SYDNEY GIRLS HIGH SCHOOL

2008

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Diagrams are NOT drawn to scale.
- All necessary working should be shown in every question.
- Start each question on a new page.

Total marks – 84

- Attempt Questions 1 – 7.
- All questions are of equal value.

This is a trial paper ONLY. It does not necessarily reflect the format or contents of the 2008 HSC Examination in this subject.
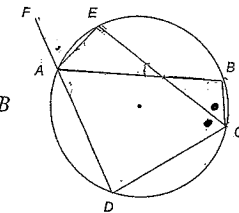

Question 1:

a) Solve for x if $\dfrac{3x}{x-1} < 2$ **[3]**

b) Find the exact value of $\displaystyle\int_0^1 xe^{-x^2}.dx$ **[2]**

c) Find $\dfrac{d}{dx}\left(\sin^{-1}\sqrt{x}\right)$ **[2]**

d) If A is the point (3,-2) and B is (-3,1) find the point P that divides AB externally in the ratio 2:1. **[2]**

e) If the limiting sum of a geometric progression is 5 and the first term is 3, find the common ratio. **[3]**

Question 2:

a) Sketch the curve $y = 3\sin^{-1} 2x$ **[2]**

b) If $2\tan\theta = 3$ and θ is acute, find the exact value of $\dfrac{3\sin\theta + 2\cos\theta}{\sec\theta - \operatorname{cosec}\theta}$ **[3]**

c) The roots of $x^3 - 3x^2 + 6x - 5 = 0$ are α, β and γ. Find the value of

 i) $(\alpha+1)+(\beta+1)+(\gamma+1)$ **[2]**

 ii) $(\alpha+1)(\beta+1)(\gamma+1)$ **[2]**

d) In the diagram, DA is extended to F and EC bisects $\angle BCD$. Prove that AE bisects $\angle FAB$ **[3]**

Question 3:

a) Find the constant term in the expansion $\left(\dfrac{2}{x} - \dfrac{x^2}{3}\right)^{12}$ [3]

b) Find the value of k if $\displaystyle\int_0^k \sqrt{1+x}\,.dx = \dfrac{14}{3}$ [3]

c) Given that $x = 2$ is an approximation to $x^3 - 9 = 0$, use one application of Newton's Method to find a better approximation.   [3]

d) Use induction to show that: $1 \times 3 \times 5 \times \text{.......x } (2n-1) = \dfrac{(2n)!}{2^n . n!}$ [3]
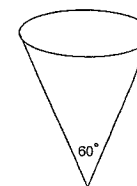
Question 4:

a)   Show that $7^n + 2$ is divisible by three for all positive integral n.   [3]

b)   Consider the polynomial $P(x) = x^3 - (k+1)x^2 + kx + 12$

     i) Find the remainder when $P(x)$ is divided by $A(x) = x - 3$ [2]

     ii) Find k if $P(x)$ is divisible by $A(x)$ [2]

     iii) For this value of k, find the zeros of $P(x)$ [2]

c)   Find $\displaystyle\int \dfrac{x.dx}{\left(25+x^2\right)^{3/2}}$ using the substitution $u = 25 + x^2$ [3]

Question 5:

a) A projectile is fired at an angle of $\tan^{-1}\left(\dfrac{5}{12}\right)$ to the horizontal with an initial velocity of $130ms^{-1}$. Using $g = 10ms^{-2}$,

     i) Derive the equations for the horizontal and vertical position of the projectile at time t.   [2]

     ii) Determine the horizontal range of the projectile.   [3]

b) A right circular cone with vertical angle $60°$ is being filled with a liquid. The liquid is being poured into the cone at a rate of $3\pi cm^3.s^{-1}$.

     i) Show that the volume of the cone is given by $V = \dfrac{\pi h^3}{9}$, where h is the height of the liquid   [1]

     ii) Find the rate the height of the liquid is increasing when that height is 9 cm.   [4]

c) Find a general solution to $\cos 2x = \sin x$ [2]

Question 6:

a) A particle is moving in a straight line with an initial velocity of $48ms^{-1}$.

