much money would be in the account after the payment of interest on the $2^{nd}$ of January 2005 if no additional deposits were made? 2

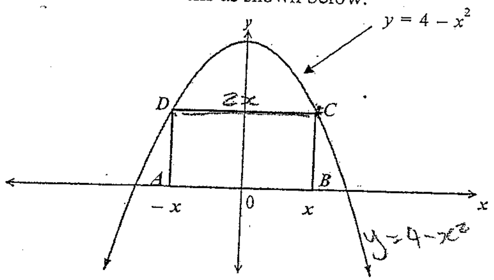
(ii) In fact, beginning on the $2^{nd}$ of January 1985, Sabrina's father deposited $200 into the savings account, and on each successive birthday. On the $2^{nd}$ of January 2005, after the payment of interest and her father's deposit, Sabrina's father withdrew all the money and gave it to her as a present. Calculate the amount of money Sabrina receives as a present. 4

(b) The ratio of heads to tails given by a biased coin is 2:3. If this coin is tossed once:

(i) What is the probability of obtaining a tail? 1

(ii) What is the probability of obtaining at least one head? 1

(c) A rectangle $ABCD$ of side $2x$ is inscribed between the parabola $y = 4 - x^2$ and the $x$-axis as shown below:




(i) Show that the area of the rectangle is $A = 2x(4 - x^2)$ units$^2$. 1

(ii) Find the value of $x$ that makes this area a maximum. 3

(d) The function $y = x^3 - 3x^2 - 9x + 1$ is defined in the domain $-4 \le x \le 5$.

(i) Find the coordinates of any turning points & determine their nature. 3

(ii) Find the coordinates of any points of inflexion. 2

(iii) Draw a neat sketch of the curve. 2

(iv) Determine the minimum value of the function in the domain $-4 \le x \le 5$ 1

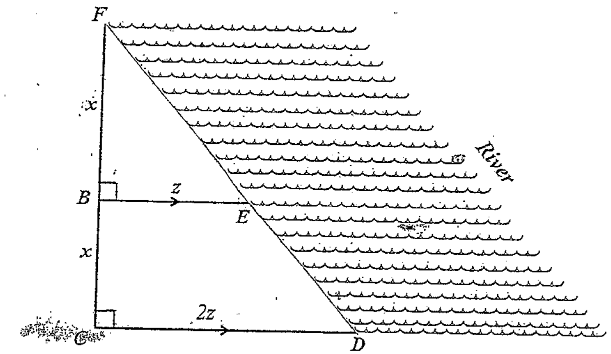
**QUESTION 5 (20 marks)** Marks

(a) The sum of $n$ terms of the series $5 - 10 + 20 - \ldots$, is equal to $-425$. Find the value of $n$. 3

(b) The first term of an arithmetic series is 9 and the last term is 44. If there are 6 terms in the series find the series. 3

(c) $p$ and $q$ are two numbers, where $p > q$ such that $p$, 6, $q$ form a geometric series and $\frac{1}{p}, \frac{5}{18}, \frac{1}{q}$ form an arithmetic series. Find the values of $p$ and $q$ which satisfy these conditions. 6

(d) Farmer McDonald who wishes to keep his animals separate, sets up his field so that fences exist at $FC$, $CD$ and $BE$ as shown in the diagram below.




$B$ is the middle of $FC$ and $CD$ is twice the length of $BE$.

(i) If $FB = x$ metres and $BE = z$ metres, write down expressions in terms of $x$ and $z$ for:

($\alpha$) the area, $A$, of the field $FCD$. 1

($\beta$) the amount of fencing, $L$, that the farmer would need. 1

(ii) If the area of the field is 1200 m$^2$, show that the length of fencing required is given by: 2

$$L = 2x + \frac{1800}{x} \text{ metres.}$$

(iii) Hence find the values of $x$ and $z$ so that the farmer uses the minimum amount of fencing. 4

END OF EXAM

SOLUTIONS TO YR 12 HSC mathematics

ASSESSMENT TASK 2 2006.

