SYD. GIRLS H.S.- SLICING No 4

1. The base of a solid is a square of side length 6cm. Find the volume of the solid if each cross section perpendicular to the diagonal of the square is:
   a) a square
   b) a semi circle
   c) a right angled isosceles triangle, hypotenuse on the base.
   d) an equilateral triangle

2. Find the volume of a square pyramid, base a units and height h units.

3. Find the volume of an elliptical pyramid, base the ellipse with axes of length $2a$ and $2b$ and height h.

4. A solid has a square base and a height H
   Cross sections of the solid parallel to the base at a height h are squares of side s (h). Find the volume of the solid in terms of H, given that:

a) $s(h) = \dfrac{1}{\sqrt{h+1}}$ b) $s(h) = \dfrac{1}{h+1}$

c) $s(h) = \sqrt{h+1}$ d) $s(h) = e^h$

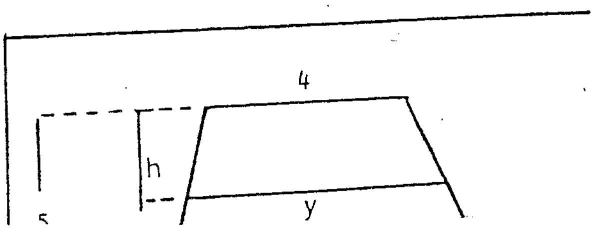
ANSWERS: 1a) $144\sqrt{2}$, b) $18\sqrt{2}\pi$, c) $36\sqrt{2}$, d) $36\sqrt{6}$

2) $\dfrac{a^2h}{3}$, 3) $\dfrac{\pi abh}{3}$, 4a) $\ln(H+1)$, b) $1 - \dfrac{1}{H+1}$,

c) $\dfrac{H^2}{2} + H$, d) $\dfrac{1}{2}(e^{2H} - 1)$

An Important Method

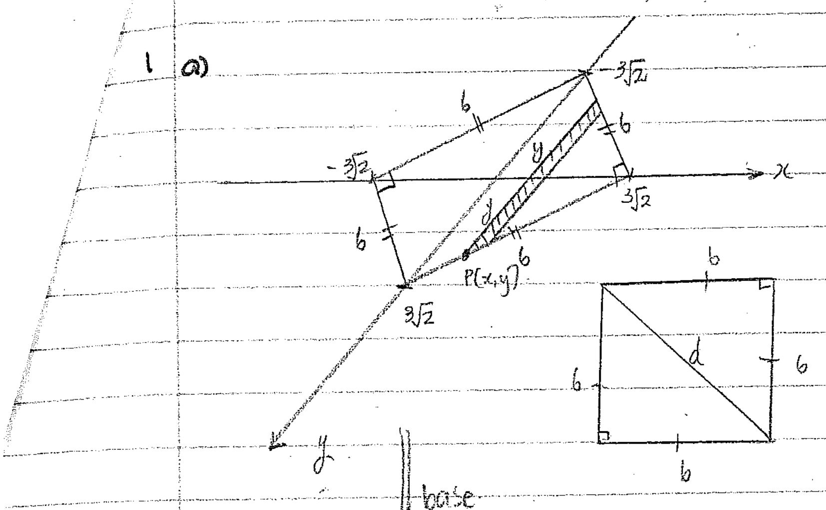
Finding lengths from the given diagram is of particular interest in calculating the required volume. An elegant method is presented below and you will be asked to derive the same result by using similar triangles.



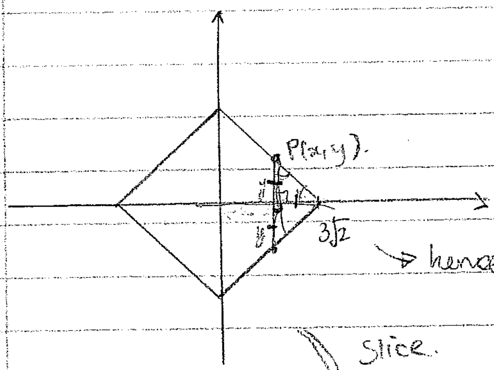
Example: (1)

Find the length y in terms of h, from the diagram which shows an isosceles

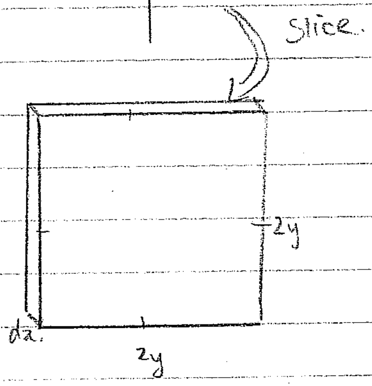
1 a)



$d = 6\sqrt{2}$.



