25-May-05

SYDNEY GIRLS — SLICING No 7 - Cylindrical Shells

1. Find the volume formed when the area enclosed by $Y = 2x^2 - x^4$ and the X axis is rotated about the Y axis.

2. Find the volume formed when the area enclosed by $y = 4x - x^2$ and $y = x^2$ is rotated about the Y axis.

3. Write down the volume of a right circular shell of height h with inner and outer radii of r and R respectively.

The region bounded by the curves $y = \dfrac{1}{x+1}$ and $y = \dfrac{1}{x+2}$

between $x = 0$ and $x = 2$ is rotated about the Y axis. Find the volume formed correct to three significant figures. (HSC 1982)

4. An egg timer has the shape of an hour glass and can be described as being the volume obtainrf by rotating $y = x + 6x^3$ about the Y axis where

$$-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$$

Find the volume correct to three significant figures.

5. Find the volume formed when the area enclosed by $y = \sin x$ and the x axis between $x = 0$ and $x = \pi$ is rotated about:
   a) Y axis
   b) the line $x = -\pi$.
   c) the line $x = 2\pi$.

6. Find the volume formed when the ellipse $x^2 + 4y^2 = 4$ is rotated about :
   a) the Y axis
   b) the line $y = 1$

7. Use cylindrical shells to show that the volume formed when $(x - 2R)^2 + y^2 = R^2$ is rotated about the Y axis is $4\pi^2R^3$. (1987 HSC)

Answers: 1) $4\pi/3$ 2) $16\pi/3$ 3) 1.81 4) 4.74 5a) $2\pi^2$ b) $6\pi^2$ c) $6\pi^2$
6a) $16\pi/3$, 6b) $4\pi^2$.

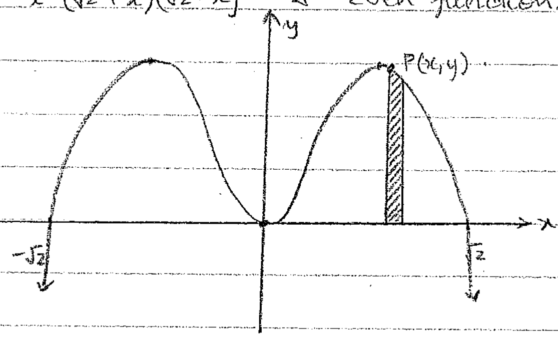
25-may-05

1. Find the volume formed when the area enclosed by $y = 2x^2 - x^4$ & the x-axis is rotated about the y-axis.

$y = 2x^2 - x^4$

$= x^2(2 - x^2)$

$= x^2(\sqrt{2}+x)(\sqrt{2}-x)$ ← Even function.



Volume of the shell:

$R = x + \Delta x$

$r = x$

$h = y$

$\left.\begin{matrix} R = x+\Delta x \\ r = x \\ h = y \end{matrix}\right\} V = \pi\left[R^2 - r^2\right]h.$

$\Delta V = \pi\left[\cancel{x^2} + 2x\Delta x + (\Delta x)^2 - \cancel{x^2}\right]y$

$= \pi\, 2x\Delta x\, y$ → $(\Delta x)^2$ is v. small

Volume of solid:

$\text{Volume} = 2\pi \int_0^{\sqrt{2}} xy\,dx.$ ⇒ $y = 2x^2 - x^4$

$= 2\pi \int_0^{\sqrt{2}} \left[2x^3 - x^5\right] dx$

$= 2\pi \left[\frac{x^4}{2} - \frac{x^6}{6}\right]_0^{\sqrt{2}}$

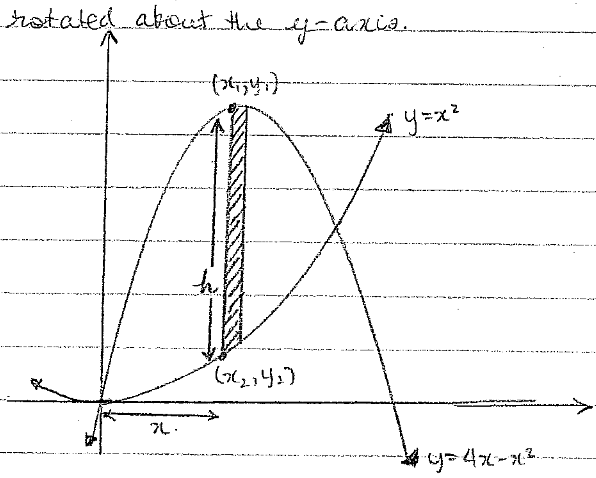
$= 2\pi\left[\frac{4}{2} - \frac{8}{6}\right]$

$= 2\pi \cdot \frac{2}{3}$

$= \frac{4\pi}{3}$ units^3

$=$

2. Find the volume formed when the area enclosed by $y = 4x - x^2$ & $y = x^2$ is rotated about the y-axis.



Point of intersection:

$x^2 = 4x - x^2$

$0 = 4x - 2x^2$

$= 2x(2 - x)$

$x = 0, \quad x = 2$

$h = y_2 - y_1$ $\qquad r = x$

$= 4x - x^2 - x^2$ $\qquad R = x + \Delta x$

$= 4x - 2x^2$

$\Delta V = \pi\left[(x+\Delta x)^2 - (x)^2\right]\left[4x - 2x^2\right]$

$= (2\pi x \Delta x)(4x - 2x^2)$ → $(\Delta x)^2$ is v small.

$= \pi\left[8x^2 - 4x^3\right]\Delta x$

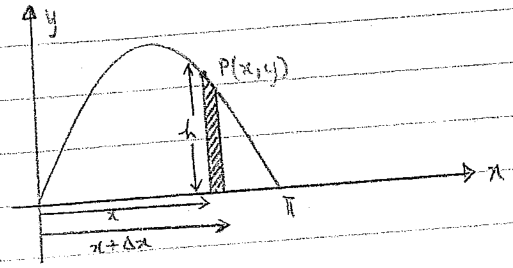
$V_{solid} = 4\pi \int_0^2 \left[2x^2 - x^3\right] dx.$

$= 4\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2$

$= 4\pi\left(5\frac{1}{3} - 4\right)$

$= \frac{16\pi}{3}$ units^3.

a) Find the volume formed when the area enclosed by $y = \sin x$ & the x-axis between $x=0$ & $x=\pi$ is rotated about the $\boxed{y\text{-axis}}$.



$$\Delta V = \pi\left[(x+\Delta x)^2 - (x)^2\right] h$$

$$= \pi\, 2x\Delta x\, y \longrightarrow (\Delta x)^2 \text{ is very small}$$

now, $y = \sin x$

$$= \pi\, 2x \sin x\, \Delta x$$

$$V_{solid} = 2\pi \int_0^{\pi} x \sin x \, dx$$

let $u = x \quad v' = \sin x$

$u' = 1 \quad v = -\cos x$

$$I = -x\cos x + \int \cos x \, dx$$

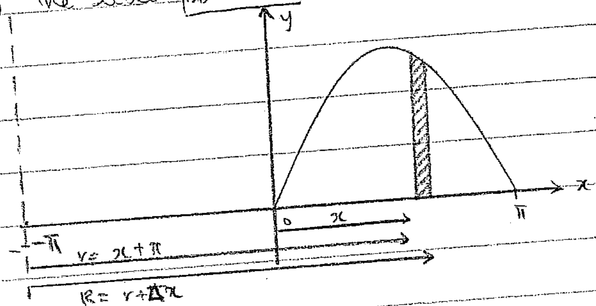
$$= -x\cos x + \sin x + c$$

$$= 2\pi\left[-x\cos x + \sin x\right]_0^{\pi}$$

$$= 2\pi\ [\pi + 0] - [0 + 0]$$

$$= 2\pi^2 \text{ units}^3$$

b) the line $\boxed{x = -\pi}$



To make the algebra easier, use r, $r + \Delta x$.