## QUESTION 1 (20 marks)

(a)(i) $P(\text{faulty}) = \frac{8}{200} = \frac{1}{25}$ (1)

(ii) No. of faulty expected $= \frac{1}{25} \times 600$
$= 24$ (1)

(b)(i) $a = 50$

$T_2 - T_1 = 43 - 50$
$= -7$ (1)

$\therefore$ common difference is $-7$.

(ii) $T_n = a + (n-1)d$
$= 50 + (n-1)-7$
$= 57 - 7n$ (2)

(iii) $-27 = 57 - 7n$
$-84 = -7n$
$12 = n$ (2)

$\therefore$ there are 12 terms

(iv) $S_{12} = \frac{12}{2}(50 - 27)$
$= 138$ (1)

(c)

(i) $P(\text{all 3 girls chosen}) = \frac{3}{5} \times \frac{4}{5} \times \frac{2}{5}$
$= \frac{24}{125}$ (1)

(ii) $P(\text{only Sarah \& Tara chosen})$
$= \frac{4}{5} \times \frac{2}{5} \times \frac{2}{5}$
$= \frac{16}{125}$ (1)

(c)(iii) $P(\text{at least 1 chosen}) = 1 - P(\text{none chosen})$
$= 1 - \left(\frac{2}{5} \times \frac{1}{5} \times \frac{3}{5}\right)$
$= 1 - \frac{6}{125}$
$= \frac{119}{125}$ (1)

(d) $S_n = 132n - 4n^2$

(i) $S_7 = 132(7) - 4(7)^2$
$= 924 - 196$
$= 728$ (1)

(ii) $S_6 = 132(6) - 4(6)^2$
$= 648$ (1)

(iii) $T_7 = S_7 - S_6$
$= 80$ (1)

(e) $y = x^3 - 3x^2 - 7x - 5$

(i) $\frac{dy}{dx} = 3x^2 - 6x - 7$ (2)

$\frac{d^2y}{dx^2} = 6x - 6$

(ii) $6x - 6 = 0$
$6x = 6$
$x = 1$ (1)

(iii) at $x = 1$ $\frac{dy}{dx} = 3(1)^2 - 6(1) - 7$
$= -10$

$\therefore m = -10$ (1)

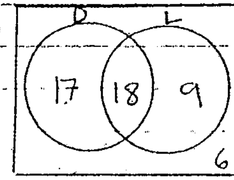
(iv) when $x = 1$ $y = 1^3 - 3(1)^2 - 7(1) - 5$
$= -14$

$y + 14 = -10(x - 1)$
$y + 14 = -10x + 10$
$10x + y + 4 = 0$

## QUESTION 2 (20 marks)

(a)

(i)




(2)

(ii) $P(\text{watch both shows}) = \frac{18}{50}$
$= \frac{9}{25}$ (1)

(iii) $P(\text{watch only Desp. Housewives}) = \frac{17}{50}$ (1)

(iv) $P(\text{watch at least 1 show}) = 1 - \frac{6}{50}$
$= \frac{22}{25}$ (1)

(b) $f(x) = 2x^3 - 3x^2 - 12x + 8$

(i) concave upwards $f''(x) > 0$

$f'(x) = 6x^2 - 6x - 12$
$f''(x) = 12x - 6$
$12x - 6 > 0$
$12x > 6$
$x > \frac{1}{2}$

(ii) concave downwards

$f''(x) < 0$
$12x - 6 < 0$
$12x < 6$
$x < \frac{1}{2}$

(c) Let $S_n$ be distance travelled to retrieve the $n$th ball

$\therefore S_1 = 20\text{m}$

$S_2 = 20 + 4$
$= 24\text{ m}$

$S_3 = 24 + 4$
$= 28\text{ m}.$

$\therefore$ we have an Arithmetic sequence with $a = 20, d = 4$

(i) if $n = 9$ find $S_9$.

