

SYDNEY GIRLS HIGH SCHOOL



2006 HSC Assessment Task 2

March 15, 2006

MATHEMATICS Extension 1ST

Year 12

Time allowed: 75 minutes

Topics: Trigonometry I & II, Polynomials

DIRECTIONS TO CANDIDATES:

- Attempt all questions
- Questions are of equal value
- There are 4 questions with part marks shown in brackets
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used
- Write on one side of the paper only

Question 1.

a) Find $\frac{dy}{dx}$ in the following

i) $y = 2 \sin(3x - 1)$

ii) $y = \sin^2 3x$

iii) $y = e^{-x} \cos x$

iv) $y = \log_e(\cos x)$

v) $y = \frac{\tan x}{x}$

[8]

b) Find the following integrals

i) $\int \sin 2x \cdot dx$

ii) $\int \cos(2 - 3x) \cdot dx$

iii) $\int \sec^2\left(\frac{x}{2}\right) \cdot dx$

iv) $\int \frac{\sin 3x}{2 - \cos 3x} \cdot dx$

v) $\int \cot x \cdot dx$

[8]

c) Find $\frac{d}{dx} \{\tan^2 x\}$ and hence evaluate $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \cdot dx$

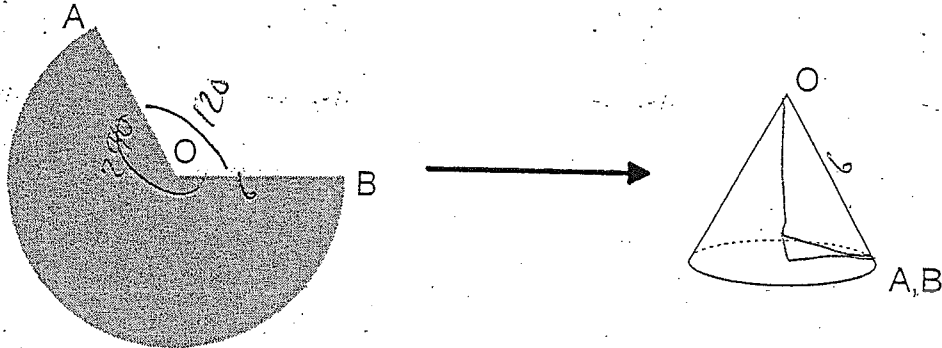
[2]

d) Evaluate $\int_0^{\frac{\pi}{4}} \cos^2 x \cdot dx$

[2]

Question 2:

- a) A circular piece of metal of radius 6 cm is cut to leave a sector with angle 240° at the centre.



Find :

- i) The area of the sector [2]
- ii) The arc length of the sector [2]
- iii) The base radius of a cone that would be formed if the sector was bent so that OA and OB were joined. [2]

b) Evaluate:

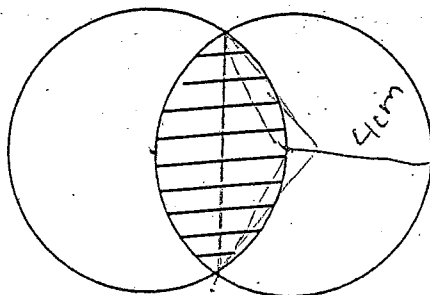
$$i) \lim_{x \rightarrow 0} \left\{ \frac{\sin 2x}{3x} \right\} \quad ii) \lim_{x \rightarrow 0} \left\{ \frac{\sin^2 2x}{x^2} \right\} \quad [4]$$

c) Sketch in the range $0 \leq x \leq 2\pi$

i) $y = 1 - \cos x$ [3]

ii) $y = 2 \cos \left(x + \frac{\pi}{4} \right)$ [3]

- d) Two circles of radius 4cm are drawn so that the centre of one is on the circumference of the other. Find to 3 significant figures, the common area of the two circles



[4]

