

Sydney Girls High School



MATHEMATICS HSC ASSESSMENT TASK 2

March 2006

Topics: Applications of the Second Derivative,
Probability, Sequences and Series.

Time Allowed: 90 minutes

Instructions:

- There are five (5) questions of equal value.
- Attempt all questions.
- Start each question on a new page.
- Show all necessary working.
- Marks may be deducted for careless or poor setting out.
- Board approved calculators may be used.

Total = 100 marks

QUESTION 1 (20 marks)

Marks

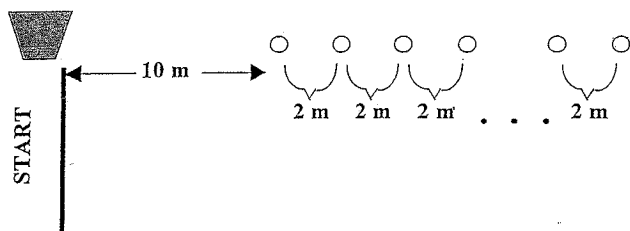
- (a) In a production line, a batch of 200 finished items are tested and 8 are found to be faulty.
- (i) What is the probability that an item selected at random is faulty? 1
- (ii) How many faulty items could be expected if a batch of 600 were tested? 1
- (b) The first three terms of an arithmetic series are 50, 43, 36.
- (i) Find the common difference. 1
- (ii) Write down a formula for the n th term. 1
- (iii) If the last term of the series is -27 , how many terms are there in the series? 2
- (iv) Find the sum of the series. 2
- (c) At Harbord High School the probability that Melissa is chosen as a prefect is $\frac{3}{5}$. The probability that Sarah is chosen as a prefect is $\frac{4}{5}$, whilst the probability that Tara is chosen as a prefect is $\frac{2}{5}$. Find the probability that out of the three girls:
- (i) all three are chosen. 1
- (ii) only Sarah and Tara are chosen. 1
- (iii) at least one of the three is chosen. 1
- (d) The sum of the first n terms of a sequence is given by $S_n = 132n - 4n^2$.
- (i) Find the sum of the first 7 terms 1
- (ii) Find the sum of the first 6 terms 1
- (iii) Hence find the 7th term. 1
- (e) If $y = x^3 - 3x^2 - 7x - 5$, find:
- (i) $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ 2
- (ii) the value of x for which $\frac{d^2y}{dx^2} = 0$ 1
- (iii) the gradient of the tangent at the point where $\frac{d^2y}{dx^2} = 0$ 1
- (iv) the equation of the tangent at the point where $\frac{d^2y}{dx^2} = 0$ 2

QUESTION 2 (20 marks)

Marks

- (a) In a group of 50 students there are 35 who watch the television show Desperate Housewives and 27 who watch Lost, while 6 watch neither.
- (i) Draw a Venn diagram using the information above. Find the probability that a student watches: 2
 - (ii) both television shows. 1
 - (iii) only Desperate Housewives. 1
 - (iv) at least one of the television shows. 1
- (b) For what value(s) of x is the function $f(x) = 2x^3 - 3x^2 - 12x + 8$:
- (i) concave upwards 4
 - (ii) concave downwards

- (c) In a game, competitors must run 10 metres from the starting point, pick up a ball, then run back to the start and place the ball in a bucket. The competitor then runs to the next ball which is 2 metres further than the first ball, picks that ball up, runs back to the start and places the ball in the bucket. This process is repeated with each successive ball 2 metres further than the previous one.



- (i) How far does Bridget run if nine balls are placed in the bucket? 2
 - (ii) If Mark runs 644 metres in total to complete the race, how many balls were there? 3
- (d) If $f(x) = x^3 + ax^2 + bx + 3$, find the value of a and b given $f'(2) = 9$ and $f''(4) = 40$. 4
- (e) Evaluate $\sum_{r=1}^8 3^r$ 2

QUESTION 3 (20 marks)

