Sydney Girls High School

2002
TRIAL HIGHER SCHOOL
CERTIFICATE EXAMINATION

Mathematics

Extension 2

This is a trial paper ONLY. It does not necessarily reflect the format or the contents of the 2002 HSC Examination Paper in this subject.

Candidate Number

General Instructions

- Reading Time - 5 mins
- Working time - 3 hours
- Attempt ALL questions
- ALL questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used.
- Diagrams are not to scale
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

Question One (15 marks)

1. Find $\int \frac{1}{x}(1+\log_e x)^4 \, dx$ [2]

2. Express $\frac{1}{x^2+3x-4}$ in the form $\frac{A}{x+4}+\frac{B}{x-1}$, hence find $\int \frac{dx}{x^2+3x-4}$ [3]

3. Find $\int \sin^3\theta \cos^2\theta \, d\theta$ [3]

4. Find $\int \frac{2x+5}{x^2+4x+5} \, dx$ [3]

5. Find $\int e^{2x} \sin 3x \, dx$ [4]

Question Two (15 marks)

1. Given $z = 2-3i$ [3]
   a) Find $\frac{1}{z}$
   b) Find iz
   c) Give a geometrical interpretation of your answer to part b)

2. Find real number x and y such that $3x+2iy-ix+5y=7+5i$ [2]

3. Given $z = 4+4\sqrt{3}i$ [3]
   a) Find $|z|$ and arg z
   b) Hence evaluate $\left(4+4\sqrt{3}i\right)^9$

4. Sketch the locus given by $\arg\left(\frac{z+i}{z-i}\right)=\frac{\pi}{2}$ [2]

5. Find the Cartesian equation of the curve represented by $\frac{(z+\overline{z})}{2}=|z|-2$ [2]

   and describe it.

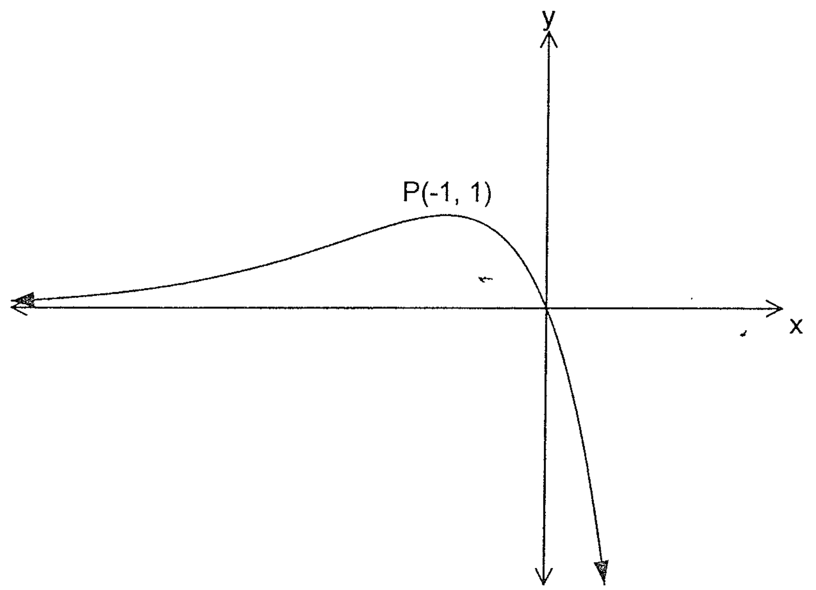
6. a) Solve the equation $z^3-1=0$ giving your answers in modulus-argument form. [3]

   b) Let w be the root of $z^3-1=0$ with the smallest positive argument
   i) Show $1+w+w^2=0$
   ii) Simplify $\left(1+w^2\right)^4$

Question Three (15 marks)

1. Given $y = f(x) = (x-2)^2(x+1)$ sketch without using calculus [5]
   a) $y = f(x)$
   b) $y = \dfrac{1}{f(x)}$
   c) $y^2 = f(x)$

2. The graph of $y = F(x)$ is shown below. The graph has a maximum turning point at P (-1, 1). [7]



Sketch on separate diagrams showing all relevant features
   a) $y = F(-x)$
   b) $y = \log_e[F(x)]$
   c) $y = e^{F(x)}$
   d) $y = [F(x)]^2$

3. Sketch the graph of $x^3 + y^3 - 1 = 0$ [3]

Question Four (15 marks)

1. The equation of a conic is $\frac{x^2}{4}+\frac{y^2}{3}=1$ find [4]
    a) The eccentricity
    b) The co-ordinates of the foci
    c) The equations of the directrices
    d) Sketch the conic showing vertices foci and directrices.

2. Find the equation of the chord of contact of the tangents to the hyperbola $x^2-16y^2=16$ from the point with coordinates (2, -4) [2]

3. Find the equation of the hyperbola with foci at $(\pm 5,0)$ and eccentricity $e=\frac{5}{4}$ [3]

4. The point $P(5\cos\theta, 3\sin\theta)$ lies on the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$. [6]
    The normal at P crosses the x-axis at Q and the y-axis at R.
    a) Derive the equation of the normal at P
    b) Show that the midpoint M of QR lies on the ellipse with equation $\frac{25x^2}{64}+\frac{9y^2}{64}=1$

Question Five (15 marks)

1. a) Sketch the graph of $g(x)=1+\dfrac{1}{x+1}$ for $x>-1$ [4]

   b) Find the equation of the inverse $g^{-1}(x)$ and sketch it on a separate set of axes.