Question 3:

a) i) Express $\sqrt{3} \sin x + \cos x$ in the form $R \sin(x + \alpha)$ [2]

ii) Hence or otherwise solve the equation:

$$\sqrt{3} \sin x + \cos x = 1 \text{ for } 0 \leq x \leq 2\pi \quad [2]$$

b) Find the acute angle between the lines $2x + y = 4$ and $3x - 2y = 6$. [3]

c) Prove the expression $\frac{\sin 2x}{1 + \cos 2x} = \tan x$ [3]

d) If $t = \tan \frac{\theta}{2}$,

i. Show how to obtain $\cos \theta$ and $\sin \theta$ in terms of t . [2]

ii. Express $\frac{1 + \sin \theta - \cos \theta}{1 + \sin \theta + \cos \theta}$ in terms of t in the simplest form. [2]

e) Show that $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$ and hence or otherwise evaluate

$$\int_0^{\frac{\pi}{6}} \cos^3 \theta \cdot d\theta \quad [3]$$

f) Express $1 + \cos x$ in terms of $\cos \frac{x}{2}$ and use the result to evaluate

$$\int_0^{\frac{\pi}{3}} \frac{dx}{1 + \cos x}$$

[3]

Question 4:

a) Find the remainder when $P(x) = 2x^3 - 3x^2 + 4x - 5$ is divided by $(x-2)$ [1]

b) Find the value of k , if $P(x) = 2x^3 + 5x^2 + kx - 6$ has a remainder of 4 when divided by $(x+2)$ [2]

c) Show that $(x-2)$ is a factor of $P(x) = x^3 + x^2 - 2x - 8$ and explain why there are no other factors [2]

d) Fully factorise the expression: $P(x) = 8x^3 + 3x^2 - 8x - 3$ [2]

e) If the polynomial $P(x) = x^3 + kx + 54$ has two equal roots, find the value of k , and all of the solutions to $P(x) = 0$ [2]

f) If the roots of $8x^3 + 4x^2 - 2x - 1 = 0$ are α, β, γ , find the value of:

i) $\alpha + \beta + \gamma$, ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$, iii) $\alpha^2 + \beta^2 + \gamma^2$ [4]

g) Solve $24x^3 - 14x^2 - 7x + 3 = 0$ given that its roots are in geometric progression [3]

h) The polynomial $3x^3 + 4x^2 - 5x - 1 = 0$ has roots: α, β, γ . Find the equation of the polynomial with roots:

i) $2\alpha, 2\beta, 2\gamma$ ii) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ [4]

.....end of paper

1 a) i) $y = 2 \sin(3x-1)$

$$\frac{dy}{dx} = 6 \cos(3x-1)$$

ii) $y = \sin^2 3x$

$$\frac{dy}{dx} = 2 \cdot 3 \cos 3x \sin 3x$$

$$= 6 \cos 3x \sin 3x$$

[OR $3 \sin 6x$]

iii) $y = e^{-x} \cos x$

$$\frac{dy}{dx} = e^{-x}(-\sin x) + \cos x(-e^{-x})$$

$$= e^{-x}(-\sin x - \cos x)$$

$$= -e^{-x}(\sin x + \cos x)$$

iv) $y = \log_e(\cos x)$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x}$$

$$= -\tan x$$

v) $y = \frac{\tan x}{x}$

$$\frac{dy}{dx} = \frac{x \cdot \sec^2 x - \tan x \cdot 1}{x^2}$$

$$= \frac{x \sec^2 x - \tan x}{x^2}$$

b) i) $\int \sin 2x dx = -\frac{1}{2} \cos 2x + C$

ii) $\int \cos(2-3x) dx = \frac{-1}{3} \sin(2-3x) + C$

iii) $\int \sec^2\left(\frac{x}{2}\right) dx = 2 \tan \frac{x}{2} + C$

iv) $\int \frac{\sin 3x}{2-\cos 3x} = \frac{\ln(2-\cos 3x)}{3} + C$

OR $= \frac{1}{3} \ln(2-\cos 3x) + C$

v) $\int \cot x dx = \int \frac{\cos x}{\sin x} dx$

$$= \ln(\sin x) + C$$

c) $\frac{d}{dx}(\tan^2 x)$

$$= 2 \sec^2 x \tan x$$

$$\therefore \int_0^{\pi/4} \tan x \sec^2 x dx = \frac{1}{2} \left[\tan^2 x \right]_0^{\pi/4}$$