(ii) hence, $y = 3\sqrt{2} - x$.



$V_{slice} = 4y^2\,dx$

$= 4(3\sqrt{2}-x)^2\,dx$

$V_{solid} = 2\int_0^{3\sqrt{2}} 4(3\sqrt{2}-x)^2\,dx$

$= 8\int_0^{3\sqrt{2}} (3\sqrt{2}-x)^2\,dx$

$= 8\int_0^{3\sqrt{2}} \left[18 - 6\sqrt{2}x + x^2\right] dx$

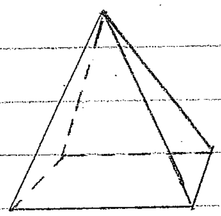
$3\sqrt{2} \cdot 3\sqrt{2} = 18$

$= 8\left[18x - \frac{\cancel{6}3\sqrt{2}x^2}{\cancel{2}} + \frac{x^3}{3}\right]_0^{3\sqrt{2}}$

$= 8\left[\cancel{54\sqrt{2}} - \cancel{54\sqrt{2}} + 18\sqrt{2}\right]$

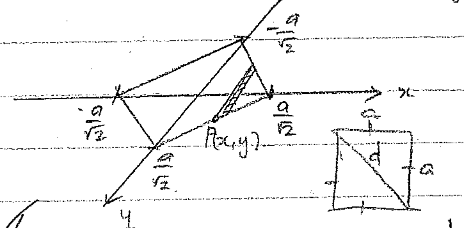
$= 144\sqrt{2}$ units^3

2.

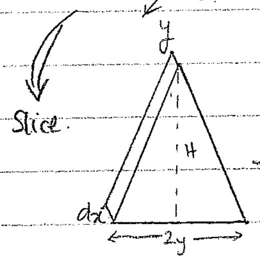


Slices perpendicular to base:

Position base so that triangular slices are parallel to the x or y axis.

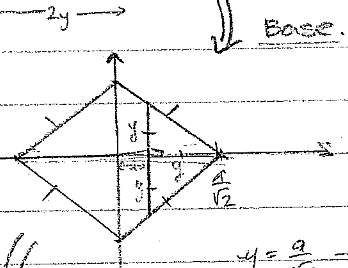


$d = a\sqrt{2}$.

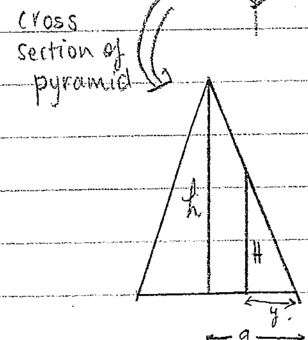


H = height of slice

h = height of pyramid.



$y = \frac{a}{\sqrt{2}} - x$.



$\frac{H}{h} = \frac{y}{\frac{a}{\sqrt{2}}}$ (sim Δ's)

$H = \frac{yh\sqrt{2}}{a}$

$V_{slice} = \frac{1}{2} \cdot 2y \cdot H \cdot dx$

$= y\,H\,dx$ ①

but $H = \frac{yh\sqrt{2}}{a}$ ⇒ sub in.

$= y \cdot \frac{y\,h\sqrt{2}}{a}\,dx$

$= \frac{y^2 h\sqrt{2}\,dx}{a}$

$= \frac{\sqrt{2}h}{a}\, y^2\, dx$

$V_{solid} = 2\,\frac{\sqrt{2}h}{a}\int_0^{\frac{a}{\sqrt2}} y^2 dx$

$= \frac{2\sqrt{2}h}{a}\int_0^{\frac{a}{\sqrt2}} \left(\frac{a}{\sqrt2}-x\right)^2 dx$

$\frac{a}{\sqrt2}\cdot\frac{a}{\sqrt2}.$

$= \frac{2\sqrt{2}h}{a}\int_0^{\frac{a}{\sqrt2}} \left[\frac{a^2}{2}-\frac{2a}{\sqrt2}x+x^2\right] dx$