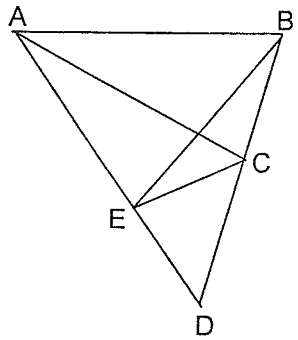
   c) Solve $g(x)=g^{-1}(x)$

2. If α,β and γ are the roots of the equation $x^3-2x^2+4x+2=0$, [2]
   Find the equation, which has roots

   a) $(\alpha-1)$, $(\beta-1)$ and $(\gamma-1)$

   b) $\dfrac{\alpha}{2}$, $\dfrac{\beta}{2}$ and $\dfrac{\gamma}{2}$

3. In the figure below $\angle AEB=\angle BCA$ [2]



   Prove $\angle BAE=\angle ECD$

4. Given that the equation $x^4-2x^3-12x^2+40x-32=0$ has a triple root, [3]
   find all the roots of this equation.

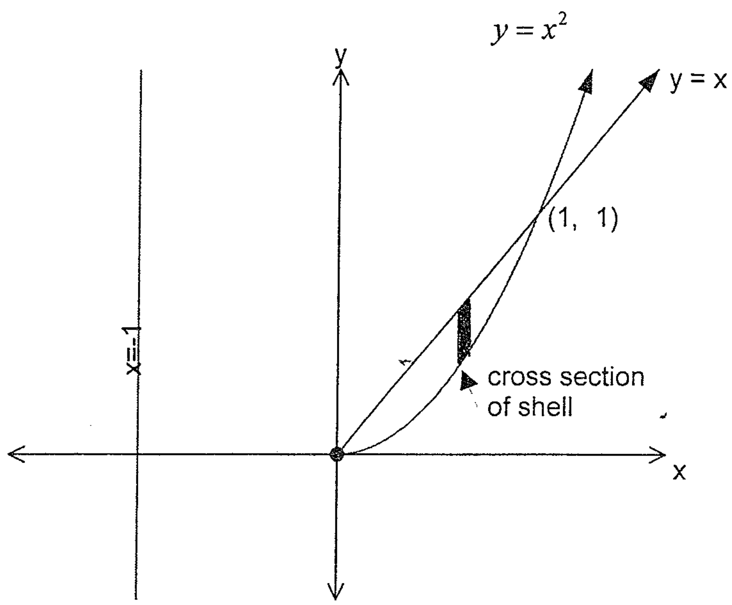
5. A solid has its base area bounded by $y=\sin x$, the x-axis, [4]
   $x=\dfrac{\pi}{4}$ and $x=\dfrac{3\pi}{4}$ Each cross section perpendicular to the x- axis is a square with one side on the base. Find the volume of the solid.

Question Six (15 marks)

1. $P(x)$ is an even monic polynomial of degree four with integer coefficients. One zero is $3i$ and the product of the zeros is -18. Factorise $P(x)$ fully over the real field. [3]

2. Prove that $\dfrac{\cos 15^0 + \cos 75^0}{\sin 15^0 - \sin 75^0} = -\sqrt{3}$ [3]

3. [6]

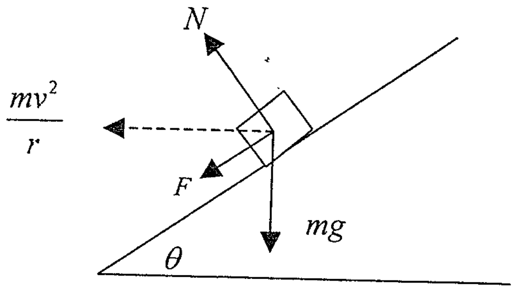


Use the method of <u>cylindrical shells</u> to calculate the volume of the solid formed when the area bounded by $y = x$ and $y = x^2$ is rotated about the line $x = -1$

4. Find and sketch the locus of z if $|w| = 1$ and $z = \dfrac{w+7}{1-w}$ [3]

Question Seven (15 marks)

1. A train line banked at an angle θ as shown below. [8]



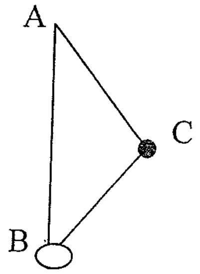
a) If the force of circular motion is given as $\frac{mv^2}{r}$, and the force due to gravity as mg, determine the components of frictional force F and the normal reaction N in terms of m, g, v, r and θ

b) A train turning a corner of radius 500 metres causes the same frictional force along the slope travelling at 30 km/h as it does travelling at 90km/h. (Note the two frictional forces are in different directions but are the same magnitude)

 i) Find the angle at which the track is banked (answer to the nearest minute)

 ii) Find the speed in km/h for which the frictional force is negligible (answer to the nearest km/h)

2. A four metre string attached at A has a 1kg mass attached at C and a 2kg metal ring attached at B. The ring at B is free to slide up and down the smooth vertical light rod descending from A. [7]



Given $AC = BC = 2$metres and $\angle BAC = 30^0$

a) Find the angular velocity of the mass at C, which is rotating about the rod AB so that the ring at B remains stationary.

b) If the mass at C is changed to a 3kg mass and retains the same angular velocity as in part a) find;

 i) The new size of $\angle BAC$ (answer to the nearest degree)

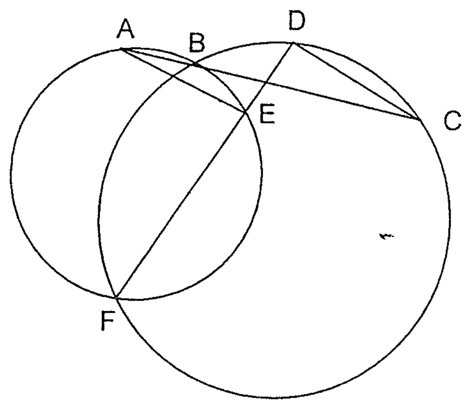
 ii) How far up the smooth rod the ring at B will rise before becoming stationary. (Answer to the nearest cm)

Question Eight (15 marks)

1. Use mathematical induction to prove that $x^{2n} - y^{2n}$ is divisible by $(x+y)$ for $n \geq 1$ (n an integer) [4]

2. If p and q are the roots of $\frac{1}{x} + \frac{1}{x+a} + \frac{1}{x+b} = 0$ and given that $a^2 + b^2 = 4ab$ prove that $p^2 + q^2 = 6pq$ [4]

3. In the figure below ABC and DEF are straight lines.
Prove AE parallel to DC [3]



4. Prove $|z-1| + |z+1| \leq 2\sqrt{2}$ given that $|z| \leq 1$ [4]

$\alpha\beta\chi\delta\varepsilon\phi\varphi\gamma\eta\iota\kappa\lambda\mu\nu o\pi\varpi\theta\vartheta\rho\sigma\varsigma\tau\upsilon\omega\xi\psi$