At any time t, the acceleration in ms^{-2} is given by $\ddot{x} = -6t$. Find

i) The distance from its initial position when it stops

[2]

ii) The velocity at which it is travelling when it returns to its initial position. $\ddot{x}t$

[2]

b) A particle moving on a horizontal line has a velocity $v(ms^{-1})$, given by
$$v^2 = 64 + 24x - 4x^2$$

i) Prove the particle moves in simple harmonic motion

[1]

ii) Find the centre of motion

[1]

iii) Write down the period and amplitude

[2]

iv) If initially, the particle is at the centre of motion and is moving to the left, write down an expression for the displacement as a function of time.

[1]

c) i) Write the expression $\sqrt{3}\cos\theta + \sin\theta$ in the form $R\cos(\theta - \alpha)$

[1]

ii) Hence or otherwise solve $\sqrt{3}\cos\theta + \sin\theta = 1$, for $0° \le \theta \le 360°$.

[2]

Question 7:

a) The maximum charge of a battery that requires recharging is 12 volts. After 1 minute on the recharger, the charge is measured as 6 volts and after a further minute the charge has risen to 8 volts. Assuming the rate of increase of the charge is given by $\dfrac{dQ}{dt} = k(12 - Q)$,

i) Show this equivalent to $Q = 12 - Ae^{-kt}$

[2]

ii) Find the value of the charge on the battery at the time it is placed on the recharger.

[2]

iii) Find the exact value of the battery after it has been on the recharger for 3 minutes.

[3]

b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. If PQ passes through the point $(4a, 0)$

i) Show that $pq = 2(p + q)$

[2]

ii) Find the locus of M, the midpoint of PQ.

[3]

Question 1:

a) Solve for x if $\dfrac{3x}{x-1} < 2$ [3]

Solution:

$$\frac{3x}{x-1} \cdot \frac{(x-1)^2}{1} < 2 \cdot \frac{(x-1)^2}{1}, \quad x \neq 1$$

$$\therefore 3x(x-1) < 2(x-1)^2$$

or $2(x-1)^2 > 3x(x-1)$

$$\therefore 2(x-1)^2 - 3x(x-1) > 0$$

$$\therefore (x-1)(2x-2-3x) > 0$$

$$\therefore (x-1)(-x-2) > 0$$

or $(x-1)(x+2) < 0$

$$\therefore -2 < x < 1$$

b) Find the exact value of $\displaystyle\int_0^1 xe^{-x^2}\,dx$ [2]

Solution:

$$\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}\int_0^1 -2xe^{-x^2}\,dx$$

$$= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$$

$$= \frac{1}{2}\left[e^{-x^2}\right]_1^0$$

$$= \frac{1}{2}\left[1 - e^{-1}\right] = \frac{1}{2}\left[1 - \frac{1}{e}\right]$$

c) Find $\dfrac{d}{dx}\left(\sin^{-1}\sqrt{x}\right)$ [2]

Solution:

$$\frac{d}{dx}\left(\sin^{-1}\sqrt{x}\right) = \frac{1}{2\sqrt{x}} \cdot \frac{1}{\sqrt{1-x}}$$

$$= \frac{1}{2\sqrt{x(1-x)}}$$

d) If A is the point (3,-2) and B is (-3,1) find the point P that divides AB externally in the ratio 2:1. [2]

Solution:

$$P(x,y) = \left(\frac{k_1 x_2 + k_2 x_1}{k_1 + k_2}, \frac{k_1 y_2 + k_2 y_1}{k_1 + k_2}\right)$$

$$= \left(\frac{2(-3) - 1(3)}{2-1}, \frac{2(1) - 1(-2)}{2-1}\right)$$

$$= \left(\frac{-6-3}{1}, \frac{2+2}{1}\right)$$

$$= (-9, 4)$$

e) If the limiting sum of a geometric progression is 5 and the first term is 3, find the common ratio. [3]