Marks

- (a) Justine is a talented soccer goal keeper. The probability that she can stop a penalty shot at goal is 70%. During a match the opposition had 3 penalty shots at goal. What is the probability that Justine stopped:
- (i) all 3 penalty shots? 1
 - (ii) exactly 1 penalty shot? 1
 - (iii) at most 2 penalty shots? 1
- (b) By considering the recurring decimal $0.4\dot{6}$ as the sum of an infinite geometric series, express the recurring decimal as a rational number. 3
- (c) If $y = (x^2 - 1)(1 + x)$ show that $x \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2x - 2 = 0$. 4
- (d) Forty tickets are sold in a raffle that has 1st, 2nd and 3rd prizes given. Ryan purchases 8 tickets.
- (i) Draw a probability tree using the information above. What is the probability that Ryan wins? 2
 - (ii) the 1st prize only 1
 - (iii) at least two prizes 2
- (e) The third term of a geometric sequence is 32 and the sixth term is 4.
- (i) Find the first term and the common ratio. 2
 - (ii) Hence find an expression for the n th term. 1
 - (iii) Which term of the sequence is equal to $\frac{1}{2}$? 2

QUESTION 4 (20 marks)

- (a) On the day of her birth, 2nd of January 1984, Sabrina's father deposited \$2000 into a savings account which earned interest at 6% per annum, compounded annually.
- (i) How much money would be in the account after the payment of interest on the 2nd of January 2005 if no additional deposits were made?
- (ii) In fact, beginning on the 2nd of January 1985, Sabrina's father deposited \$200 into the savings account, and on each successive birthday. On the 2nd of January 2005, after the payment of interest and her father's deposit, Sabrina's father withdrew all the money and gave it to her as a present. Calculate the amount of money Sabrina receives as a present.

Marks

2

4

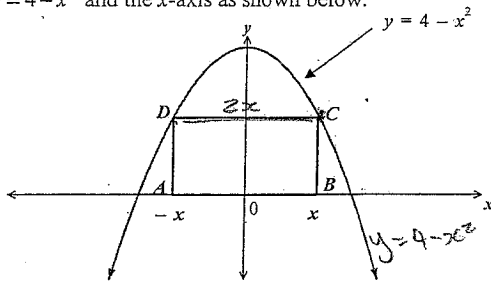
- (b) The ratio of heads to tails given by a biased coin is 2:3. If this coin is tossed once:

- (i) What is the probability of obtaining a tail?
- (ii) What is the probability of obtaining at least one head?

1

1

- (c) A rectangle $ABCD$ of side $2x$ is inscribed between the parabola $y = 4 - x^2$ and the x -axis as shown below:



- (i) Show that the area of the rectangle is $A = 2x(4 - x^2)$ units².
- (ii) Find the value of x that makes this area a maximum.

1

3

- (d) The function $y = x^3 - 3x^2 - 9x + 1$ is defined in the domain $-4 \leq x \leq 5$.

- (i) Find the coordinates of any turning points & determine their nature.
- (ii) Find the coordinates of any points of inflexion.
- (iii) Draw a neat sketch of the curve.
- (iv) Determine the minimum value of the function in the domain

3

2

2

1

$-4 \leq x \leq 5$

QUESTION 5 (20 marks)

- (a) The sum of n terms of the series $5 - 10 + 20 - \dots$, is equal to -425 .

Find the value of n .

3

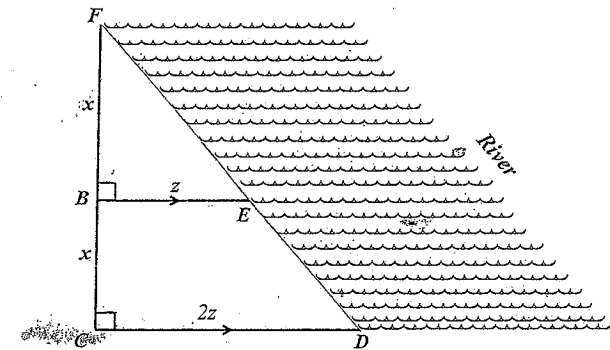
- (b) The first term of an arithmetic series is 9 and the last term is 44. If there are 6 terms in the series find the series.

3

- (c) p and q are two numbers, where $p > q$ such that $p, 6, q$ form a geometric series and $\frac{1}{p}, \frac{5}{18}, \frac{1}{q}$ form an arithmetic series. Find the values of p and q which satisfy these conditions.

6

- (d) Farmer McDonald who wishes to keep his animals separate, sets up his field so that fences exist at FC, CD and BE as shown in the diagram below.



B is the middle of FC and CD is twice the length of BE .