Soln's S.G.H.S. Extn 2 Trial 2002

Question One

1. $I=\int \frac{1}{x}(1+\log_e x)^4\,dx$

let $u = 1+\log_e x$

$du = \frac{dx}{x}$

$I = \int u^4\,du$

$= \frac{1}{5}u^5 + c$

$= \frac{1}{5}(1+\log_e x)^5 + c$ (2)

2. $\frac{1}{x^2+3x-4} = \frac{A}{x+4} + \frac{B}{x-1}$

$1 \equiv A(x-1) + B(x+4)$

put $x=1$, $B=\frac{1}{5}$

put $x=-4$, $A=-\frac{1}{5}$

$\int \frac{dx}{x^2+3x-4} = \frac{1}{5}\int\frac{dx}{x-1} - \frac{1}{5}\int\frac{dx}{x+4}$

$= \frac{1}{5}\log_e(x-1) - \frac{1}{5}\log_e(x+4) + c$

$= \frac{1}{5}\log_e\left(\frac{x-1}{x+4}\right) + c$ (3)

3. $I = \int \sin^3\theta\cos^2\theta\,d\theta$

$= \int (1-\cos^2\theta)(\cos^2\theta)\sin\theta\,d\theta$

put $u=\cos\theta$, $du = -\sin\theta\,d\theta$

$I = -\int (1-u^2)(u^2)\,du$

$= -\int u^2 - u^4\,du$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + c$

$= \frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta + c$ (3)

4. $I = \int \frac{2x+5}{x^2+4x+5}\,dx$

$= \int \frac{2x+4}{x^2+4x+5}\,dx + \int\frac{1}{x^2+4x+4+1}\,dx$

$= \int \frac{2x+4}{x^2+4x+5}\,dx + \int\frac{dx}{(x+2)^2+1}$

$= \log_e(x^2+4x+5) + \tan^{-1}(x+2) + C$ (3)

5. $I = \int e^{2x}\sin 3x\,dx$

let $u=\sin 3x$, $\dot{v}=e^{2x}$, $\dot{u}=3\cos 3x$, $v=\frac{1}{2}e^{2x}$

$I = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}I_1$

let $u=\cos 3x$, $\dot{v}=e^{2x}$, $\dot{u}=-3\sin 3x$, $v=\frac{1}{2}e^{2x}$

$I_1 = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx$

$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}I$

$\therefore I = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\left[\frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}I\right]$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}I$

$\frac{13}{4}I = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x$

$I = \frac{2}{13}e^{2x}\sin 3x - \frac{3}{13}e^{2x}\cos 3x$

$I = \frac{1}{13}(2e^{2x}\sin 3x - 3e^{2x}\cos 3x) + C$ (4)

Question Two

1. $z = 2-3i$

a) $\frac{1}{2-3i}\times\frac{2+3i}{2+3i} = \frac{1}{13}(2+3i)$

b) $iz = i(2-3i)$

$= 3+2i$ (3)

c) rotation of z through $90°$ anticlockwise

2. $3x+2iy-ix+5y = 7+5i$

$3x+5y+i(2y-x) = 7+5i$

equate real, imaginary

$3x+5y=7$ ①

$-x+2y=5$ ②

②×3 $-3x+6y=15$ ③

①+③ $11y=22$

$y=2$, $x=-1$ (2)

3. $z = 4+4\sqrt{3}i$

a) $|z| = \sqrt{16+48}$

$= 8$

b) $\arg z = \tan^{-1}\frac{4\sqrt{3}}{4}$

$= \tan^{-1}\sqrt{3}$

$= \frac{\pi}{3}$

b) $(4+4\sqrt{3}i)^9 = \left(8 \text{ cis } \frac{\pi}{3}\right)^9$

$= 2^{27} \text{ cis } 3\pi$

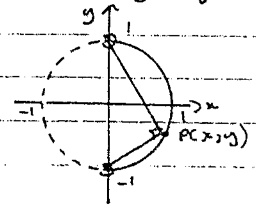
$= 2^{27}(-1)$

$= -2^{27}$

3

4. $\arg\left(\frac{z+i}{z-i}\right) = \frac{\pi}{2}$

ie $\arg(z+i) - \arg(z-i) = \frac{\pi}{2}$



2

5. $\frac{1}{2}[x+iy+x-iy] = \sqrt{x^2+y^2} - 2$

$x+2 = \sqrt{x^2+y^2}$

$\cancel{x^2} + 4x + 4 = \cancel{x^2} + y^2$

$y^2 = 4(x+1)$ ie parabola vertex $(-1,0)$

Major Axis x focus $(0,0)$

2

6. a) $z^3 = 1$

$= \text{cis } 0$

$z^3 = \text{cis }(2\pi k)$, $k = 0, 1, 2$

$z = \text{cis }\left(\frac{2\pi k}{3}\right)$

$z_0 = \text{cis } 0$, $z_1 = \text{cis } \frac{2\pi}{3}$, $z_2 = \text{cis } \frac{4\pi}{3}$

$= 1$ $\quad = w$ $\quad = \text{cis }\left(-\frac{2\pi}{3}\right)$

$= w^2$

b) i) $1 + w + w^2 =$ sum of roots

$= \frac{-0}{1} \quad \left(\frac{-b}{a}\right)$

$= 0$ [or sum G.P.]

