Sydney Girls High School

2009

MATHEMATICS

EXTENSION 1

[December 2008]

YEAR 12

ASSESSMENT TASK 1

Time Allowed: 75 minutes

Topics: Logarithms and Exponentials, Integration, Locus and The Parabola

General Instructions:

- There are FIVE (5) Questions which are of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Start each Question on a new page.
- Write on one side of the paper only.
- Diagrams are NOT to scale.
- Board-approved calculators may be used.
- Write your student number clearly at the top of each question and clearly number each question.

Total: 75 marks

QUESTION ONE (15 marks)

a) Find

i) $\int (2x+5)\, dx$ (2)

ii) $\int (3x-1)^5 dx$ (2)

iii) $\int e^{3x-1} dx$ (1)

iv) $\int \frac{x^3}{\sqrt{x}}\, dx$ (2)

b) Find the coordinates of the centre and radius of the circle with equation (3)

$$x^2+y^2-8x+2y-8=0$$

c) The table shows the value of $f(x)$ for 5 values of x. Use Simpson's rule with 5 function values to find an approximation for $\int_1^9 f(x)\, dx$, correct to 2 decimal places. (3)

x	1	3	5	7	9
$f(x)$	5	9	2	-1	-6

d) Differentiate with respect to x (2)

$$y = x\log_e x$$

QUESTION TWO (15 marks)

a) Find the equation of the tangent to the curve $y=e^{3x}+1$ at the point where $x=\log_e 2$. Leave your answer in general form. (3)

b) Solve $2\log_e x=\log_e(2x+3)$ (3)

c) The point $P(k,2)$ lies on the parabola $x^2=12y$.
Find the distance from P to the focus, S, of the parabola. (3)

d) Evaluate $\int_1^4\left(x-\frac{2}{x}\right)^2 dx$ (4)

e) Express as a single logarithm in its simplest form (2)

$$\log_e 2+2\log_e 18-\frac{3}{2}\log_e 36$$

QUESTION THREE (15 marks)

a) Evaluate $\int_0^2 \frac{4x}{x^2+4}\,dx$ (3)

b) Differentiate with respect to x

$$y=\log_e\left(\frac{5x+1}{2x^2-3}\right)$$ (3)

c) i) Sketch the parabola $y=x^2-6x$ and the line $y=4x$ on the same diagram, showing the coordinates of their points of intersection. (4)

ii) Find the area bounded by these two graphs. (2)

d) Differentiate with respect to x

i) $\frac{e^x}{1-x}$ (2)

ii) $\log_e x^3$ (1)

QUESTION FOUR (15 marks)

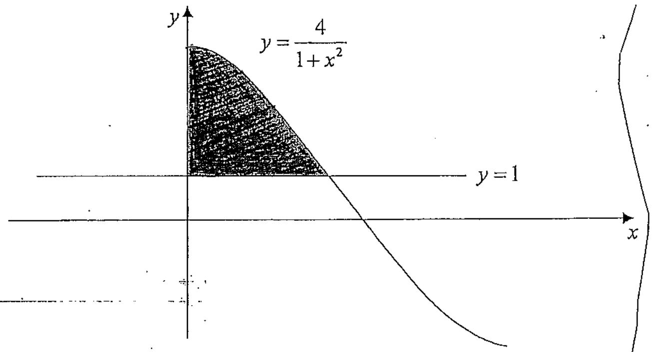
a) For the parabola $x^2 - 4x = 8y + 4$ write down the

i) Coordinates of the vertex (2)
ii) Focal length (2)
iii) Equation of the directrix (1)
iv) Coordinates of the focus (1)

b) The shaded region makes a revolution about the y - axis. (4)

i) Show that the volume of the resulting solid is given by $V = \pi \int_1^4 \left(\frac{4}{y} - 1 \right) dy$

ii) Find the exact volume of the solid



c) A function $y = f(x)$ passes through $(0,1)$ and has a derivative $f'(x) = 12e^{-2x}$.