Solution:

$$S_\infty = \frac{a}{1-r}$$
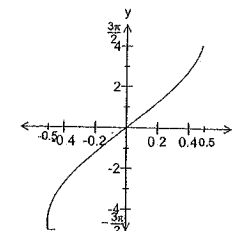
$$5 = \frac{3}{1-r}$$

$$5 - 5r = 3$$

$$2 = 5r$$

$$\therefore r = \frac{2}{5}$$

Question 2:

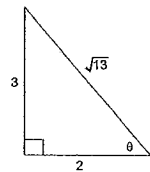**a)** Sketch the curve $y = 3\sin^{-1} 2x$ [2]

Solution:

b) If $2\tan\theta = 3$ and θ is acute, find the exact value of $\dfrac{3\sin\theta + 2\cos\theta}{\sec\theta - \csc\theta}$ *[3]*

Solution:

$$\frac{3\sin\theta + 2\cos\theta}{\sec\theta - \csc\theta} = \frac{3\left(\dfrac{3}{\sqrt{13}}\right) + 2\left(\dfrac{2}{\sqrt{13}}\right)}{\dfrac{\sqrt{13}}{2} - \dfrac{\sqrt{13}}{3}}$$

$$= \frac{\dfrac{9}{\sqrt{13}} + \dfrac{4}{\sqrt{13}}}{\dfrac{3\sqrt{13}}{6} - \dfrac{2\sqrt{13}}{6}}$$

$$= \frac{13}{\sqrt{13}} \div \frac{\sqrt{13}}{6} = 6$$

(right triangle with hypotenuse $\sqrt{13}$, opposite side 3, adjacent side 2, angle θ)

c) The roots of $x^3 - 3x^2 + 6x - 5 = 0$ are α, β and γ. Find the value of

  i) $(\alpha+1) + (\beta+1) + (\gamma+1)$ *[2]*

Solution: $\alpha + \beta + \gamma = -\dfrac{b}{a} = 3$

$$\therefore (\alpha+1) + (\beta+1) + (\gamma+1) = 3 + 3 = 6$$

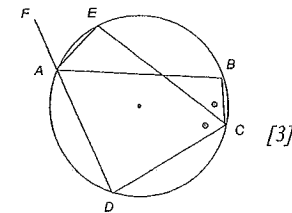 **ii)** $(\alpha+1)(\beta+1)(\gamma+1)$ *[2]*

Solution:

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 6$$

$$\alpha\beta\gamma = -\frac{d}{a} = 5$$

$$(\alpha+1)(\beta+1)(\gamma+1) = \alpha\beta\lambda + \alpha\beta + \beta\gamma + \gamma\alpha + \alpha + \beta + \gamma + 1$$
$$= 5 + 6 + 3 + 1 = 15$$

e) In the diagram, DA is extended to F and EC bisects $\angle BCD$. Prove that AE bisects $\angle FAB$ *[3]*

Solution:

$\angle ECD = \angle EAF$ (exterior angle of cyclic quad)

$\angle ECB = \angle EAB$ (angles on the same arc)

Since $\angle ECB = \angle ECD$ (given), $\angle EAB = \angle EAF$

$\therefore$ AE bisects $\angle FAB$

Question 3:

a) Find the constant term in the expansion $\left(\dfrac{2}{x} - \dfrac{x^2}{3}\right)^{12}$ *[3]*

Solution:

$$U_{r+1} = {}^n C_r a^{n-r} b^r = {}^{12}C_r \left(\frac{2}{x}\right)^{12-r}\left(-\frac{x^2}{3}\right)^r$$

The x coefficient is $x^{r-12} x^{2r} = x^{3r-12}$, so $x = 4$

$$U_5 = {}^{12}C_4 \left(\frac{2}{x}\right)^8 \left(-\frac{x^2}{3}\right)^4 = \frac{495 \times 256}{81} = \frac{14080}{9}$$

b) Find the value of k if $\displaystyle\int_0^k \sqrt{1+x}\,dx = \dfrac{14}{3}$ *[3]*