ii) $(1+w^2)^4 = (-w)^4$

$= w^4$

$= w^3 \cdot w$

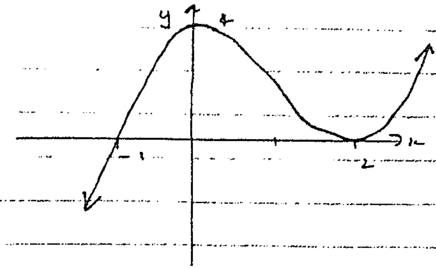
$= w$ since $w^3 = 1$

$z^3 = 1 \therefore w^3 = 1$

3

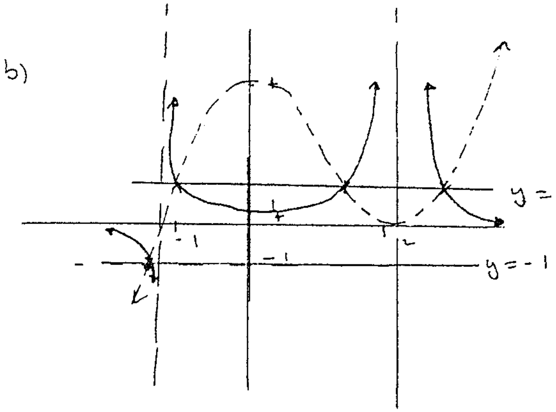
Question Three

1. a) $y = f(x) = (x-2)^2(x+1)$



1

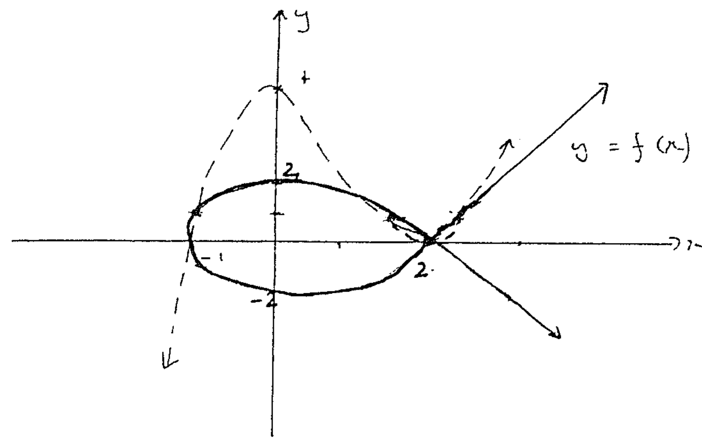
b)



$y = \frac{1}{f(x)}$

2

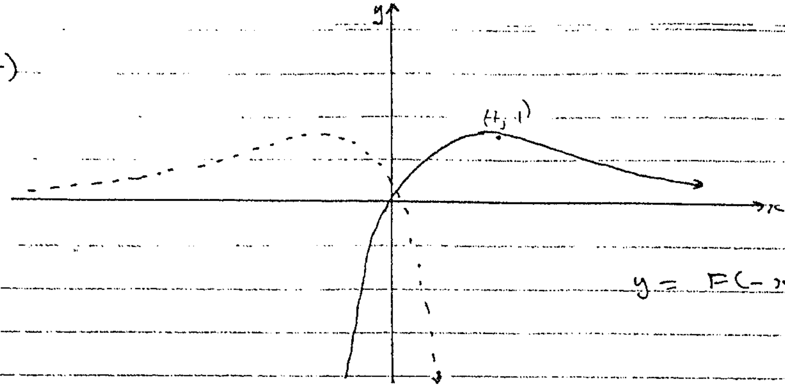
c)



$y^2 = f(x)$

2

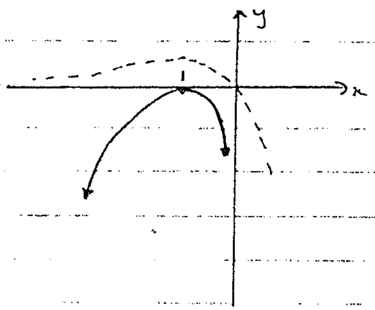
2. a)



$y = F(-x)$

1

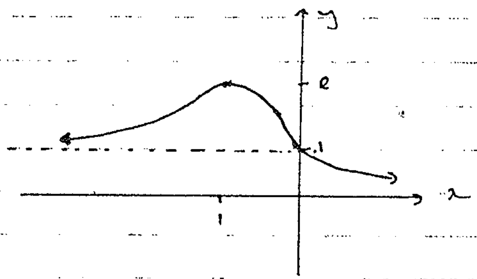
b)



$y = \log_e [F(x)]$

2

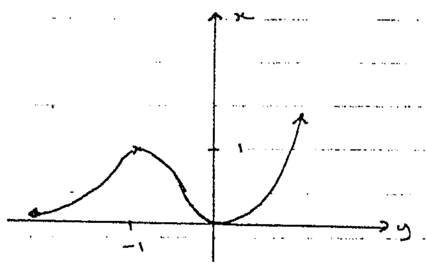
c)



$y = e^{F(x)}$

2

d)



$y = [F(x)]^2$

2

3. $x^3 + y^3 - 1 = 0$ Intercepts $x = 1$, $y = 1$

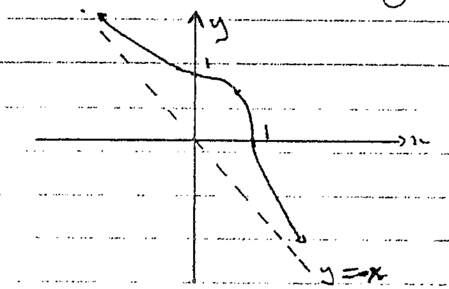
as $x \to \infty$ $y \to -x$. asymptote: $y = -x$

x and y can be interchanged $\therefore$ symmetrical $y = x$

Also, consider: $3x^2 + 3y^2 \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{x^2}{y^2}$ when $x = 0$ $\rightarrow$ horizontal tan

when $y = 0$ $\rightarrow$ vertical tan



3

Question Four

1. $\frac{x^2}{4} + \frac{y^2}{3} = 1$ $a = 2$, $b = \sqrt{3}$

a) $b^2 = a^2(1 - e^2)$

$3 = 4(1 - e^2)$

$3 = 4 - 4e^2$

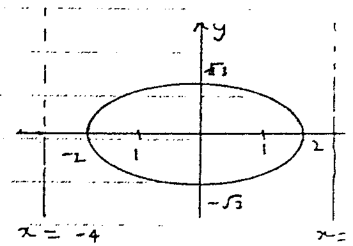
$-1 = -4e^2$ $e = \frac{1}{2}$

b) Foci $(\pm ae, 0)$

$(\pm 1, 0)$

c) Directrices $x = \pm \frac{a}{e}$

$= \pm 4$



2. $x^2 - 16y^2 = 16$

$\frac{x^2}{16} - y^2 = 1$ $(2, -4)$ $a^2 = 16$, $b^2 = 1$

$\frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$

$\frac{2x}{16} + \frac{4y}{1} = 1$

$2x + 64y = 16$

$x + 32y = 8$

3. Foci at $(\pm 5, 0)$, $e = 5/4$

$\therefore ae = 5$

$a \times \frac{5}{4} = 5 \quad \therefore a = 4$

and $b^2 = a^2(e^2 - 1)$

$= 16\left(\frac{25}{16} - 1\right)$

$b^2 = 9$

ie $\frac{x^2}{16} - \frac{y^2}{9} = 1$

(3)