- (i) If $FB = x$ metres and $BE = z$ metres, write down expressions in terms of x and z for:

(α) the area, A , of the field FCD .

1

(β) the amount of fencing, L , that the farmer would need.

1

- (ii) If the area of the field is 1200 m^2 , show that the length of fencing required is given by:

2

$$L = 2x + \frac{1800}{x} \text{ metres.}$$

- (iii) Hence find the values of x and z so that the farmer uses the minimum amount of fencing.

4

END OF EXAM

SOLUTIONS TO YR 12 HSC mathematics

ASSESSMENT TASK 2 2006.

QUESTION 1 (20 marks)

(a)(i) $P(\text{faulty}) = \frac{8}{200} = \frac{1}{25}$ (1)

(ii) No. of faulty expected = $\frac{1}{25} \times 600 = 24$ (1)

(b) $n_2 = 50$

$T_2 - T_1 = 43 - 50 = -7$ (1)

\therefore common difference is -7 .

(ii) $T_n = a + (n-1)d = 50 + (n-1)(-7) = 57 - 7n$ (2)

(iii) $-27 = 57 - 7n$
 $-84 = -7n$
 $12 = n$ (2)

\therefore there are 12 terms

(iv) $S_{12} = \frac{12}{2} (50 - 27) = 138$ (1)

(c)

(i) $P(\text{all 3 girls chosen}) = \frac{3}{5} \times \frac{4}{5} \times \frac{2}{5} = \frac{24}{125}$ (1)

(ii) $P(\text{only Sarah \& Tara chosen}) = \frac{4}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{16}{125}$ (1)

(c)(iii) $P(\text{at least 1 chosen}) = 1 - P(\text{none chosen}) = 1 - \left(\frac{2}{5} \times \frac{1}{5} \times \frac{3}{5}\right) = 1 - \frac{6}{125} = \frac{119}{125}$ (1)

(d) $S_n = 132n - 4n^2$
 (i) $S_7 = 132(7) - 4(7)^2 = 924 - 196 = 728$ (1)

(ii) $S_6 = 132(6) - 4(6)^2 = 648$ (1)

(iii) $T_7 = S_7 - S_6 = 80$ (1)

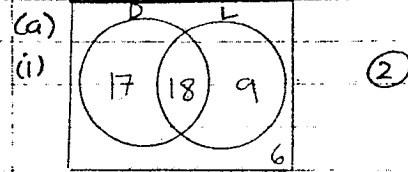
(e) $y = x^3 - 3x^2 - 7x - 5$
 (i) $\frac{dy}{dx} = 3x^2 - 6x - 7$ (2)
 $\frac{d^2y}{dx^2} = 6x - 6$

(ii) $6x - 6 = 0$
 $6x = 6$
 $x = 1$ (1)

(iii) at $x = 1$ $\frac{dy}{dx} = 3(1)^2 - 6(1) - 7 = -10$
 $\therefore m = -10$ (1)

(iv) when $x = 1$ $y = 1^3 - 3(1)^2 - 7(1) - 5 = -14$
 $y + 14 = -10(x - 1)$
 $y + 14 = -10x + 10$
 $10x + y + 4 = 0$

QUESTION 2 (20 marks)



(ii) $P(\text{watch both shows}) = \frac{18}{50} = \frac{9}{25}$ (1)

(iii) $P(\text{watch only Desp. Housewives}) = \frac{17}{50}$ (1)

(iv) $P(\text{watch at least 1 show}) = 1 - \frac{6}{50} = \frac{22}{25}$ (1)

(b) $f(x) = 2x^3 - 3x^2 - 12x + 8$

(i) concave upwards $f''(x) > 0$
 $f'(x) = 6x^2 - 6x - 12$
 $f''(x) = 12x - 6$
 $12x - 6 > 0$
 $12x > 6$
 $x > \frac{1}{2}$

(ii) concave downwards $f''(x) < 0$
 $12x - 6 < 0$
 $12x < 6$
 $x < \frac{1}{2}$

(c) let S_n be distance travelled to retrieve the n th ball
 $\therefore S_1 = 20\text{m}$
 $S_2 = 20 + 4 = 24\text{m}$

$S_3 = 24 + 4 = 28\text{m}$

\therefore we have an arithmetic sequence with $a = 20, d = 4$

(i) if $n = 9$ find S_9 .
 $S_9 = \frac{9}{2} [40 + 8(9)] = 324$ (2)