$S_9 = \frac{9}{2}[40 + 8(4)]$ (2)
$= 324$

(ii) find $n$ when $S_n = 644$

$644 = \frac{n}{2}[40 + (n-1)4]$
$1288 = 40n + 4n^2 - 4n$
$0 = 4n^2 + 36n - 1288$
$0 = n^2 + 9n - 322$
$0 = (n + 23)(n - 14)$
$14, -23 = n$

But $n$ must be positive

$\therefore n = 14$

(3) i.e. there were 14 balls

(d). $f(x) = x^3 + ax^2 + bx + 3$

$f'(x) = 3x^2 + 2ax + b$
$f''(x) = 6x + 2a$
$f'(2) = 9$
$9 = 12 + 4a + b$
$-3 = 4a + b$
$f''(4) = 40$
$40 = 6(4) + 2a$
$16 = 2a$
$8 = a$ (4)

$\therefore b = -3 - 4(8)$
$= -35$

QUESTION 2 (cont)

(e) Evaluate $\sum_{1}^{8} 3^r$

$\sum_{1}^{8} 3^r = 3^1 + 3^2 + 3^3 + \ldots + 3^8$

This is a G.S with $a = 3$ & $r = 3$

$$S_8 = \frac{3(3^8 - 1)}{3-1}$$
$$= \frac{3(6560)}{2}$$
$$= 9840 \quad (2)$$

QUESTION 3 (20 marks)

(a)(i)

$P(\text{stop all 3 shots}) = \frac{7}{10} \times \frac{7}{10} \times \frac{7}{10}$
$= \frac{343}{1000} \quad (1)$

(ii) $P(\text{stops exactly 1 shot}) = 3\left[\frac{7}{10} \times \left(\frac{3}{10}\right)^2\right]$
$= \frac{189}{1000} \quad (1)$

(iii) $P(\text{stops at most 2 shots})$
$= P(\text{none or 1 or 2 stopped})$

$= 1 - P(3 \text{ stopped})$
$= 1 - \frac{343}{1000}$
$= \frac{657}{1000} \quad (1)$

(b) $0.\dot{4}\dot{6} = 0.464646\ldots$

$= \frac{46}{100} + \frac{46}{10000} + \frac{46}{1000000}$

$= 46\left[\frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} + \ldots\right]$

G.S. with a limiting sum

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{\frac{1}{10^2}}{1 - \frac{1}{10^2}}$$
$$= \frac{1}{99}$$

$\therefore 0.\dot{4}\dot{6} = \frac{46}{99} \quad (3)$

(c) $y = (x^2 - 1)(1 + x)$
$= x^2 + x^3 - 1 - x$

$\frac{dy}{dx} = 2x + 3x^2 - 1$

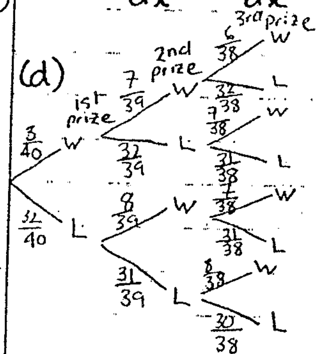
$\frac{d^2y}{dx^2} = 2 + 6x$

$\therefore LHS = x(2 + 6x) - 2(2x + 3x^2 - 1) + 2x - 2$

$= 2x + 6x^2 - 4x - 6x^2 + 2 + 2x - 2$
$= 0$
$= RHS$

$\therefore x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2x - 2 = 0 \quad (4)$

(d)




sample space

WWW
WWL
WLW
WLL
LWW
LWL
LLW
LLL

(2)

(i) $P(\text{wins } 1^{st} \text{ prize only}) = P(WLL)$
$= \frac{8}{40} \times \frac{32}{39} \times \frac{31}{38}$
$= \frac{496}{3705} \quad (1)$

QUESTION 3 (cont)

(d)(iii) $P(\text{at least 2 prizes})$
$= P(WWL) + P(WLW) + P(LWW) + P(WWW)$
$= \left(\frac{8}{40} \times \frac{7}{39} \times \frac{32}{38}\right) + \left(\frac{8}{40} \times \frac{32}{39} \times \frac{7}{38}\right) + \left(\frac{32}{40} \times \frac{8}{39} \times \frac{7}{38}\right) + \left(\frac{8}{40} \times \frac{7}{39} \times \frac{6}{38}\right)$

$= \frac{119}{1235} \quad (2)$

(e)(i) $T_3 = 32 \quad T_6 = 4$

$T_n = ar^{n-1}$

$32 = ar^2 \ldots (1)$

$4 = ar^5 \ldots (2)$

$(2) \div (1) \quad \frac{1}{8} = r^3$

$\frac{1}{2} = r$

$32 = a\left(\frac{1}{2}\right)^2$

$32 = \frac{a}{4}$

$128 = a \quad (2)$

$\therefore$ 1st term is 128 & common ratio is $\frac{1}{2}$.