4. a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

$\frac{2x}{25} + \frac{2y}{9}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-9x}{25y}$

at P $\quad m_1 = \frac{-45\cos\theta}{75\sin\theta}$

$= \frac{-3\cos\theta}{5\sin\theta}$

$\therefore$ m of normal $\quad m_2 = \frac{5\sin\theta}{3\cos\theta}$

Eqn. of Normal $\quad y - 3\sin\theta = \frac{5\sin\theta}{3\cos\theta}(x - 5\cos\theta)$

$3y\cos\theta - 9\sin\theta\cos\theta = 5x\sin\theta - 25\sin\theta\cos\theta$

ie $5x\sin\theta - 3y\cos\theta - 16\sin\theta\cos\theta = 0$.

b) at Q $\quad y = 0 \quad 5x\sin\theta - 16\sin\theta\cos\theta = 0$

$x = \frac{16\cos\theta}{5}$

ie $Q\left(\frac{16\cos\theta}{5}, 0\right)$

at R $\quad x = 0, \; -3y\cos\theta = 16\sin\theta\cos\theta$

$y = \frac{16\sin\theta}{-3}$

ie $R\left(0, \frac{16\sin\theta}{-3}\right)$

Then coords M $\left(\frac{8\cos\theta}{5}, \; \frac{-8\sin\theta}{3}\right)$

subst in $\frac{25x^2}{64} + \frac{9y^2}{64} = 1$

LHS $= \frac{25 \times 64\cos^2\theta}{64 \times 25} + \frac{9 \times 64\sin^2\theta}{64 \times 9}$

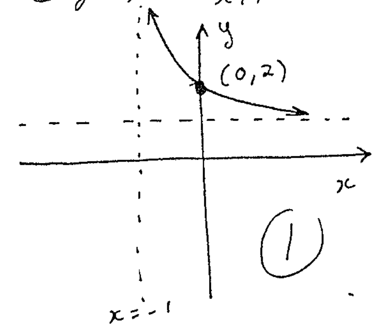
$= 1$

M lies on ellipse

(6)

Question 5

1. (a) $g(x) = 1 + \frac{1}{x+1} \quad x > -1$



(1)

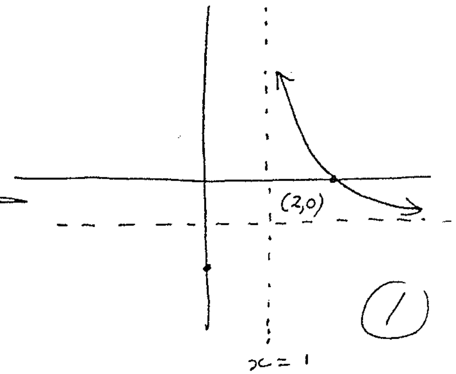
(b) $x = 1 + \frac{1}{y+1} \Rightarrow x - 1 = \frac{1}{y+1}$

$y + 1 = \frac{1}{x-1}$

$y = \frac{1}{x-1}$

(1)

$g^{-1}(x) = \frac{1}{x-1} - 1$



(1)

(c) $g(x) = g^{-1}(x)$.

$1 + \frac{1}{x+1} = \frac{1}{x-1} - 1$

$2 + \frac{1}{x+1} = \frac{1}{x-1}$

$\frac{2x+3}{x+1} = \frac{1}{x-1}$

$(2x+3)(x-1) = x + 1$

$2x^2 + 3x - 2x - 3 = x + 1$

$2x^2 = 4$

$x^2 = 2$

$x = \pm\sqrt{2}$

But $x > -1 \quad \therefore x = \sqrt{2}$

(1)

2. (a) $x^3 - 2x^2 + 4x + 2 = 0$. Roots α, β, γ.

Eqn $\alpha - 1, \beta - 1, \gamma - 1 \qquad y = x - 1 \qquad x = y + 1$

$(y+1)^3 - 2(y+1)^2 + 4(y+1) + 2 = 0$.

$y^3 + 3y^2 + 3y + 1 - 2y^2 - 4y - 2 + 4y + 4 + 2 = 0$.

$y^3 + y^2 + 3y + 5 = 0$

$\therefore x^3 + x^2 + 3x + 5 = 0$

(1)

(b) $y = \frac{\alpha}{2} \Rightarrow y = \frac{x}{2} \Rightarrow x = 2y$.

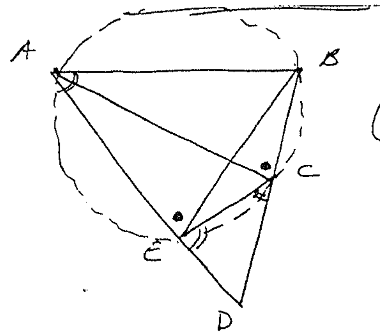
$\therefore (2y)^3 - 2(2y)^2 + 4(2y) + 2 = 0$

$8y^3 - 8y^2 + 8y + 2 = 0$.

$\therefore$ eqn $\quad 4x^3 - 4x^2 + 4x + 1 = 0$

(1)

3.



$ABCE$ lie on a circle
(1) (arc AB supports equal $\angle$'s at the circumference)
$\angle AEB = \angle BCA$ (data)
$\therefore \angle BAE = \angle ECD$ (exterior angle of a cyclic quad)
(1)

---

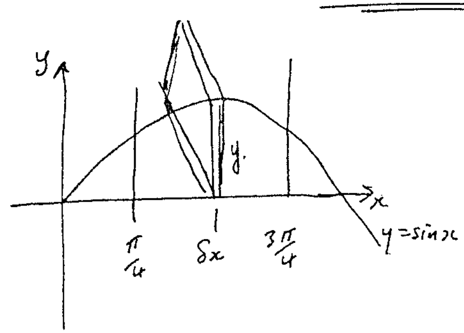
4. $x^4 - 2x^3 - 12x^2 + 40x - 32 = 0$ (triple root)
$4x^3 - 6x^2 - 24x + 40 = 0$ (double root)
$12x^2 - 12x - 24 = 0$ (single root).
(1) $x^2 - x - 2 = 0.$
$(x+1)(x-2) = 0.$

Test $x = -1$ $\quad P(-1) = (-1)^4 - 2(-1)^3 - 12(-1)^2 + 40(-1) - 32 \neq 0$
$\therefore x = -1$ not a triple root.