$$= \frac{1}{2} (1 - 0)$$

$$= \frac{1}{2}$$

d) $\int_0^{\pi/4} \cos^2 x dx = \frac{1}{2} \int_0^{\pi/4} (1 + \cos 2x) dx$

$$= \frac{1}{2} \left[x + \frac{1}{2} \sin 2x \right]_0^{\pi/4}$$

$$= \frac{1}{2} \left[\frac{\pi}{4} + \frac{1}{2} \cdot 1 \right] - 0$$

$$= \frac{\pi}{8} + \frac{1}{4} \text{ OR } \frac{1}{8}(\pi + 2)$$

2 a) i) $A = \frac{1}{2} \pi r^2 \theta$

$$240^\circ = \frac{4\pi}{3}$$

$$\therefore A = \frac{1}{2} \times \frac{6}{36} \times \frac{4\pi}{3}$$

$$= 24\pi$$

$$\therefore \text{Area} = 24\pi \text{ cm}^2$$

ii) $l = \pi r \theta$

$$= 8 \times \frac{4\pi}{3}$$

$$= 8\pi$$

$$\therefore \text{length} = 8\pi \text{ cm.}$$

iii) $\frac{1}{2} 8\pi \text{ cm}$ is the base radius of a cone

$$2\pi r = 8\pi$$

$$\therefore r = 4$$

$$\therefore \text{radius} = 4 \text{ cm.}$$

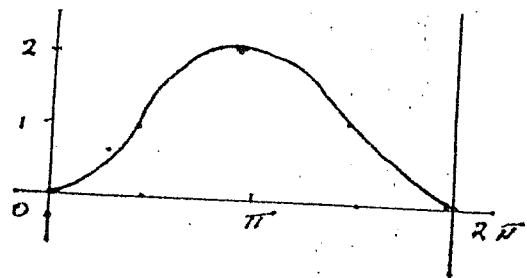
b) $\lim_{x \rightarrow 0} \left\{ \frac{\sin 2x}{3x} \right\} \rightarrow \frac{2x}{3x} = \frac{2}{3}$

($\sin 2x \doteq 2x$ as $x \rightarrow 0$)

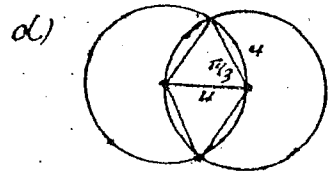
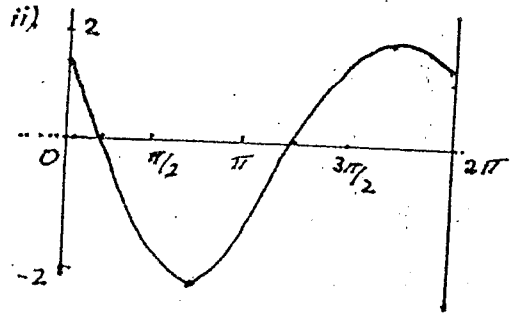
$$\lim_{x \rightarrow 0} \left\{ \frac{\sin^2 2x}{x^2} \right\} = \lim_{x \rightarrow 0} \left\{ \frac{\sin 2x}{x} \cdot \frac{\sin 2x}{x} \right\}$$

as $x \rightarrow 0$, $\sin 2x \rightarrow 2x$

$$\therefore \lim_{x \rightarrow 0} \left(\frac{\sin^2 2x}{x^2} \right) \rightarrow \frac{2x}{x} \cdot \frac{2x}{x}$$



$$y = 1 - \cos x$$



$$A_{\Delta} = \frac{1}{2} ab \sin C$$

$$A_{\text{seg}} = \frac{1}{2} r^2 (\theta - \sin \theta)$$

$A = 2 \Delta's + 4 \text{ segments}$

$$= 2 \times \frac{1}{2} \cdot 4 \cdot 4 \cdot \frac{\sqrt{3}}{2} + 4 \times \frac{1}{2} \cdot 16 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right)$$