(ii) $T_n = 128\left(\frac{1}{2}\right)^{n-1} \quad (1)$

(iii) find $n$ when $T_n = \frac{1}{2}$

$\frac{1}{2} = 128\left(\frac{1}{2}\right)^{n-1}$

$\frac{1}{256} = \left(\frac{1}{2}\right)^{n-1}$

$\left(\frac{1}{2}\right)^8 = \left(\frac{1}{2}\right)^{n-1}$

Equating powers

$8 = n - 1$

$9 = n$

$\therefore$ the $9^{th}$ term of the sequence is $\frac{1}{2}$.

QUESTION 4 (20 marks)

(a)(i) $n = 21 \quad R = 1.06 \quad P = 2000$

$A = 2000(1.06)^{21}$
$= \$6799.13 \quad (2)$

(ii) $A = 2000(1.06)^{21} + 200(1.06)^{20} + 200(1.06)^{19} + 200(1.06)^{18} + \ldots + 200(1.06)^2 + 200(1.06) + 200$

Now $200 + 200(1.06)^1 + \ldots + 200(1.06)^{20}$
$= 200\left[1 + (1.06)^1 + (1.06)^2 + \ldots + (1.06)^{20}\right]$

this is a G.P with $a = 1$ $r = 1.06$ and $n = 21$

$\therefore 200 \times \left[\frac{1(1.06^{21} - 1)}{1.06 - 1}\right]$

$= \$7998.55$

$\therefore$ Total amount of present $= \$6799.13 + \$7998.55$
$= \$14797.68$

(b)(i) $P(\text{tail}) = \frac{3}{5} \quad (1)$

(ii) $P(\text{at least 1 head}) = 1 - P(\text{no hea}$
$= 1 - \frac{3}{5}$
$= \frac{2}{5} \quad (1)$

## QUESTION 4 (cont)

(c)(i) $AB = 2x$ and $BC = 4 - x^2$

$A = 2x(4 - x^2)$ ①

(ii) $A = 8x - 2x^3$

$\frac{dA}{dx} = 8 - 6x^2$

$\frac{d^2A}{dx^2} = -12x$

For stat. pts $\frac{dA}{dx} = 0$ ③

$8 - 6x^2 = 0$

$8 = 6x^2$

$\frac{4}{3} = x^2$

$x = \pm\sqrt{\frac{4}{3}}$

if $x = \sqrt{\frac{4}{3}}$ $\frac{d^2A}{dx^2} = -12\left(\sqrt{\frac{4}{3}}\right) = \frac{-24}{\sqrt{3}} < 0$

$\therefore$ when $x = \sqrt{\frac{4}{3}}$ the area is a maximum.

(d)(i) $y = x^3 - 3x^2 - 9x + 1$

For stat. pts $y' = 0$.

$y' = 3x^2 - 6x - 9$

$= (3x - 9)(x + 1)$

$x = 3, -1$

at $x = 3$ $y = -26$

at $x = -1$ $y = 6$

$\therefore$ stat pts are $(3, -26)$ & $(-1, 6)$.

$y'' = 6x - 6$

at $x = 3$ $y'' = 6(3) - 6$

$= 12 > 0$

$\therefore (3, -26)$ is a min. turning pt.