(1) Test $x = +2$ $\quad P(2) = 2^4 - 2.(2)^3 - 12.(2^2) + 40(2) - 32 = 0$
$\therefore x = 2$ is triple root.

$\therefore P(x) = (x-2)^3 Q(x) = x^4 - 2x^3 - 12x^2 + 40x - 32.$
$= (x-2)^3(x+a)$ and $-8a = -32$ $\therefore \underline{a = 4}$
(1) $\therefore P(x) = \underline{(x-2)^3(x+4)}$ Roots $x = -4, 2, 2, 2$

5.



Area of a slice is y^2
Volume of a slice
$\delta V = y^2 . \delta x .$ (1)
Volume of the solid
$V \doteqdot \lim_{\delta x} \sum_{\frac{\pi}{4}}^{\frac{3\pi}{4}} y^2 . \delta x$ (1)

$y = \sin x$
$y^2 = \sin^2 x .$

$V = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} y^2 . dx .$

$\sin^2 x = \frac{1}{2}\{1 - \cos 2x\}$ (1)

$\therefore V = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (1 - \cos 2x) . dx .$

$V = \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{1}{2}\left[\frac{3\pi}{4} - \frac{1}{2}\times -1\right] - \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\times 1\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} + 1\right] = \frac{\pi}{4} + \frac{1}{2}$

Q6(i) $P(x)$ even, monic, degree 4.
(1) $\alpha = 3i$ $\quad \therefore \beta = -3i$ (by rule of conjugates)
Also $\alpha\beta\gamma\delta = -18.$ $\qquad x = 3i$
(1) But $\alpha\beta = 9$ $\quad \therefore \gamma\delta = -2.$ $\qquad x = -3i$
$\therefore (x^2 + 9)$

But Monic and Even:
$\therefore P(x) = (x^2 + 9)(x^2 - 2)$
(1) $= (x^2 + 9).(x - \sqrt{2})(x + \sqrt{2})$

(ii) $\dfrac{\cos 15^\circ + \cos 75^\circ}{\sin 15^\circ - \sin 75^\circ} = \dfrac{2\cos\frac{15+75}{2}\cos\frac{15-75}{2}}{2\cos\frac{15+75}{2}\sin\frac{15-75}{2}}$

$= \dfrac{2\cos 45\cos(-30)}{2\cos 45\sin(-30)}$ (3)

$= \dfrac{1}{\tan(-30)} = -\dfrac{1}{\tan 30} = -\dfrac{1}{\frac{1}{\sqrt{3}}}$

$= -\sqrt{3}$

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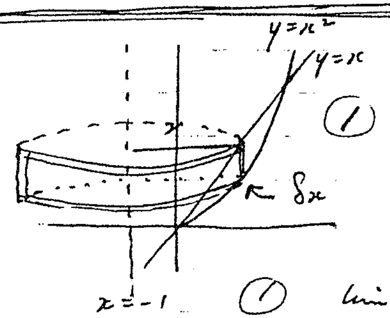
(3)



Volume of a Typical Shell
(1) $\Delta V = \pi(R^2 - r^2) \times$ height
$R = r + \delta x$
$\therefore \Delta V = \pi((r + \delta x)^2 - r^2)h.$
$\Delta V = \pi(2r\,\delta x + (\delta x)^2)h.$
(1) lim as $\delta x \to 0$ $(\delta x)^2$ is negligible
$\therefore \Delta V \doteqdot 2\pi r h . \delta x .$

Now. $r = 1 + x$ and $h = x - x^2.$ (1)
(1) $\therefore V \doteqdot \lim_{\delta x \to 0} \sum_0^1 2\pi r h . \delta x . =$

(1) $= \int_0^1 2\pi(1+x)(x - x^2).dx = \int_0^1 2\pi(x - x^3).dx$

$= 2\pi\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 2\pi\left[\frac{1}{2} - \frac{1}{4}\right] - 0$
$= 2\pi\left[\frac{1}{4}\right]$
(1) $= \underline{\frac{\pi}{2}}$ c.u.

$|w| = 1 \qquad z = \dfrac{w+7}{1-w}$

$z - zw = w + 7$

$z - 7 = w + wz \quad \Rightarrow \quad z - 7 = w(1+z)$

$\therefore \ w = \dfrac{z-7}{z+1} \qquad |w| = 1.$

1

$\therefore \ 1 = \left|\dfrac{z-7}{z+1}\right| \quad \Rightarrow \quad |z+1| = |z-7|$

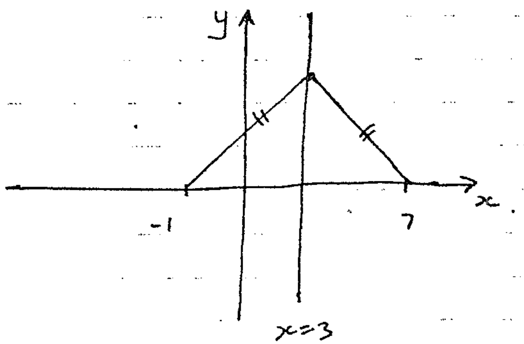
Now $(x+1)^2 + y^2 = (x-7)^2 + y^2$

$x^2 + 2x + 1 + y^2 = x^2 - 14x + 49 + y^2$

$16x = 48$

1

$\underline{\underline{x = 3}}$



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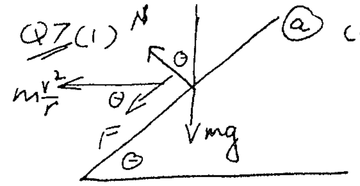
Q7(1)



(a) (i) Resolving vertically:

$N\cos\theta = F\sin\theta + mg. \quad —①$

Resolving horizontally:

$N\sin\theta + F\cos\theta = m\dfrac{v^2}{r} \quad —②$

$① \times \cos\theta \qquad ② \sin\theta.$

$N\cos^2\theta = F\sin\theta\cos\theta + mg\cos\theta.$

$N\sin^2\theta + F\sin\theta\cos\theta = m\dfrac{v^2}{r}\sin\theta.$ 2

$\therefore \ N = mg\cos\theta + m\dfrac{v^2}{r}\sin\theta. \quad *$

$① \times \sin\theta \qquad ② \cos\theta$

$N\sin\theta\cos\theta = F\sin^2\theta + mg\sin\theta$

$N\sin\theta\cos\theta + F\cos^2\theta = m\dfrac{v^2}{r}\cos\theta.$ 2

$\therefore \ F + mg\sin\theta = m\dfrac{v^2}{r}\cos\theta$

$F = m\dfrac{v^2}{r}\cos\theta - mg\sin\theta \quad *$

(b) (i) $30\text{km/hr} \Rightarrow V_1 = \dfrac{30\times1000}{3600}\text{m/s} \qquad 90\text{km/hr} = \dfrac{90\times1000}{3600}$

For V_1 and V_2 $\quad F_1 + F_2 = 0$

$\therefore \ m\left(\dfrac{V_1^2}{r}\cos\theta + \dfrac{V_2^2}{r}\cos\theta\right) = m(g\sin\theta + g\sin\theta).$

$\therefore \ \cos\theta\left(\dfrac{V_1^2}{r} + \dfrac{V_2^2}{r}\right) = 2g\sin\theta.$

$V_1 =$

$V_2 =$

$\therefore \ \cos\theta = \left(\dfrac{\left(\frac{100}{12}\right)^2 + \left(\frac{300}{12}\right)^2}{500}\right) = 20\sin\theta.$

3

$\therefore \ \tan\theta = \left(\dfrac{\left(\frac{25}{3}\right)^2 + \left(\frac{75}{3}\right)^2}{500\times20}\right) \Rightarrow \underline{\underline{\theta = 3°58'}}$

(ii) for $F = 0 \qquad m\dfrac{v^2}{r}\cos\theta = mg\sin\theta$

$v^2 = rg.\tan\theta$

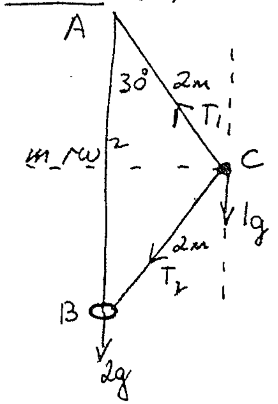
$v^2 = 500\times20\times\tan\theta.$

$v = \dfrac{55.9}{3}\text{m/s} \quad \doteqdot \underline{18.6}\text{ m/s}$

$v \doteqdot 67\text{km/hr}$

1

Q Seven (a)



(a) Resolving vertically at C

$T_1\cos 30^\circ = T_2\cos 30 + 1g.$

$\frac{\sqrt3}{2}T_1 = \frac{\sqrt3}{2}T_2 + 1g.$

$\sqrt3 T_1 = \sqrt3 T_2 + 2g \quad -①$

$\sin\theta = \frac{r}{2}$

$\sin 30 = \frac{r}{2}$

$\frac12 = \frac{r}{2}$

$\therefore r = 1$

Resolving horizontally at C

$T_1\sin 30^\circ + T_2 \sin 30^\circ = mr\omega^2$

$\frac{T_1}{2} + \frac{T_2}{2} = 1\times 1\times\omega^2.$

$\therefore T_1 + T_2 = 2\omega^2 \quad -②.$

Resolving vertically at B

$T_2\cos 30 = 2g. \quad -③.$

$\therefore T_1 \cos 30 = 3g$ from ①

Sub in ② $T_2 = \frac{2g}{\cos 30}$ $T_1 = \frac{3g}{\cos 30}$

$= \frac{3g}{\cos 30} + \frac{2g}{\cos 30} = 2\omega^2.$ 3

$\frac{5g}{\cos 30} = 2\omega^2.$

$\frac{5g}{\frac{\sqrt3}{2}\times 2} = \omega^2 \Rightarrow \omega = \sqrt{\frac{5g}{\sqrt3}}$ rad/sec

$\omega \doteq 5.373$ rad/sec

(b) Change mass at C to 3kg and angle to θ.

(i) Vertically at C

$T_1\cos\theta = T_2\cos\theta + 3g.$

Vertically at B

$T_2\cos\theta = 2g.$

$\therefore T_1\cos\theta = 5g.$

Horizontally at C

$T_1\sin\theta + T_2\sin\theta = mr\omega^2$

where $r = 2\sin\theta$

$\therefore T_1\cancel{\sin\theta} + T_2\cancel{\sin\theta} = 3\times 2\cancel{\sin\theta}\times\frac{5g}{\sqrt3}$

$\therefore \frac{5g}{\cos\theta} + \frac{2g}{\cos\theta} = 6\times\frac{5g}{\sqrt3}$

$\frac{7g}{\cos\theta} = \frac{30g}{\sqrt3}.$

$\cos\theta = \frac{7\sqrt3}{30}$

$\theta \doteq 66^\circ$ 3

(ii) Initially AB

$= 2\times 2\cos 30 = 4\times\frac{\sqrt3}{2}$

$3.464 = 2\sqrt3$ m

Now AB

$= 2\times 2\cos 66^\circ$

$= 2\times\frac{7\sqrt3}{30} = 1.617$

$\therefore$ B will rise

$3.464 - 1.617$

$= 1.85$ m

or 185 cm

Q8 (i)

Prove by mathematical induction $x^{2n} - y^{2n}$ is divisible by $(x+y)$ for $n \geq 1$.

Step 1: Prove true for $n=1$

$x^2 - y^2 = (x-y)(x+y)$ ($\frac12$)

$\therefore$ true for $n=1$.