$$\Delta V = \pi\left[(r+\Delta x)^2 - (r)^2\right] y$$

$$= \pi\left[2r\Delta x\right] y$$

$$= 2\pi\left[\pi + x\right] \sin x \cdot \Delta x$$

$$V_{solid} = 2\pi \int_0^{\pi} (\pi + x)\sin x \, dx$$

$u = \pi + x \quad v' = \sin x$

$u' = 1 \quad v = -\cos x$

$$I = -(\pi + x)\cos x + \int \cos x \, dx$$

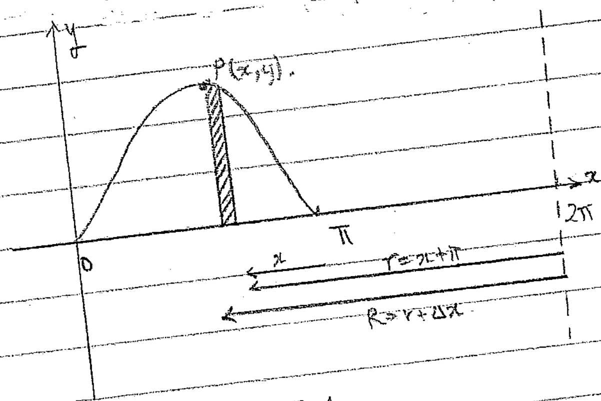
$$= -(\pi + x)\cos x + \sin x + c$$

$$= 2\pi\left[-(\pi + x)\cos x + \sin x\right]_0^{\pi}$$

$$= 2\pi\left(\left[2\pi + 0\right] - \left[-\pi\right]\right)$$

$$= 6\pi^2 \text{ units}^3$$

The line $x = 2\pi$.



$$\Delta V = \pi\left[R^2 - r^2\right]h$$

$$= \pi\left[(r+\Delta x)^2 - r^2\right]y$$

$$= \pi\left[2r\Delta x\right]y \quad \rightarrow (\Delta x)^2 \text{ is v. small.}$$

$$V_{solid} = 2\pi \int_0^{\pi} (x+\pi)\sin x \, dx$$

let $u = x+\pi$ $\quad v' = \sin x$

$u' = 1$ $\quad v = -\cos x$

$$I = -(x+\pi)\cos x + \int \cos x \, dx$$

$$= -(x+\pi)\cos x + \sin x + C$$

$$= 2\pi\left[-(x+\pi)\cos x + \sin x\right]_0^{\pi}$$

$$= 2\pi\left((2\pi+0) - (-\pi+0)\right)$$

$$= 2\pi \cdot 3\pi$$

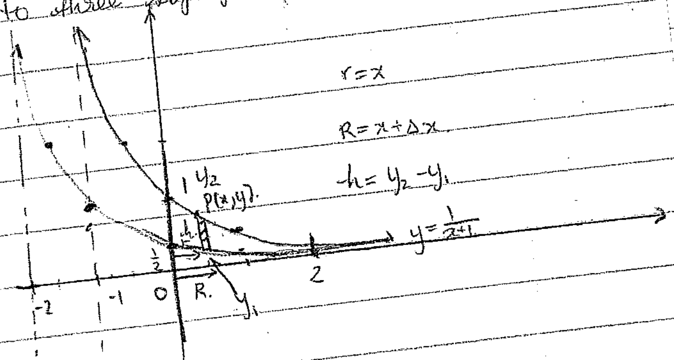
$$= 6\pi^2 \text{ units}^3$$

3. Write down the volume of a right circular shell of height h with inner & outer radii of r & R respectively.

The region bounded by the curves:

$$y = \frac{1}{x+1} \quad \& \quad y = \frac{1}{x+2}$$

between $x=0$ & $x=2$ is rotated about the y-axis. Find the volume formed to three significant figures.