$$= 8\sqrt{3} + 32 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right)$$

$$= \frac{32\pi}{3} - 8\sqrt{3}$$

$$\text{Area} \doteq 19.7 \text{ cm}^2$$

$$R \sin(x+\alpha) + R \cos(x+\alpha) = R \sin(x+\alpha)$$

$$R \sin(x+\alpha) = R \sin \alpha \cos x + R \cos \alpha \sin x$$

$$\therefore R \cos \alpha = \sqrt{3}, R \sin \alpha = 1$$

$$\therefore R^2 = 3+1, R = 2$$

$$\tan \alpha = \frac{1}{\sqrt{3}}, \alpha = \frac{\pi}{6}$$

$$\therefore \sqrt{3} \sin x + \cos x = 2 \sin(x + \pi/6)$$

$$\text{ii) } 2 \sin(x + \pi/6) = 1$$

$$\therefore \sin(x + \pi/6) = \frac{1}{2}$$

$$\therefore x + \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$$

$$\therefore x = 0, 2\pi/3, 2\pi$$

$$\text{b) } 2x + y = 4, \quad y = -2x + 4, \quad m_1 = -2$$

$$3x - 2y = 6, \quad y = \frac{3}{2}x - 3, \quad m_2 = 3/2$$

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{-2 - 3/2}{1 - 3(3/2)} \right|$$

$$= \left| \frac{-3\frac{1}{2}}{-\frac{1}{2}} \right|$$

$$= 7/4$$

$$\therefore \theta \doteq 60^{\circ}15'$$

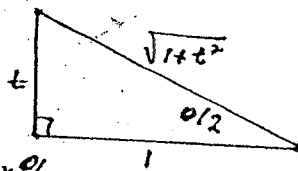
$$\text{c) } \frac{\sin 2x}{1 + \cos 2x} = \tan x$$

$$\text{LHS} = \frac{2 \sin x \cos x}{1 + \cos^2 x - \sin^2 x}$$

$$= \frac{2 \sin x \cos x}{\cos^2 x + \cos^2 x}$$

$$= \frac{2 \sin x \cos x}{2 \cos^2 x} = \tan x = \text{RHS}$$

$$\text{d) } t = \tan \theta/2$$



$$\cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}$$

$$= \frac{1-t^2}{1+t^2} - \frac{t^2}{1+t^2}$$

$$= \frac{1-t^2-t^2}{1+t^2}$$

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$= 2 \cdot \frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}}$$

$$= \frac{2t}{1+t^2}$$

$$\text{ii) } \frac{1 + \sin \theta - \cos \theta}{1 + \sin \theta + \cos \theta} = \frac{1 + \frac{2t}{1+t^2} - \left(\frac{1-t^2}{1+t^2} \right)}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}$$

$$= \frac{1+t^2+2t-1+t^2}{1+t^2+2t+1-t^2}$$

$$= \frac{2t+2t^2}{2+2t}$$

$$= \frac{2t(1+t)}{2(1+t)}$$

$$= t$$

$$\text{e) } \cos 3\theta = \cos(2\theta + \theta)$$

$$= \cos 2\theta \cos \theta - \sin 2\theta \sin \theta$$

$$= \cos \theta (\cos^2 \theta - \sin^2 \theta) - 2 \sin^2 \theta \cos \theta$$

$$= \cos^3 \theta - \cos \theta \sin^2 \theta - 2 \cos \theta \sin^2 \theta$$

$$= \cos^3 \theta - 3 \cos \theta (1 - \cos^2 \theta)$$

$$= \cos^3 \theta - 3 \cos \theta + 3 \cos^3 \theta$$

$$= 4 \cos^3 \theta - 3 \cos \theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\therefore 4\cos^3\theta = 3\cos\theta + \cos 3\theta$$

$$\cos 3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$$

$$\therefore \int_0^{\pi/6} \cos^3\theta \cdot d\theta = \left[\frac{3}{4}\sin\theta + \frac{1}{12}\sin 3\theta \right]_0^{\pi/6}$$

$$\left[\frac{3}{4}\sin\frac{\pi}{6} + \frac{1}{12}\sin\frac{\pi}{2} \right]$$

$$= \frac{3}{4} \times \frac{1}{2} + \frac{1}{12} \times 1$$

$$= \frac{3}{8} + \frac{1}{12}$$

$$= \frac{11}{24}$$

$$f) 1 + \cos x = 1 + \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$$

$$= \cos^2 \frac{x}{2} + \cos^2 \frac{x}{2}$$

$$= 2\cos^2 \frac{x}{2}$$

$$\therefore \int_0^{\pi/3} \frac{dx}{1 + \cos x} = \int_0^{\pi/3} \frac{dx}{2\cos^2 \frac{x}{2}}$$

$$= \int_0^{\pi/3} \frac{1}{2} \sec^2 \frac{x}{2} dx$$

$$= \left[\tan \frac{x}{2} \right]_0^{\pi/3} = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$Q4. a) P(x) = 2x^3 - 3x^2 + 4x - 5$$

$$P(2) = 16 - 12 + 8 - 5$$

$$= 24 - 17$$

$$= 7$$

$$b) P(x) = 2x^3 + 5x^2 + kx - 6$$

$$P(-2) = 4$$

$$\therefore 4 = -16 + 20 - 2k - 6$$

$$4 = -2k - 2$$

$$2k = -6, \quad k = -3$$

$$c) P(x) = x^3 + x^2 - 2x - 8$$

$$P(2) = 8 + 4 - 4 - 8$$

$$= 12 - 12 = 0$$

$\therefore (x-2)$ is a factor.