Solution:

$$\left[\frac{2(1+x)^{3/2}}{3}\right]_0^k = \frac{14}{3}$$

$$\therefore \left[(1+x)^{3/2}\right]_0^k = 7$$

$$\therefore (1+k)^{3/2} - 1 = 7$$

$$(1+k)^{3/2} = 8$$

$$1 + k = 4 \quad \rightarrow \quad k = 3$$

c) Given that $x = 2$ is an approximation to $x^3 - 9 = 0$, use one application of Newton's Method to find a better approximation.

[3]

Solution:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$f(x) = x^3 - 9 \quad \rightarrow \quad f(2) = 8 - 9 = -1$$

$$f'(x) = 3x^2 \quad \rightarrow \quad f'(2) = 12$$

$$x_1 = 2 - \frac{f(3)}{f'(3)} = 2 - \frac{-1}{12} = 2\frac{1}{12}$$

d) Use induction to show that: $1 \times 3 \times 5 \times \ldots\ldots\times (2n-1) = \dfrac{(2n)!}{2^n . n!}$

[3]

Solution:

at $n = 1, LHS = 1$

$$RHS = \frac{2!}{2^1} = 1 = LHS \quad \rightarrow \quad \therefore \text{ true for } n = 1$$

assume true for $n = k \quad \rightarrow \quad 1 \times 3 \times 5 \times \ldots\ldots\times (2k-1) = \dfrac{(2k)!}{2^k . k!}$

prove for $n = k+1$, $i.e.$ prove $1 \times 3 \times 5 \times \ldots\ldots\times (2k-1)(2k+1) = \dfrac{(2k+2)!}{2^{k+1} . (k+1)!}$

$$LHS = 1 \times 3 \times 5 \times \ldots\ldots\times (2k-1)(2k+1)$$

$$= \frac{(2k)!}{2^k . k!}(2k+1) \qquad \text{[using assumption]}$$

$$= \frac{(2k)!}{2^k . k!}\frac{(2k+1)}{1}\left(\frac{2k+2}{2k+2}\right)$$

$$= \frac{(2k)!}{2^k . k!}\frac{(2k+1)(2k+2)}{2(k+1)}$$

$$= \frac{(2k+2)!}{2^{k+1} . (k+1)!}$$

$$= RHS$$

$\therefore$ if true for $n = k$, it is true for $n = k+1$.
Since true for $n = 1$, it is true for $n = 2$, then $n = 3$ and so on for all positive integral values of n.

Question 4:

a)   Show that $7^n + 2$ is divisible by three for all positive integral n.

[3]

Solution:

at $n = 1$, $7^1 + 2 = 9 = 3 \times 3$, so is true for $n = 1$

assume true for $n = k \quad \rightarrow \quad 7^k + 2 = 3M$ (M an integer)

prove for $n = k+1$

$$7^{k+1} + 2 = 7 \times 7^k + 2$$

$$= 7 \times (3M - 2) + 2$$

$$= 21M - 14 + 2$$

$$= 21M - 12$$

$$= 3(7M - 4)$$

$\therefore$ if true for $n = k$, it is true for $n = k+1$.
Since true for $n = 1$, it is true for $n = 2$, then $n = 3$ and so on for all positive integral values of n.

b)   Consider the polynomial $P(x) = x^3 - (k+1)x^2 + kx + 12$

     i) Find the remainder when $P(x)$ is divided by $A(x) = x - 3$ **[2]**

Solution:

$$P(3) = 27 - 9k - 9 + 3k + 12$$

$$= 30 - 6k$$

     ii) Find k if $P(x)$ is divisible by $A(x)$ **[2]**

Solution:

If $P(3) = 0, 30 - 6k = 0$

and $k = 5$

     iii) For this value of k, find the zeros of $P(x)$ **[2]**

Solution:

$$P(x) = x^3 - 6x^2 + 5x + 12$$

$$= (x-3)(ax^2 + bx + c) \quad \text{as } (x-3) \text{ is a factor}$$

$$= (x-3)(x^2 - 3x - 4)$$

$$= (x-3)(x-4)(x+1)$$

$$\therefore x = -1, 3, 4$$

c) Find $\int \dfrac{x \, dx}{\left(25+x^2\right)^{3/2}}$ using the substitution $u = 25 + x^2$

[3]

Solution:

$$\text{let } u = 25 + x^2,$$
$$\therefore du = 2x \, dx$$
$$I = \frac{1}{2}\int \frac{du}{u^{3/2}} = \frac{1}{2}\int u^{-3/2} \, du$$
$$= \frac{1}{2} \cdot 2u^{-1/2} + c = \frac{1}{\sqrt{u}} + c$$
$$= \frac{1}{\sqrt{25+x^2}} + c$$

Question 5:

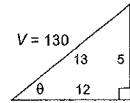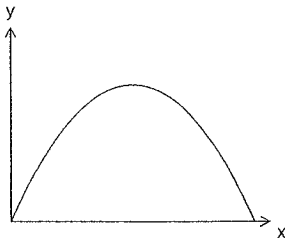a) A projectile is fired at an angle of $\tan^{-1}\left(\dfrac{5}{12}\right)$ to the horizontal with an initial velocity of $130 ms^{-1}$. Using $g = 10 ms^{-2}$,

i) Derive the equations for the horizontal and vertical position of the projectile at time t.

[2]

ii) Determine the horizontal range of the projectile.

[3]

Solution:

$$\ddot{x} = 0, \qquad \ddot{y} = -10$$
$$\dot{x} = c_1, \qquad \dot{y} = -10t + c_2$$

at $t = 0, \dot{x} = V\cos\theta, \qquad \dot{y} = V\sin\theta$
$$= 130\left(\frac{12}{13}\right), \qquad = 130\left(\frac{5}{13}\right)$$
$$= 120 = c_1, \qquad = 50 = c_2$$
$$\therefore \dot{x} = 120, \qquad \dot{y} = -10t + 50$$
$$\therefore x = 120t + c_3, \qquad y = -5t^2 + 50t + c_4$$
at $t = 0, x = 0 = c_3, y = 0 = c_4$
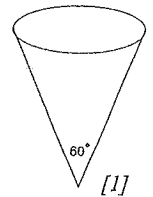$$\therefore x = 120t, \qquad y = -5t^2 + 50t$$

ii) at $y = 0, \; t = 0,10$
range at $t = 10$
sub into $x = 120t$ and range is 600 metres

b) A right circular cone with vertical angle $60°$ is being filled with a liquid. The liquid is being poured into the cone at a rate of $3\pi cm^3.s^{-1}$.

i) Show that the volume of the cone is given by $V = \dfrac{\pi h^3}{9}$, where h is the height of the liquid

[1]

ii) Find the rate the height of the liquid is increasing when that height is $9 \; cm$.

[4]
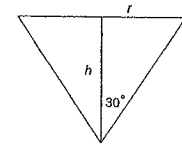
Solution:

i) $V = \dfrac{1}{3}\pi r^2 h \qquad \tan 30^0 = \dfrac{r}{h} \to \dfrac{1}{\sqrt{3}} = \dfrac{r}{h}$ and $r = \dfrac{h}{\sqrt{3}}$

$$\therefore V = \frac{\pi}{3}\left(\frac{h}{\sqrt{3}}\right)^2 (h) = \frac{\pi h^3}{9}$$

ii) $\dfrac{dV}{dt} = \dfrac{dV}{dh} \cdot \dfrac{dh}{dt} \qquad$ and $\qquad \dfrac{dV}{dh} = \dfrac{\pi h^2}{3}$

$$\therefore \frac{dV}{dt} = \frac{\pi h^2}{3} \cdot \frac{dh}{dt}$$

$$\therefore 3\pi = \frac{\pi(81)}{3} \cdot \frac{dh}{dt} \quad \to \quad \frac{dh}{dt} = \frac{1}{9} \, cm.\sec^{-1}$$

c) Find a general solution to $\cos 2x = \sin x$

[2]