$r = x$

$R = x + \Delta x$

$h = y_2 - y_1$

$$\Delta V = \pi(R^2 - r^2)h$$

$$= \pi\left((x+\Delta x)^2 - x^2\right)(y_2 - y_1)$$

$$= \pi(2x\Delta x)(y_2 - y_1) \longrightarrow (\Delta x)^2 \text{ is v small}$$

$$V_{solid} = 2\pi \int_0^2 x(y_2 - y_1)\, dx$$

$$= 2\pi \int_0^2 \frac{x\, dx}{(x+1)(x+2)}$$

$$I: \frac{x\,dx}{(x+1)(x+2)} = \frac{A\,dx}{x+1} + \frac{B\,dx}{x+2}$$

$$A(x+2) + B(x+1) = x$$

let $x = -1$

$A = -1$

let $x = -2$

$-B = -2$

$B = 2$

$$I = -\int \frac{dx}{x+1} + 2\int \frac{dx}{x+2}$$

$$= \log_e\left|\frac{(x+2)^2}{x+1}\right| + C$$

$$V_{solid} = 2\pi\left[\log_e\left|\frac{(x+2)^2}{x+1}\right|\right]_0^2$$

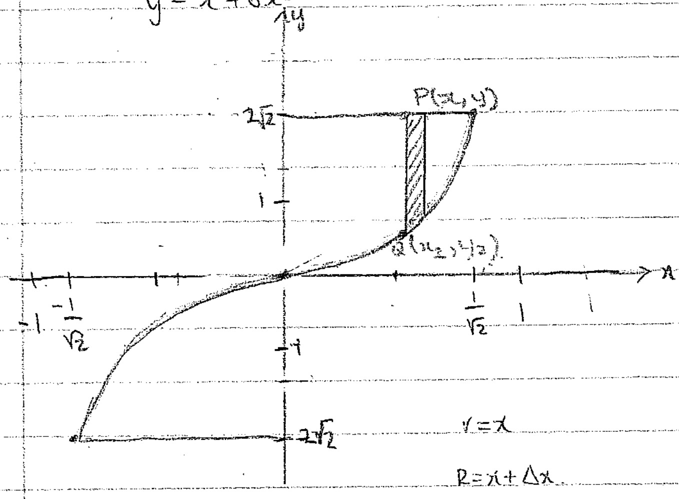
$= 2\pi\left[\log_e\left|\frac{16}{3}\right| - \log_e\left|\frac{4}{1}\right|\right]$

$= 1.8076\ldots$

$= 1.81$ (3 sig fig)

4. An egg timer has the shape of an hour glass & can be described as being the volume obtained by rotating $y = x+6x^3$ about the y-axis where $-\frac{1}{\sqrt2} \le x \le \frac{1}{\sqrt2}$. Find the volume correct to 3 significant figures.

$y = x + 6x^3$



$r = x$

$R = x + \Delta x$

$h = y - (x+6x^3)$

$= 2\sqrt2 - (x+6x^3)$

$\Delta V = \pi(R^2 - r^2)h$

$= \pi((x+\Delta x)^2 - (x)^2)(2\sqrt2 - (x+6x^3))$

$= \pi(2x\Delta x)(2\sqrt2 - (x+6x^3)) \rightarrow (\Delta x)^2$ is v. small.