Find the equation of the function. (2)

d) Find the equation of the normal to $y = e^{2x}$ at the point where $y = 1$. (3)

QUESTION FIVE (15 marks)

a) Find $\int x e^{5x^2+2}\, dx$ (2)

b) i) Find the derivative of $(\log_e 6x)^3$ and hence, (1)

ii) Evaluate $\int_3^5 \frac{(\log_e 6x)^2}{x}\, dx$, correct to 1 decimal place. (3)

c) Differentiate $y = 6^{2x}$ with respect to x (1)

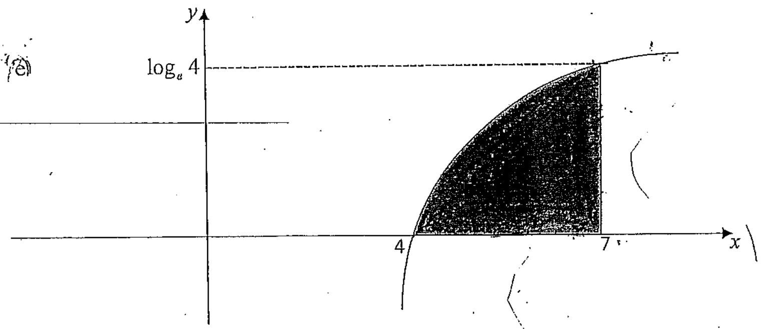
d) The Parabola $x^2 = 8y$ is reflected in its latus rectum.

The latus rectum is a focal chord parallel to the directrix.

i) Find the equation of the new parabola. (2)

ii) The point $P(12,18)$ lies on the first parabola. Finds its coordinates after it has been reflected. (2)

e) (4)



In the diagram, the shaded region is bounded by $y = \log_e(x-3)$, the x-axis and the line $x = 7$. Find the exact value of the area of the shaded region.

THE END

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1. (a)(i) $\int 2x+5\, dx = x^2+5x+C$ ✓

(ii) $\int (3x-1)^5\, dx = \dfrac{\frac{1}{3}(3x-1)^6}{6} + C$

$= \dfrac{(3x-1)^6}{18} + C$ ✓

(iii) $\int e^{3x-1}\, dx = \dfrac{e^{3x-1}}{3} + C$ ✓

(iv) $\int \dfrac{x^3}{\sqrt{x}}\, dx = \int x^{3-\frac{1}{2}}\, dx$ ✓

$= \int x^{5/2}\, dx$

$= \dfrac{2x^{7/2}}{7} + C$ ✓

(b) $x^2 - 8x + y^2 + 2y - 8 = 0$

$x^2 - 8x + 16 + y^2 + 2y + 1 = 17 + 8$ ✓

$(x-4)^2 + (y+1)^2 = 25$ ✓

Circle with centre $(4,-1)$ ✓ and radius $= 5$ ✓ units

(c)

$A \doteqdot \frac{3-1}{3}\left(1\times5 + 4\times9 + 2\times2 + 4\times -1 + 1\times -6\right)$ ✓

$= \frac{2}{3}(5+36+4-4-6)$

$= \frac{2}{3}(35)$ ✓

$= \frac{70}{3}$

(d) $y = x\log_e x$

$\dfrac{dy}{dx} = \log_e x + \dfrac{x}{x}$ ✓

$= \log_e x + 1$ ✓

2. (a) $y = e^{3x}+1$

$\dfrac{dy}{dx} = 3e^{3x}$ ✓

at $x = \log_e 2$,

$\dfrac{dy}{dx} = 3\times e^{\log_e 8}$

$= 3\times 8$

$= 24$ ✓

At $(\log_e 2, 9)$, ✓

$(y-9) = 24(x-\log_e 2)$ ✓

$24x - y + (9 - 24\log_e 2) = 0$

(b) $\log_e x^2 = \log_e(2x+3)$ ✓

$x^2 = 2x+3$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$ ✓

$x = 3$ or $x = -1$ ✓

(c) At $(k, 2)$

$x^2 = 12y$

$x^2 = 12 \times 2$

$x^2 = 24$

$x = \sqrt{24}$ ✓

$12 = 4a$

$a = 3$ ✓

The focus of the parabola $x^2 = 12y$ is at $(0, 3)$

Distance from $P(\sqrt{24}, 2)$ to $S(0, 3)$ is:

$D = \sqrt{(3-2)^2 + (0-\sqrt{24})^2}$

$= \sqrt{1 + 24}$

$= \pm 5$ ✓

$= 5$ units as $D > 0$

(d) $\int_1^4 \left(x - \frac{2}{x}\right)^2 dx = \int_1^4 \left[x^2 - \frac{2x}{x} - \frac{2x}{x} + \frac{4}{x^2}\right] dx$ ✓

