S. G. H. S.

Ext n 2

2003

Question 1.

a) Evaluate i) $\int_0^2 \frac{x}{x^2+4}.dx$

ii) $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin 2x.\cos x.dx$

iii) $\int_1^2 x^2 \log_e x.dx$

b) Let n be a positive integer, and let $I_n = \int_1^2 (\log_e x)^n.dx$.

prove that $I_n = 2(\log_e 2)^n - n.I_{n-1}$ and hence evaluate

$\int_1^2 (\log_e x)^3.dx$ as a polynomial in $\log_e 2$

Question 2.

a) i) Find $\sqrt{-3-4i}$

ii) Solve the equation $x^2 - 3x + 3 + i = 0$ over the complex field

b) i) Show that there are two complex numbers z such that

$|z-2-i| = 1$ and $\arg z = \frac{\pi}{4}$,

ii) Find the moduli of the two values of z found in part i)

c) A point P representing the complex number z moves in the Argand Diagram so this it lies in the region defined by:

$|z-1| \leq |z-i|$ and $|z-2-2i| \leq 1$

i) Indicate on a sketch, the region within which P lies

ii) If P describes the boundary of the region, find

α) the value of $|z|$ when arg z has its smallest value

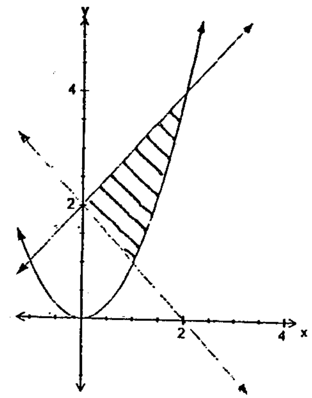
β) the values of z in the form a + ib when $\arg(z-1) = \frac{\pi}{4}$

Question 3:

a) The adjacent diagam shows the area enclosed by $y = 2-x$, $y = 2+x$ and $y = x^2$. The area is to be rotated about the Y axis.

i) Find the shaded area

ii) Find the volume that is formed when it is rotated



b) A satellite moves in a circular orbit of radius 8000km, making 12 revolutions per day. Find:

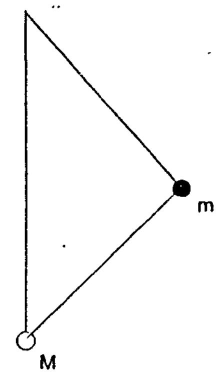
i) the velocity of the satellite

ii) the centripetal force acting on the satellite if the mass of that satellite is 500kg.

c) A particle of mass m is attached to a fixed point O by a string of length one metre, and by another string of the same length to a small ring of mass M which can slide on a smooth vertical wire underneath O.

If m describes a horizontal circle with constant angular velocity w, prove that its depth below O is $\left(\frac{m+2M}{mw^2}\right)g$,

where g is acceleration due to gravity



Questions 4:

a) (i) Sketch $\frac{x^2}{4} + \frac{y^2}{9} = 2$, indicating the centre, foci and directrices

(ii) If $P(2\sqrt{2}\cos\theta, 3\sqrt{3}\sin\theta)$ lies on the ellipse find

α) The equation of the normal at P

β) The value of θ to the nearest degree if the normal passes through the point $(-2\sqrt{2}, 0)$

b) $P\left(3p, \frac{3}{p}\right)$ and $Q\left(3q, \frac{3}{q}\right)$ lie on the hyperbola $xy = 9$.

(i) Find the equation of the tangent at P

(ii) Find the point of intersection T, of the tangents at P and Q.

(iii) If the chord of contact from T passes through the point (0,2) find the locus of T.