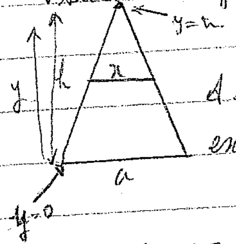
$= \frac{2\sqrt{2}h}{a}\left[\frac{xa^2}{2}-\frac{\cancel{2}ax^2}{\cancel{2}\sqrt2}+\frac{x^3}{3}\right]_0^{\frac{a}{\sqrt2}}$

$= \frac{2\sqrt{2}h}{a}\left[\frac{a^3}{6\sqrt2}-\frac{a^3}{2\sqrt2}+\frac{a^3}{2\sqrt2}\right]$

$= \frac{2\sqrt{2}h}{a}\cdot\frac{a^3}{6\sqrt2}$

$= \frac{1}{3}a^2h.$

$V_{slice} = x^2 dy.$



A linear relationship exists between x & y.

$\therefore x = my + b.$

when $y=0$, $x=a$.

$\therefore a = b.$

$\therefore x = my + a.$ sub.

when $y=h$, $x=0$

$0 = mh + a.$

$m = \frac{-a}{h}.$

$\therefore x = a - \frac{ay}{h}$

$= \frac{a}{h}(h-y).$

$V_{slice} = x^2 dy.$

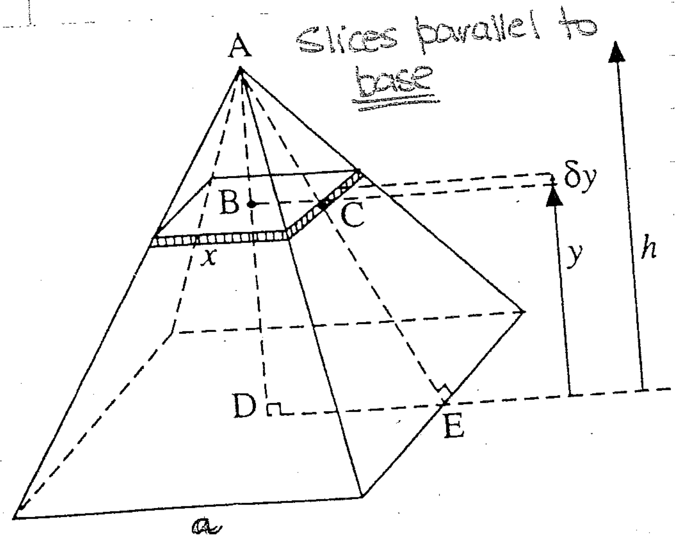
$V_{solid} = \int_0^h \left[\frac{a}{h}(h-y)\right]^2 dy.$

$= \frac{a^2}{h^2}\int_0^h (h^2-2hy+y^2)\,dy$

$= \frac{a^2}{h^2}\left[h^2y - \frac{2hy^2}{2}+\frac{y^3}{3}\right]_0^h$

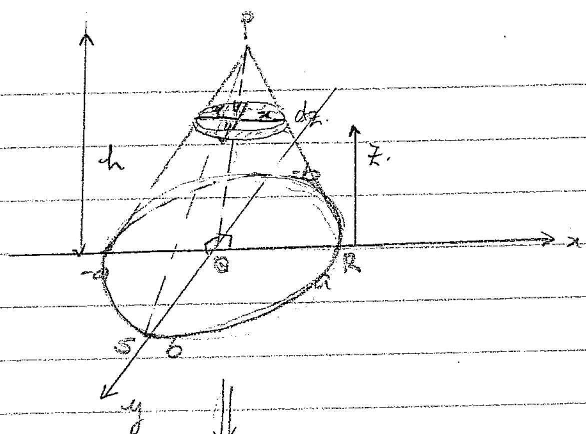
$= \frac{a^2}{h^2}\left[h^3 - h^3 + \frac{h^3}{3}\right]$

$= \frac{a^2h}{3}.$

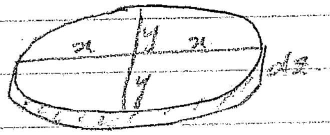


Slicing the pyramid parallel to its base gives square slices of side x and width δy, where y is the height of the slice above the base. E, C are midpoints of the sides of the base and slice respectively.

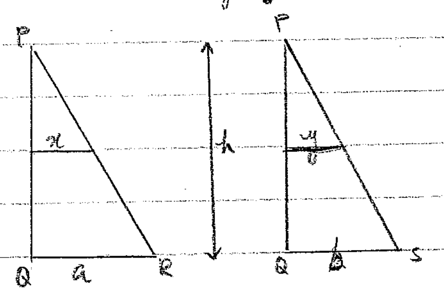
3.