$= \left[\frac{x^3}{3} - \frac{4}{x} - 4x\right]_1^4$ ✓

$= \frac{64}{3} - 1 - 16 - \left(\frac{1}{3} - 4 - 4\right)$ ✓

$= \frac{63}{3} - 9$

$= 21 - 9$

$= 12$ ✓

(e) $\log_e 2 + 2\log_e 18 - \frac{3}{2}\log_e 36$

$= \log_e 2 + \log_e 18^2 - \log_e 6^3$ ✓

$= \log_e\left(\frac{2 \times 324}{216}\right)$ ✓

$= \log_e 3$ ✓

3. (a) $\int_0^2 \frac{4x}{x^2+4}\, dx = 2\int_0^2 \frac{2x}{x^2+4}\, dx$

$= 2\left[\ln(x^2+4)\right]_0^2$ ✓

$= 2\left[\ln 8 - \ln 4\right]$ ✓

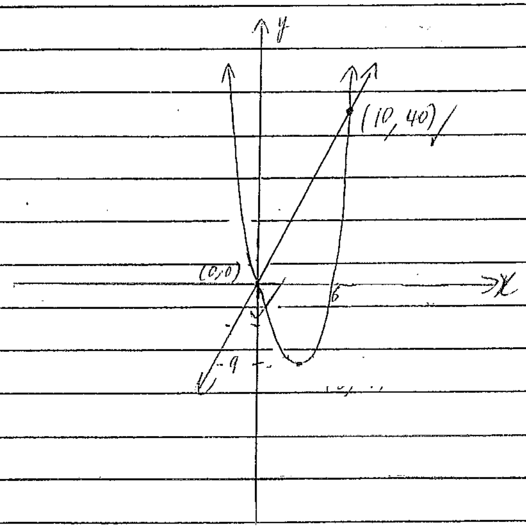
$= 2\ln 2$ ✓

(b) $y = \log_e\left(\frac{5x+1}{2x^2-3}\right)$ ✓

$y = \log_e(5x+1) - \log_e(2x^2-3)$

$\frac{dy}{dx} = \frac{5}{5x+1} - \frac{4x}{2x^2-3}$ ✓

(c)(i) $y = x^2 - 6x$ $\qquad y = 4x$



(ii) $A = \int_0^{10} 4x - (x^2 - 6x)\, dx$ ✓

$= \int_0^{10} -x^2 + 10x\, dx$

$= \int_{10}^{0} x^2 - 10x\, dx$ ✓

$= \left[\frac{x^3}{3} - 5x^2 \right]_{10}^{0}$

$= 0 - \left(\frac{1000}{3} - 500 \right)$ ✓

$= \frac{500}{3}$ units^2 ✓

(d) (i) $y = \dfrac{e^x}{1-x}$ $\qquad \dfrac{vu' - uv'}{v^2}$

$\dfrac{dy}{dx} = \dfrac{(1-x)e^x - (-e^x)}{(1-x)^2}$

$= \dfrac{2e^x - xe^x}{(1-x)^2}$ ✓

$= \dfrac{(2-x)e^x}{(1-x)^2}$ ✓

(ii) $y = \log_e x^3 = 3 \ln x$

$y' = \frac{3}{x}$

$\dfrac{dy}{dx} = \dfrac{3\cancel{x^2}}{x^3}$

4. (a) (i) $x^2 - 4x = 8y + 4$

$x^2 - 4x + 4 = 8y + 8$ ✓

$(x-2)^2 = 8(y+1)$ ✓

It follows that the vertex is at $(2, -1)$ ✓

(ii) $4a = 8$

$a = 2$ ✓

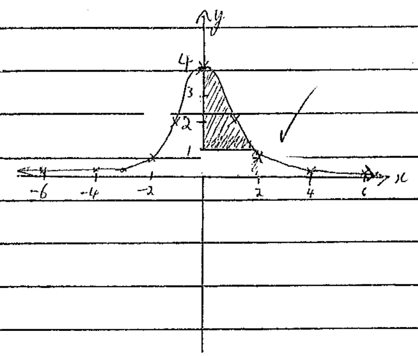
$\therefore$ The focal length is 2 ✓

(iii) Directrix at $y = -1 - a$

$\therefore y = -3$ ✓

(iv) Focus at $(2, 1)$ ✓

(b) As the diagram is wrong, one is drawn here:



✓

(i) $y = \frac{4}{1+x^2}$

$x^2+1 = \frac{4}{y}$

$x^2 = \frac{4}{y} - 1$ ✓

$V = \pi\int_1^4 x^2\,dy$

$= \pi\int_1^4 \frac{4}{y} - 1\ dy$ ✓

(ii) $V = \pi\left[4\log_e y - y\right]_1^4$ ✓

$= \pi\left[(4\log_e 4 - 4) - (0-1)\right]$

$= \pi(4\log_e 4 - 3)$ units^3 ✓

(c) $f'(x) = 12e^{-2x}$

$f(x) = -6e^{-2x} + C$

Sub point $(0,1)$ ✓

$1 = -6e^{-2\times 0} + C$

$C = 7$ ✓

$\therefore f(x) = -6e^{-2x} + 7$

5.(a) $\int x e^{5x^2+2}\,dx$

$= \frac{1}{10}\int 10x e^{5x^2+2}\,dx$ ✓

$= \frac{1}{10}\left(e^{5x^2+2}\right) + C$ ✓

(b) (i) $y = (\log_e 6x)^3$

$\frac{dy}{dx} = 3(\log_e 6x)^2 \times \frac{\cancel{6}}{\cancel{6}x}$

$= \frac{3(\log_e 6x)^2}{\cancel{2}x}$

(ii) $\int_3^5 \frac{(\log_e 6x)^2}{x}\,dx$

$= \cancel{\int_3^5 \frac{(\log_e 6x)^3}{3}}\,dx$

$= \frac{1}{3}\left[(\log_e 6x)^3\right]_3^5$

$= \frac{1}{3}\left[(\log_e 30)^3 - (\log_e 18)^3\right]$

$= \frac{1}{3} \times 15.199\ldots$

$= \cancel{30.4}$ ~~...~~ 5.1 (to 1dp).

(c) $y = 6^{2x}$

$= e^{\ln 6^{2x}}$ ✓

$= e^{2x\ln 6}$ ✓

$\frac{dy}{dx} = 2\ln 6 \cdot 6^{2x}$

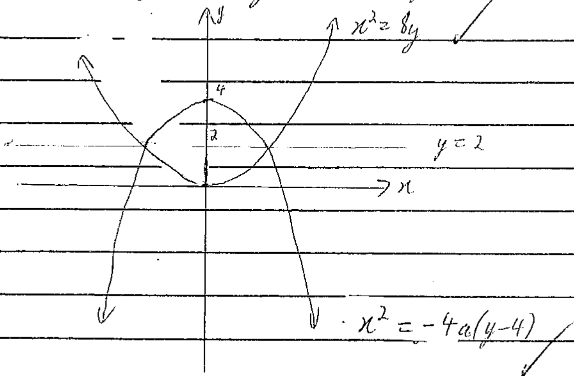
$= 6^{2x} \cdot 2\ln 6$ ✓

(d) $x^2 = 8y$

$4a = 8 \qquad a = 2$

(i) latus rectum: $y = 2$ ✓

So $x^2 = 8y$ is reflected on $y = 2$



Thus the equation of the new parabola is $x^2 = -4a.(y-4)$

$x^2 = -8(y-4)$

(ii) The x value is the same, so subbing into the new equation:

$12^2 = -8y + 32$ ✓

$144 - 32 = -8y$

$y = -14$

∴ the new coordinates are $(12, -14)$

(e)

$y = \log_e(x-3)$

$e^y = x - 3$

$x = e^y + 3$ ✓

$A = 7.\log_e 4 - \int_0^{\log_e 4} x . dy$

$= 7\log_e 4 - \int_0^{\log_e 4} e^y + 3\, dy$ ✓

$= 7\log_e 4 - \left[e^y + 3y \right]_0^{\log_e 4}$

$= 7\log_e 4 - \left[4 + 3\log_e 4 - 1 \right]$ ✓

$= \log_e 4^7 - \log_e 4^3 - 3$

$= \log_e 256 - 3$ units^2 ✓