Solution:

$$\therefore 1 - 2\sin^2 x = \sin x$$
$$\therefore 2\sin^2 x + \sin x - 1 = 0$$
$$\therefore (2\sin x - 1)(\sin x + 1) = 0$$
$$\therefore \sin x = \frac{1}{2} \text{ or } \sin x = -1$$
$$\therefore x = n\pi + (-1)^n \left(\frac{\pi}{6}\right) \text{ or } \frac{\pi}{2}(4n+3)$$

Note: This may also be answered in degrees .

Question 6:

a) A particle is moving in a straight line with an initial velocity of $48ms^{-1}$.

At any time t, the acceleration in ms^{-2} is given by $\ddot{x} = -6t$. Find

i) The distance from its initial position when it stops

[2]

ii) The velocity as which it is travelling when it returns to its initial position.

[2]

Solution:

$$\ddot{x} = -6t$$
$$\dot{x} = -3t^2 + c$$
at $t = 0, \dot{x} = 48 = c$
$$\therefore \dot{x} = -3t^2 + 48$$
$$x = -t^3 + 48t + c_2$$
at $t = 0, x = 0 = c_2$
$$\therefore x = -t^3 + 48t$$
at $\dot{x} = 0, 3(-t^2 + 16) = 0 \rightarrow t = 4$
at $t = 4, x = -4^3 + 48 \times 4 = 128$ Distance is $128m$
at $x = 0, -t^3 + 48t = 0 \rightarrow t = \sqrt{48} = 4\sqrt{3}$
at $t = 4\sqrt{3}, v = -3\left(4\sqrt{3}\right)^2 + 48 = -96$ Velocity is $-96ms^{-1}$

b) A particle moving on a horizontal line has a velocity $v(ms^{-1})$, given by
$$v^2 = 64 + 24x - 4x^2$$

i) Prove the particle moves in simple harmonic motion

[1]

ii) Find the centre of motion

[1]

iii) Write down the period and amplitude

[2]

iv) If initially, the particle is at the centre of motion and is moving to the left, write down an expression for the displacement as a function of time.

[1]

Solution:

i) $\frac{1}{2}v^2 = 32 + 12x - 2x^2$

$\ddot{x} = \frac{d}{dx}\left(\frac{v^2}{2}\right) = 12 - 4x = -4(x-3)$ which is the form of SHM

ii) At $\ddot{x} = 0, x = 3$

iii) $n^2 = 4 \rightarrow n = 2, \therefore T = \frac{2\pi}{2} = \pi$

At $v = 0, 4x^2 - 24x - 64 = 0 \rightarrow 4(x-8)(x+2) = 0$

Particle moves from -2 to 8, so amplitude is 5

iv) $x = 3 - 5\sin 2t$

c)   i) Write the expression $\sqrt{3}\cos\theta + \sin\theta$ in the form $R\cos(\theta - \alpha)$

[1]

ii) Hence or otherwise solve $\sqrt{3}\cos\theta + \sin\theta = 1$, for $0° \leq \theta \leq 360°$.

[2]

Solution:

i) $R\cos(\theta - \alpha) = R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$\sqrt{3}\cos\theta + \sin\theta = 2\left(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right)$, $\therefore R = 2, \theta = \frac{\pi}{6}$

$\therefore R\cos(\theta - \alpha) = 2\cos\left(\theta - \frac{\pi}{6}\right)$

ii) $2\cos\left(\theta - \frac{\pi}{6}\right) = 1 \rightarrow \cos\left(\theta - \frac{\pi}{6}\right) = \frac{1}{2}$

$\therefore \theta - \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3} \rightarrow \theta = \frac{\pi}{2}, \frac{11\pi}{6}$

Question 7:

a) The maximum charge of a battery that requires recharging is 12 volts. After 1 minute on the recharger, the charge is measured as 6 volts and after a further minute the charge has risen to 8 volts. Assuming the rate of increase of the charge is given by $\frac{dQ}{dt} = k(12 - Q)$,

    i) Show this equivalent to $Q = 12 - Ae^{-kt}$

                         [2]

    ii) Find the value of the charge on the battery at the time it is placed on the recharger.