Slice:



$V_{slice} = \pi xy \cdot dz$



$\boxed{z = mx + c}$ $\qquad$ $\boxed{y = zm + c}$

when $z=0, x=a.$ $\qquad$ when $z=0, y=b.$

$\therefore 0 = am + c. \Rightarrow m = \frac{-c}{a}.$ $\qquad$ $b = c.$

when $z=h, x=0$ $\qquad$ when $z=h, y=0.$

$\therefore h = c$ $\qquad$ $0 = \cancel{hm+c}.$

$z = mx + h.$ $\qquad$ $0 = mh + c.$

$z = \frac{-cx}{a} + h.$ $\qquad$ $m = \frac{-c}{h}$

$z = h - \frac{hx}{a}.$ $\qquad$ $\therefore y = \frac{-bz}{h} + b$

$x = a - \frac{az}{h}.$

$y = b - \frac{bz}{h}$

$V_{solid} = \pi \int_0^h \left(a - \frac{az}{h}\right)\left(b - \frac{bz}{h}\right) dz$

$= \pi \int_0^h \frac{a}{h}(h-z) \cdot \frac{b}{h}(h-z)\, dz$

$= \frac{\pi ab}{h^2} \int_0^h (h-z)^2 dz$

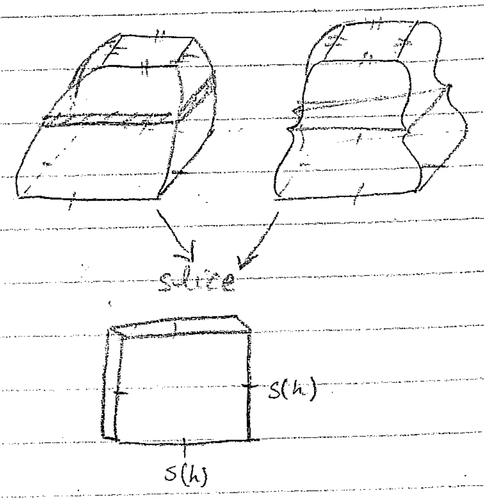
$= \frac{\pi ab}{h^2} \int_0^h \left[h^2 - 2hz + z^2\right] dz$

$= \frac{\pi ab}{h^2} \left[h^2 z - h^2 z^2 + \frac{z^3}{3}\right]_0^h$

$= \frac{\pi ab}{h^2} \cdot \frac{h^3}{3}$

$= \frac{\pi}{3} abh \text{ units}^3$

4.a)



$V_{slice} = \{s(h)\}^2 dh$

$V_{solid} = \int_0^H \{s(h)\}^2 dh$

a) $\rightarrow$ when $s(h) = \frac{1}{\sqrt{h+1}}$

$V_{solid} = \int_0^H \left\{\frac{1}{\sqrt{h+1}}\right\}^2 dh$

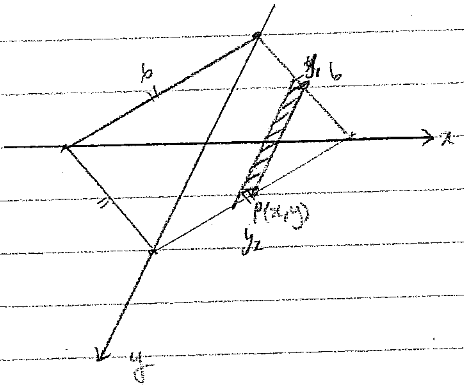
$= \int_0^H \left\{\frac{1}{h+1}\right\} dh$

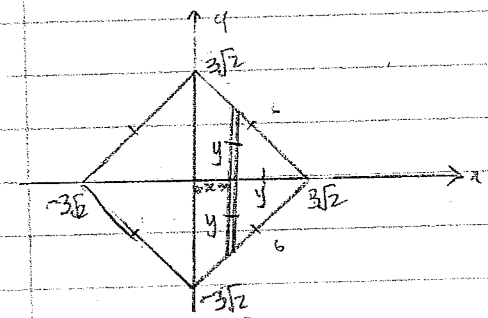
$= \left[\log_e |h+1|\right]_0^H$

$= \log_e |H+1| + \log_e 1$

$= \log_e |H+1| \text{ units}^3$

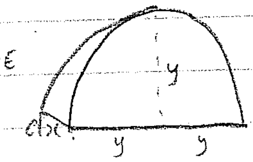
b)





$y = \sqrt{3}/_2 - x$.