(ii) find n when $S_n = 644$
 $644 = \frac{n}{2} [40 + (n-1)4]$
 $1288 = 40n + 4n^2 - 4n$
 $0 = 4n^2 + 36n - 1288$
 $0 = n^2 + 9n - 322$
 $0 = (n + 23)(n - 14)$
 $14, -23 = n$

But n must be positive

$\therefore n = 14$
 (3) i.e. there were 14 balls

(d) $f(x) = x^3 + ax^2 + bx + 3$
 $f'(x) = 3x^2 + 2ax + b$
 $f''(x) = 6x + 2a$
 $f'(2) = 9$
 $9 = 12 + 4a + b$
 $-3 = 4a + b$
 $f''(4) = 40$
 $40 = 6(4) + 2a$
 $16 = 2a$
 $8 = a$

$\therefore b = -3 - 4(8) = -35$ (4)

QUESTION 2 (cont)

(e) Evaluate $\sum_{r=1}^8 3^r$

$$\sum_{r=1}^8 3^r = 3^1 + 3^2 + 3^3 + \dots + 3^8$$

This is a G.S with $a=3$
 $r=3$

$$S_8 = \frac{3(3^8 - 1)}{3 - 1}$$

$$= \frac{3(6560)}{2}$$

$$= 9840 \quad (2)$$

QUESTION 3 (20 marks)

(a)(i)

$$P(\text{stop all 3 shots}) = \frac{7}{10} \times \frac{7}{10} \times \frac{7}{10} = \frac{343}{1000} \quad (1)$$

$$(ii) P(\text{stops exactly 1 shot}) = 3 \left[\frac{7}{10} \times \left(\frac{3}{10}\right)^2 \right] = \frac{189}{1000} \quad (1)$$

(ii) $P(\text{stops at most 2 shots}) = P(\text{none or 1 or 2 stopped})$

$$= 1 - P(3 \text{ stopped})$$

$$= 1 - \frac{343}{1000} = \frac{657}{1000} \quad (1)$$

(b) $0.46 = 0.464646\dots$

$$= \frac{46}{100} + \frac{46}{10000} + \frac{46}{1000000} + \dots$$

$$= 46 \left[\frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} + \dots \right]$$

\uparrow G.S. with a limiting sum

$$S_{\infty} = \frac{a}{1-r} = \frac{1}{1-\frac{1}{10}} = \frac{1}{\frac{9}{10}} = \frac{10}{9}$$

$$\therefore 0.46 = \frac{46}{99} \quad (3)$$

$$(c) y = (x^2 - 1)(1 + x) = x^2 + x^3 - 1 - x$$

$$\frac{dy}{dx} = 2x + 3x^2 - 1$$

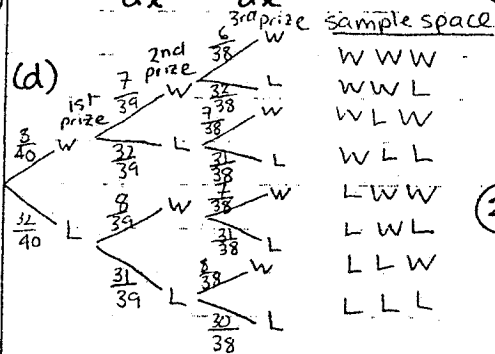
$$\frac{d^2y}{dx^2} = 2 + 6x$$

$$\therefore \text{LHS} = x(2 + 6x) - 2(2x + 3x^2 - 1) + 2x - 2$$

$$= 2x + 6x^2 - 4x - 6x^2 + 2 + 2x - 2 = 0$$

$= \text{RHS}$

$$\therefore x \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2x - 2 = 0 \quad (4)$$



$$(i) P(\text{wins 1st prize only}) = P(WLL) = \frac{8}{40} \times \frac{32}{39} \times \frac{31}{38}$$

$$= \frac{496}{3705} \quad (1)$$

QUESTION 3 (cont)