$V\text{ solid} = 2\pi \int_0^{\frac{1}{2\sqrt2}} 2x(2\sqrt2 - (x+6x^3))\,dx$

$= 2\pi \int_0^{\frac{1}{\sqrt2}} (4\sqrt2 x - 2x^2 - 12x^4)\,dx$

$= 2\pi \left[2\sqrt2 x^2 - \frac{2}{3}x^3 - \frac{12}{5}x^5\right]_0^{\frac{1}{\sqrt2}}$

$= 2\pi\left[\sqrt2 - \frac{1}{3\sqrt2} - \frac{3}{5\sqrt2}\right]$

$= 4.7391\ldots$

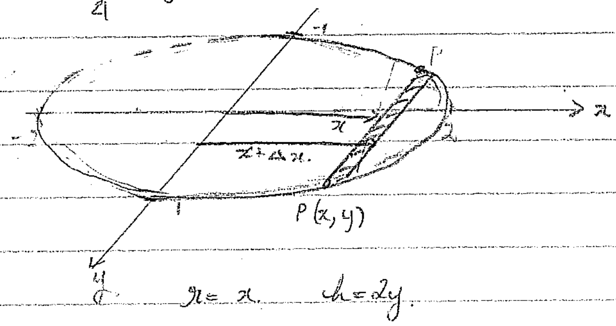
$= 4.74$ (3 sig fig)

6. Find the volume formed when the ellipse $x^2+4y^2=4$ is rotated about

a) the y-axis.

$x^2+4y^2=4$

$= \frac{x^2}{4} + y^2 = 1.$



$r = x \quad h = 2y$

$R = x + \Delta x$

$\Delta V = \frac{\pi}{\cancel{2}}(\cancel{x^2} + 2x\Delta x + (\Delta x)^2 - \cancel{x^2})2y$

$= \frac{\pi}{\cancel{2}}(2x\Delta x)2y \rightarrow (\Delta x)^2$ is v. small

now, $x^2+4y^2=4$

$4y^2 = 4-x^2$

$y^2 = \frac{4-x^2}{4}$

$y = \pm\frac{\sqrt{4-x^2}}{2}$

$V\text{ solid} = \pi\int_0^2 (2x\sqrt{4-x^2})\,dx$

$= 2\pi\int_0^2 x\sqrt{4-x^2}\,dx$

$= -\pi\int_0^2 -2x(4-x^2)^{1/2}\,dx$

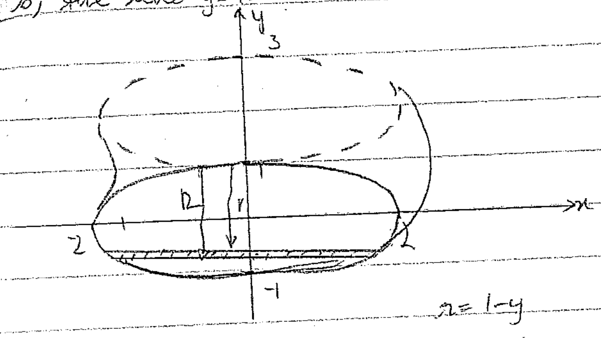
$= -\pi\left[\frac{2}{3}(4-x^2)^{3/2}\right]_0^2$

$= \frac{-2\pi}{3}\left[(4-x^2)^{3/2}\right]_0^2$

$= \frac{-2\pi}{3}(0-8)$

$= \frac{16\pi}{3}$ units^3

b) the line $y=1$.



$x = 1-y$

$R = r + \Delta y$

$\Delta V = \pi\left[(r+\Delta y)^2 - r^2\right] . 2x$

$= \pi\left(\cancel{r^2} + 2r\Delta y + (\Delta y)^2 - \cancel{r^2}\right)2x$

$= \pi\, 2x .\, 2r\,\Delta y \rightarrow (\Delta y)^2$ is very small

$= 4\pi\, x\,(1-y)\,\Delta y$

~~$= 4\pi(x-xy)\Delta y$~~

$x^2 + 4y^2 = 4$

$4y^2 = 4 - x^2$

$y^2 = \dfrac{4-x^2}{4}$

$y = \dfrac{\sqrt{4-x^2}}{2}$

$V_{solid} =$

$4\pi \int_{-1}^{1} x(1-y)\,dy$

$= 4\pi$