at $x = -1$

$y'' = 6(-1) - 6$ ③

$= -12 < 0$

$\therefore$ max turning pt

(ii) For inflexions

$\frac{d^2y}{dx^2} = 0$

$6x - 6 = 0$

$6x = 6$

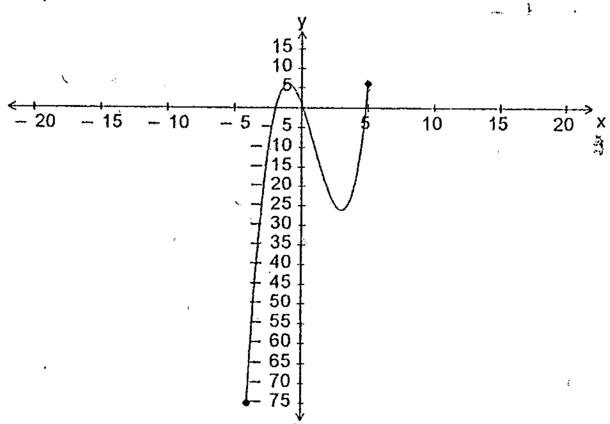
$x = 1$ $y = -10$

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $\frac{d^2y}{dx^2}$ | $-6$ | 0 | 6 |

②

since concavity changes $(1, -10)$ is a point of inflexion.

(iii)




(iv) when $x = -4$

$y = (-4)^3 - 3(-4)^2 - 9(-4) + 1$

$= -75$

## QUESTION 5 (20 marks)

(a) $5 - 10 + 20 - \ldots$

$a = 5$ $r = -2$ $S_n = -425$

find $n$.

$S_n = \frac{a(1 - r^n)}{1 - r}$

$-425 = \frac{5[1 - (-2)^n]}{1 + 2}$

$-1275 = 5 - 5(-2)^n$

$-1280 = -5(-2)^n$

$256 = (-2)^n$

$(-2)^8 = (-2)^n$

equating powers ③

$8 = n$

(b) $9, T_2, T_3, T_4, T_5, 44$

$a = 9$ $T_6 = 44$

$a + 5d = 44$

$9 + 5d = 44$

$5d = 35$ ③

$d = 7$

$\therefore T_2 = 16,\ T_3 = 23,\ T_4 = 30$

$T_5 = 37$

$\therefore 9, 16, 23, 30, 37, 44.$

(c) $p, 6, q$ are in G.P

$\frac{6}{p} = \frac{q}{6}$

i.e. $pq = 36$ ...①

$\frac{1}{p}, \frac{5}{18}, \frac{1}{q}$ are in A.P

$\frac{5}{18} - \frac{1}{p} = \frac{1}{q} - \frac{5}{18}$

$\frac{5}{9} = \frac{1}{p} + \frac{1}{q}$ ⑥

$= \frac{p + q}{pq}$

$\frac{5}{9} = \frac{p + q}{36}$ using ①

$20 = p + q$

$p = 20 - q$ & $pq = 36$

$(20 - q)q = 36$

$20q - q^2 = 36$

$0 = q^2 - 20q + 36$

$0 = (q - 18)(q - 2)$

$\therefore 2, 18 = q$

But $p > q$

$\therefore p = 18,\ q = 2.$

(d)(i)(α) $A = \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 2z \times 2x$ ①

$= 2xz$

(β) $L = 2x + 3z$ ①

(ii) If $A = 1200$

$1200 = 2xz$

$z = \frac{600}{x}$ ②

$L = 2x + 3z$

$= 2x + 3\left(\frac{600}{x}\right)$

$= 2x + \frac{1800}{x}$ m

(iii) $\frac{dL}{dx} = 2 - 1800x^{-2}$

$= 2 - \frac{1800}{x^2}$

For stat. pts $\frac{dL}{dx} = 0$

$2 - \frac{1800}{x^2} = 0$

$2 = \frac{1800}{x^2}$

$x^2 = 900$

QUESTION 5 (cont)

$\therefore x = \pm 30$

$\frac{d^2L}{dx^2} = 3600x^{-3}$

$= \frac{3600}{x^3}$

when $x = 30$

$\frac{d^2L}{dx^2} = \frac{3600}{(30)^3}$ (4)

$= \frac{2}{15}$

$> 0$ $\therefore$ minimum

When $x = -30$

$\frac{d^2L}{dx^2} = -\frac{2}{15}$

$< 0$

$\therefore$ maximum

when $x = 30$ $z = \frac{600}{30}$

$= 20$

$\therefore x = 30\text{ m}$ & $z = 20\text{ m}$