(d)(iii) $P(\text{at least 2 prizes}) = P(WWL) + P(WLW) + P(LWW) + P(WWW)$

$$= \left(\frac{8}{40} \times \frac{7}{39} \times \frac{32}{38}\right) + \left(\frac{8}{40} \times \frac{32}{39} \times \frac{7}{38}\right) + \left(\frac{32}{40} \times \frac{7}{39} \times \frac{7}{38}\right) + \left(\frac{8}{40} \times \frac{7}{39} \times \frac{6}{38}\right)$$

$$= \frac{119}{1235} \quad (2)$$

(e)(i) $T_3 = 32$ $T_6 = 4$

$$T_n = ar^{n-1}$$

$$32 = ar^2 \dots (1)$$

$$4 = ar^5 \dots (2)$$

$$(2) \div (1) \quad \frac{1}{8} = r^3$$

$$\frac{1}{2} = r$$

$$32 = a \left(\frac{1}{2}\right)^2$$

$$32 = \frac{a}{4} \quad (2)$$

$$128 = a$$

\therefore 1st term is 128 & common ratio is $\frac{1}{2}$.

$$(ii) T_n = 128 \left(\frac{1}{2}\right)^{n-1} \quad (1)$$

(iii) Find n when $T_n = \frac{1}{2}$

$$\frac{1}{2} = 128 \left(\frac{1}{2}\right)^{n-1}$$

$$\frac{1}{256} = \left(\frac{1}{2}\right)^{n-1}$$

$$\left(\frac{1}{2}\right)^8 = \left(\frac{1}{2}\right)^{n-1}$$

Equating powers

$$8 = n - 1$$

$$9 = n$$

\therefore the 9th term of the sequence is $\frac{1}{2}$.

QUESTION 4 (20 marks)

$$(a)(i) n=21 \quad R=1.06 \quad P=2000$$

$$A = 2000(1.06)^{21}$$

$$= \$6799.13 \quad (2)$$

$$(ii) A = 2000(1.06)^{21} + 200(1.06)^{20} + 200(1.06)^{19} + 200(1.06)^{18} + \dots + 200(1.06)^2 + 200(1.06) + 200$$

$$\text{Now } 200 + 200(1.06)^1 + \dots + 200(1.06)^{20} = 200 \left[1 + (1.06)^1 + (1.06)^2 + \dots + (1.06)^{20} \right]$$

\uparrow
 This is a G.P with $a=1$ $r=1.06$ and $n=21$

$$\therefore 200 \times \left[\frac{1(1.06^{21} - 1)}{1.06 - 1} \right]$$

$$= \$7998.55$$

\therefore Total amount of present = $\$6799.13 + \$7998.55 = \$14797.68$

$$(b)(i) P(\text{tail}) = \frac{3}{5} \quad (1)$$

$$(ii) P(\text{at least 1 head}) = 1 - P(\text{none}) = 1 - \frac{3}{5} = \frac{2}{5} \quad (1)$$

QUESTION 4 (cont)

(c)(i) $AB = 2x$ and $BC = 4 - x^2$
 $A = 2x(4 - x^2)$ (1)

(ii) $A = 8x - 2x^3$

$\frac{dA}{dx} = 8 - 6x^2$

$\frac{d^2A}{dx^2} = -12x$

For stat. pts $\frac{dA}{dx} = 0$ (3)

$8 - 6x^2 = 0$

$8 = 6x^2$

$\frac{4}{3} = x^2$

$x = \pm \sqrt{\frac{4}{3}}$

if $x = \sqrt{\frac{4}{3}}$, $\frac{d^2A}{dx^2} = -12\left(\sqrt{\frac{4}{3}}\right) = -\frac{24}{\sqrt{3}} < 0$

\therefore when $x = \sqrt{\frac{4}{3}}$ the area is a maximum. (iii)

(d)(i) $y = x^3 - 3x^2 - 9x + 1$

for stat. pts $y' = 0$.

$y' = 3x^2 - 6x - 9 = (3x - 9)(x + 1)$

$x = 3, -1$

at $x = 3$ $y = -26$

at $x = -1$ $y = 6$

\therefore stat pts are $(3, -26)$ & $(-1, 6)$.