SLICE



$V_{slice} = \frac{1}{2}\pi y^2 . dx$

$= \frac{\pi}{2}(3\sqrt{2} - x)^2 dx$

$V_{solid} = \pi \int_0^{3\sqrt{2}} (3\sqrt{2} - x)^2 dx$

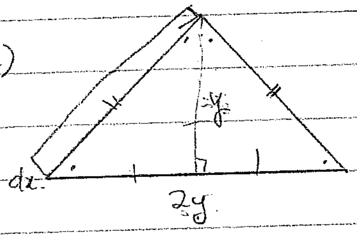
$= \pi \int_0^{3\sqrt{2}} (18 - 6\sqrt{2}x + x^2)$

$= \pi \left[18x - 3\sqrt{2}x^2 + \frac{x^3}{3}\right]_0^{3\sqrt{2}}$

$= \pi \left[54\sqrt{2} - 54\sqrt{2} + 18\sqrt{2}\right]$

$= 18\pi\sqrt{2}$ units³

c)



$V_{slice} = \frac{1}{2} bh . dx$

$= \frac{1}{2} . 2y^2 . dx$

$= y^2 dx$.

$V_{solid} = 2\int_0^{3\sqrt{2}} y^2 dx \quad (y = 3\sqrt{2} - x)$

$= 2\int_0^{3\sqrt{2}} (3\sqrt{2} - x)^2 dx$

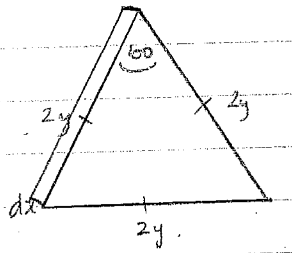
$= 2\int_0^{3\sqrt{2}} (18 - 6\sqrt{2}x + x^2) dx$

$= 2\left[18x - 3\sqrt{2}x^2 + \frac{x^3}{3}\right]_0^{3\sqrt{2}}$

$= 2\left[\cancel{54\sqrt{2}} - \cancel{54\sqrt{2}} + 6 . 3\sqrt{2}\right]$

$= 36\sqrt{2}$ units³

d)



$V_{slice} = \frac{1}{\cancel{2}} \cancel{4}y^2 . \frac{\sqrt{3}}{2} . dx$

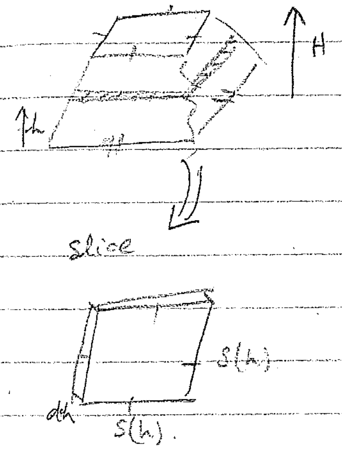
$= \sqrt{3}y^2 dx$

$V_{solid} = 2\sqrt{3}\int_0^{3\sqrt{2}} y^2 dx$

$= 2\sqrt{3}\left[18\sqrt{2}\right]$

$= 36\sqrt{6}$ units³

4b)



$V_{slice} = \{S(h)\}^2 dh.$

$V_{solid} = \int_0^H \{S(h)\}^2 dh.$ where $S(h) = \frac{1}{h+1}.$

$= \int_0^H \left(\frac{1}{h+1}\right)^2 dh$

$= \int_0^H \left(\frac{1}{h^2+2h+1}\right) dh.$

$= \int_0^H \frac{dh}{h^2+2h+1.}$

$h^2+2h+1 = (h+1)^2$

c) $V_{slice} = \{S(h)\}^2 dh.$

$V_{solid} = \int_0^H \{S(h)\}^2 dh.$

$= \int_0^H \{\sqrt{H+1}\}^2 dh$

$= \int_0^H (h+1)\, dh$

$= \left[\frac{h^2}{2} + h\right]_0^H$

$= \frac{H^2}{2} + H.$

d) $V_{slice} = \{S(h)\}^2 dh.$

$V_{solid} = \int_0^H [S(h)]^2 dh$

$= \int_0^H e^{2h}.dh.$

$= \left[\frac{1}{2} e^{2h}\right]_0^H$

$= \frac{1}{2}\left[e^{2H} - 1\right].$