Step 2: Assume true for $n=k$ where k is a +ve intege[r]

($\frac12$) i.e. $x^{2k} - y^{2k} = (x+y)M$ where M is positive multip[le]

Now: Prove true for $n = k+1$

$\therefore x^{2(k+1)} - y^{2(k+1)} = x^{2k+2} - y^{2k+2}$

(1) $= x^2.x^{2k} - y^{2k+2} = x^2.x^{2k} - x^2.y^{2k} + x^2y^{2k}$

$= x^2(x^{2k} - y^{2k}) - y^2.y^{2k}$

$+ y^{2k}(x^2 - y^2).$

($\frac12$) Now: (from the assumption) $= x^2.M(x+y) + y^{2k}(x-y)(x$

(1) $= (x+y)\{x^2M + (x-y)y^{2k}\}.$

which is divisible by $(x+y)$.

Step 3 Now the statement is true for $n=k+1$ if true for $n=k$. Statement is true for $n=1$, so is

($\frac12$) true for $n = 1+1 = 2$ and $n = 2+1 = 3$ and so on for all +ve integer values of n

(2) $\frac1x + \frac{1}{x+a} + \frac{1}{x+b} = 0 \Rightarrow (x+a)(x+b) + x(x+b) + x(x+a$

$\Rightarrow 3x^2 + x(2a+2b) + ab = 0$

Roots are $p+q$.

(1) $\therefore pq = \frac{ab}{3}$ and $(p+q) = -\frac{2(a+b)}{3}$

Now $(p+q)^2 = p^2 + q^2 + 2pq$

(1) $\therefore p^2+q^2 = (p+q)^2 - 2pq$

$= \frac{8ab}{3} - \frac{2ab}{3} = \frac{6ab}{3}$

$= 2ab.$

$(p+q)^2 = \frac{4(a^2+b^2+2}{9}$

But $a^2+b^2 =$

(1) $\therefore (p+q)^2 = \frac{4(6ab)}{9} = \frac{8}{}$

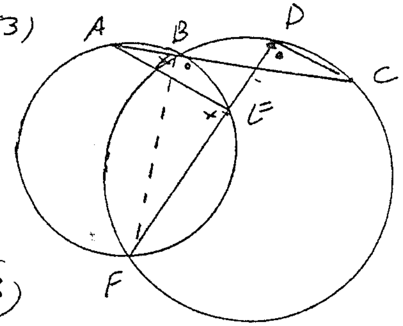
and $2pq = \frac{2ab}{3}.$

But $pq = \frac{ab}{3}$

$\therefore 6pq = 2ab.$

(1) Hence: $p^2 + q^2 = 6pq$ (both equal $2ab$)

Q8(3)



(3)

Construction: join BF.

$\angle ABE = \angle AEF$ ($\angle$'s standing on arc AF)

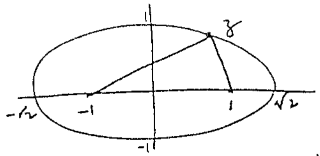
$\angle FBC = \angle FDC$ ($\angle$'s standing on arc FC)

But $\angle ABE + \angle FBC = 180°$ (suppl.

$\therefore \angle AED = \angle FDC$ (supp of $\angle AEF$

Now $AE \parallel DC$ (pair of alt $\angle$'s equal

Q8 4. $|z-1| + |z+1| = 2\sqrt{2}$



$PS + PS' = 2a$

$\therefore a = \sqrt{2}$

$(\pm ae, 0) = (\pm 1, 0)$

$\therefore ae = 1$

$\sqrt{2}e = 1$

$\therefore e = \frac{1}{\sqrt{2}}$

$b^2 = a^2(1-e^2)$

$b^2 = 2\left(1 - \frac{1}{2}\right)$

$= 2\left(\frac{1}{2}\right)$

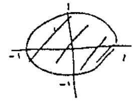
$b^2 = 1$

$\Rightarrow \frac{x^2}{2} + y^2 = 1$

RTP:

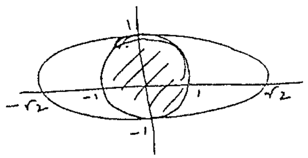
That z represents a pt on or inside the ellipse, $\frac{x^2}{2} + y^2 = 1$

now, $|z| \le 1$



z represents any pt. on or inside circle

$\therefore$ if z lies on/inside the circle centre $(0,0)$ radius 1, then it also lies on/inside the ellipse $\frac{x^2}{2} + \frac{y^2}{1} = 1$



OR

4. $|z-1| + |z+1| \le 2\sqrt{2}$

$|z| \le 1$

$\sqrt{x^2+y^2} \le 1$

$\therefore x^2 + y^2 \le 1$

$LHS = |(x-1)+iy| + |(x+1)+iy|$

$= \sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2}$

$= \sqrt{x^2 - 2x + 1 + y^2} + \sqrt{x^2 + 2x + 1 + y^2}$

$= \sqrt{x^2 + y^2 - 2x + 1} + \sqrt{x^2 + y^2 + 2x + 1}$.

$\le \sqrt{1 - 2x + 1} + \sqrt{1 + 2x + 1}$

$= \sqrt{2 - 2x} + \sqrt{2 + 2x}$.

$= \sqrt{2}.\sqrt{1-x} + \sqrt{2}\sqrt{1+x}$.

$= \sqrt{2}\left[\sqrt{1-x} + \sqrt{1+x}\right]$.

$= \sqrt{2}\left[\sqrt{\left(\sqrt{1-x} + \sqrt{1+x}\right)^2}\right]$

$= \sqrt{2}\left[\sqrt{1 - x + 1 + x + 2\sqrt{(1-x)}\sqrt{(1+x)}}\right]$

$= \sqrt{2}\left[\sqrt{2 + 2\sqrt{1-x^2}}\right]$

$= \sqrt{2} \times \sqrt{2}\sqrt{1 + \sqrt{1-x^2}}$

$= 2\sqrt{1 + \sqrt{1-x^2}}$

$\le 2\sqrt{1+1}$

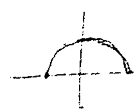
$= 2\sqrt{2}$

$= RHS$

$\therefore |z-1| + |z+1| \le 2\sqrt{2}$

x

$x^2 + y^2 \le 1$



and $\sqrt{1-x^2}$ max ≤ 1 when $x = 0$ $\sqrt{1-x^2} = 1$