$y'' = 6x - 6$

at $x = 3$ $y'' = 6(3) - 6 = 12 > 0$

$\therefore (3, -26)$ is a min. turning pt.

at $x = -1$

$y'' = 6(-1) - 6 = -12 < 0$ (3)

\therefore max turning pt

(ii) For inflexions

$\frac{d^2y}{dx^2} = 0$

$6x - 6 = 0$

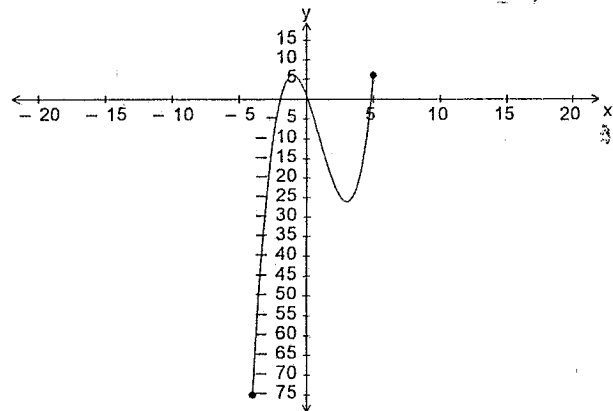
$6x = 6$

$x = 1$ $y = -10$

x	0	1	2
$\frac{d^2y}{dx^2}$	-6	0	6

(2)

Since concavity changes $(1, -10)$ is a point of inflexion.



(iv) when $x = -4$

$y = (-4)^3 - 3(-4)^2 - 9(-4) + 1 = -75$

QUESTION 5 (20 marks)

(a) $5 - 10 + 20 - \dots$

$a = 5$ $r = -2$ $S_n = -425$

find n .

$S_n = \frac{a(1-r^n)}{1-r}$

$-425 = \frac{5[1 - (-2)^n]}{1+2}$

$-1275 = 5 - 5(-2)^n$

$-1280 = -5(-2)^n$

$256 = (-2)^n$

$(-2)^8 = (-2)^n$

equating powers (3)

$8 = n$

(b) $9, T_2, T_3, T_4, T_5, 44$

$a = 9$ $T_6 = 44$

$a + 5d = 44$

$9 + 5d = 44$

$5d = 35$

$d = 7$

$\therefore T_2 = 16, T_3 = 23, T_4 = 30$

$T_5 = 37$

$\therefore 9, 16, 23, 30, 37, 44.$

(c) $p, 6, q$ are in G.P.

$\frac{6}{p} = \frac{q}{6}$

i.e. $pq = 36 \dots (1)$

$\frac{1}{p}, \frac{5}{18}, \frac{1}{q}$ are in A.P.

$\frac{5}{18} - \frac{1}{p} = \frac{1}{q} - \frac{5}{18}$

$\frac{5}{q} = \frac{1}{p} + \frac{1}{q}$ (6)

$= \frac{p+q}{pq}$

$\frac{5}{q} = \frac{p+q}{36}$ using (1)

$20 = p+q$

$p = 20 - q$ & $pq = 36$

$(20 - q)q = 36$

$20q - q^2 = 36$

$0 = q^2 - 20q + 36$

$0 = (q - 18)(q - 2)$

$2, 18 = q$

But $p > q$

$\therefore p = 18, q = 2.$

(d) (i) $A = \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 2z \times 2x$ (1)

$= 2xz$

(ii) $L = 2x + 3z$ (1)

(ii) If $A = 1200$

$1200 = 2xz$

$z = \frac{600}{x}$

$L = 2x + 3z$

$= 2x + 3\left(\frac{600}{x}\right)$

$= 2x + \frac{1800}{x}$ m (2)

(iii) $\frac{dL}{dx} = 2 - 1800x^{-2}$

$= 2 - \frac{1800}{x^2}$

For stat. pts $\frac{dL}{dx} = 0$

$2 - \frac{1800}{x^2} = 0$

$2 = \frac{1800}{x^2}$

$x^2 = 900$

QUESTIONS (cont)

$$\therefore x = \pm 30$$

$$\frac{d^2L}{dx^2} = 3600x^{-3}$$

$$\frac{d^2L}{dx^2} = \frac{3600}{x^3}$$

When $x = 30$

$$\frac{d^2L}{dx^2} = \frac{3600}{(30)^3}$$

$$= \frac{2}{15}$$

$> 0 \therefore$ minimum

When $x = -30$

$$\frac{d^2L}{dx^2} = -\frac{2}{15}$$

$$< 0$$

\therefore maximum

$$\text{When } x = 30 \quad z = \frac{600}{30}$$

$$= 20$$

$$\therefore x = 30 \text{ m \& } z = 20 \text{ m}$$