                         [2]

    iii) Find the exact value of the battery after it has been on the recharger for 3 minutes.
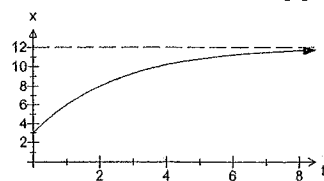
                         [3]

Solution:

i) $Q = 12 - Ae^{-kt}$

$\frac{dQ}{dt} = k(Ae^{-kt})$

and since $Ae^{-kt} = 12 - Q$, $\frac{dQ}{dt} = k(12 - Q)$

ii) $Q = 12 - Ae^{-kt}$

at $t = 1, Q = 6 \rightarrow \therefore 6 = 12 - Ae^{-k}$ and $Ae^{-k} = 6$ or $A = 6e^{k}$

at $t = 2, Q = 8 \rightarrow \therefore 8 = 12 - Ae^{-2k}$ and $Ae^{-2k} = 4$ or $A = 4e^{2k}$

Equating for A, $6e^{k} = 4e^{2k} \rightarrow \frac{3}{2} = 3^{k}$

Since $A = 6e^{k}$, $A = 6(\frac{3}{2}) = 9$ and $Q = 12 - 9e^{-kt}$

At $t = 0, Q = 12 - 9e^{0} = 3$

iii) At $t = 3, Q = 12 - 9e^{-3k}$

        $= 12 - 9(e^{-k})^{3}$

Since $e^{k} = \frac{3}{2}, e^{-k} = \frac{2}{3}$

and $Q = 12 - 9(\frac{2}{3})^{3} = 9\frac{1}{3}$

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b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. If PQ passes through the point $(4a, 0)$

    i) Show that $pq = 2(p + q)$

                         [2]

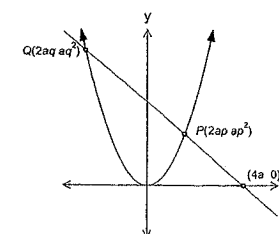   ii) Find the locus of M, the midpoint of PQ.

                         [3]

Solution:

i) $m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq} = \frac{a(p-q)(p+q)}{2a(p-q)} = \frac{p+q}{2}$

$\therefore$ Eqn PQ is $y - ap^2 = \frac{p+q}{2}(x - 2ap)$

$y - ap^2 = \left(\frac{p+q}{2}\right)x - ap^2 - apq \rightarrow y = \left(\frac{p+q}{2}\right)x - apq$

at $(4a, 0)$, $0 = \left(\frac{p+q}{2}\right)4a - apq \rightarrow pq = 2(p+q)$

ii) Midpoint PQ is $a(p+q), \frac{a}{2}(p^2 + q^2)$

Let $x = a(p+q)$, $y = \frac{a}{2}(p^2 + q^2)$

$\frac{x}{a} = p + q \rightarrow \frac{2x}{a} = 2(p+q)$, $\therefore \frac{2x}{a} = pq$

Write y as $y = \frac{a}{2}\{(p+q)^2 - 2pq\}$

$\therefore y = \frac{a}{2}\left\{\left(\frac{x}{a}\right)^2 - 2\left(\frac{2x}{a}\right)\right\}$

$= \frac{a}{2}\left\{\frac{x^2}{a^2} - \frac{4x}{a}\right\}$

$= \frac{x^2}{2a} - 2x$

$\therefore 2ay = x^2 - 4ax$

or $2ay + 4a^2 = x^2 - 4ax + 4a^2 \rightarrow (x - 2a)^2 = 2a(y + 2a)$