$$\therefore P(x) = (x-2)(x^2 + 3x + 4)$$

$$\text{For } x^2 + 3x + 4 \quad b^2 - 4ac = 9 - 16$$

$$= -7$$

$$< 0$$

\therefore there are no other factors.

$$d) P(x) = 8x^3 + 3x^2 - 8x - 3$$

$$P(1) = 8 + 3 - 8 - 3 = 0$$

$$\therefore P(x) = (x-1)(8x^2 + 11x + 3)$$

$$= (x-1)(x+1)(8x+3)$$

$$e) P(x) = x^3 + kx + 54$$

Let the roots be α, α, β

$$2\alpha + \beta = 0 \Rightarrow \beta = -2\alpha$$

$$\alpha^2 \beta = -54$$

$$\therefore -2\alpha^3 = -54$$

$$\alpha^3 = 27$$

$$\alpha = 3$$

$$\therefore P(3) = 27 + 3k + 54 = 0$$

$$3k = -81$$

$$k = -27$$

$$\therefore P(x) = x^3 - 27x + 54$$

\therefore Roots are $3, 3, -6$

$$f) 8x^3 + 4x^2 - 2x - 1 = 0$$

$$i) \alpha + \beta + \gamma = -b/a = -1/2$$

$$ii) \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{-\alpha - \beta - \gamma}{\alpha\beta\gamma}$$

$$= -\frac{1}{4} \div \frac{1}{8} = -2$$

$$iii) \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= \frac{1}{4} - 2(-\frac{1}{4})$$

$$= 3/4$$

$$g) 24x^3 - 14x^2 - 7x + 3 = 0$$

Let roots be $\frac{a}{r}, a, ar$

$$\text{Product, } a^3 = \frac{-3}{24}, \quad \therefore a = -\frac{1}{2}$$

$$\therefore P(x) = (2x+1)(12x^2 - 13x + 3)$$

$$= (2x+1)(4x-3)(3x-1)$$

$$\therefore \text{If } P(x) = 0, \quad x = \frac{1}{3}, -\frac{1}{2}, \frac{3}{4}$$

$$h) 3x^3 + 4x^2 - 5x - 1 = 0 \text{ has roots } \alpha, \beta, \gamma$$

i.e. Polynomial is $(x-\alpha)(x-\beta)(x-\gamma) = 0$

i) If roots are $2\alpha, 2\beta, 2\gamma$

$$\text{then } (x-2\alpha)(x-2\beta)(x-2\gamma) = 0$$

$$\therefore \left(\frac{x}{2} - \alpha\right)\left(\frac{x}{2} - \beta\right)\left(\frac{x}{2} - \gamma\right) = 0$$

$$\Rightarrow 3\left(\frac{x}{2}\right)^3 + 4\left(\frac{x}{2}\right)^2 - 5\left(\frac{x}{2}\right) - 1 = 0$$

$$\therefore 3x^3 + 8x^2 - 20x - 8 = 0$$

ii) If roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ then

$$\left(x - \frac{1}{\alpha}\right)\left(x - \frac{1}{\beta}\right)\left(x - \frac{1}{\gamma}\right) = 0$$

$$\text{OR } \left(\frac{1}{x} - \alpha\right)\left(\frac{1}{x} - \beta\right)\left(\frac{1}{x} - \gamma\right) = 0$$

$$\therefore 3\left(\frac{1}{x}\right)^3 + 4\left(\frac{1}{x}\right)^2 - 5\left(\frac{1}{x}\right) - 1 = 0$$

$$\therefore 3 + 4x - 5x^2 - x^3 = 0$$

$$\text{OR } x^3 + 5x^2 - 4x - 3 = 0$$