Question 5.

a) Given $f(x) = \frac{7x}{(x^2+3)(x+2)}$

i) Express f(x) as a sum of partial fractions

ii) Evaluate $\int_0^1 f(x).dx$

b) Without the use of calculus, sketch the following curves

i) $y = \frac{x(x-2)}{x-1}$ ii) $y = \frac{x(x-1)}{x-2}$

c) Consider $y = \frac{x^3}{(x-1)^2}$

i) Determine the asymptotes

ii) Determine the stationary points

iii) Sketch the curve showing any important features

Question 6.

a) (i) Show that if a polynomial P(x) has a root b of multiplicity m, the the polynomial P'(x) has the root b with multiplicity m-1

(ii) Given that $Q(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$ has a zero of multiplicity 2, solve the equation $Q(x) = 0$ over the complex field

b) If α, β and γ are the roots of $3x^3 + 4x^2 + 5x + 1 = 0$, find the value of $\frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\gamma^2\alpha^2}$

c) i) If $x^2 - 3x + 4 = 0$, show that $x^4 = 3x - 20$

ii) Hence or otherwise, find the equation with roots α^4 and β^4 if the roots of $x^2 - 3x + 4 = 0$ are α and β.

Question 7:

a) Show that the volume of the largest cylinder that can be cut from a solid sphere of radius r cm is $\frac{4\pi r^3}{3\sqrt{3}}$ cm^3

b) (i) Find the five roots of $z^5 = 1$ and write them in mod-arg form.

(ii) Show that when these five roots are plotted on an Argand Diagram, they form the vertices of a regular pentagon of area $\frac{5}{2}\sin\frac{2\pi}{5}$

(iii) Factorise $z^5 - 1$ over the real field

(iv) Deduce that $\cos\frac{2\pi}{5}$ is a root of the equation $4x^2 + 2x - 1 = 0$ and hence find the exact value of $\cos\frac{2\pi}{5}$.

Question 8:

a) A particle of mass m falls from rest at a height h above the earth's surface, against a resistance kv per unit mass when its speed is v; k being a positive constant.

(i) Show that its equation of motion may be written in the form

$$v\frac{dv}{dx} = g - kv$$

(ii) If the particle reaches the surface of the earth with speed V, show that

$$\log_e\left(1 - \frac{kV}{g}\right) + \frac{kV}{g} + \frac{k^2h}{g} = 0$$

b) (i) A particle P is projected from a point O on horizontal ground, with speed V at an angle $\theta = \tan^{-1}\left(\frac{1}{3}\right)$. The particle passes through the point with co-ordinates $\left(3a, \frac{3a}{4}\right)$. Show that $v^2 = 20ga$.

(ii) A particle Q is projected from the same point O at the instant when P reaches its maximum height. It strikes the ground at the same place and time as P strikes the ground. Show that the speed of projection of Q is $\sqrt{\frac{145ga}{2}}$ and find the tangent of the angle of projection.

Soln's + Scale Extn 2 2003

Question 1

$\int_0^2 \frac{x}{x^2+4}\,dx = \frac{1}{2}\int_0^2 \frac{2x}{x^2+4}\,dx$

$= \frac{1}{2}\left[\log_e(x^2+4)\right]_0^2$

$= \frac{1}{2}\left[\log_e 8 - \log_e 4\right]$ ③

$= \underline{\underline{\frac{1}{2}\log_e 2}}$

$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin 2x \cos x \, dx = \underline{\underline{0}}$

as fn odd [sin 2x odd, cos x even]

③

$\int_1^2 x^2 \log_e x \, dx$

let $u = \log_e x$, $v' = x^2$, $u' = \frac{1}{x}$, $v = \frac{x^3}{3}$

$I = \frac{x^3}{3}\log_e x - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx$

$= \frac{x^3}{3}\log_e x - \frac{1}{3}\int x^2\,dx$

$= \underline{\frac{x^3}{3}\log_e x - \frac{x^3}{9}}$ 3

$\int_1^2 x^2 \log_e x \, dx = \left[\frac{x^3}{3}\log_e x - \frac{x^3}{9}\right]_1^2$

$= \left[\frac{8}{3}\log_e 2 - \frac{8}{9}\right] - \left[\frac{1}{3}\log_e 1 - \frac{1}{9}\right]$

$= \underline{\underline{\frac{8}{3}\log_e 2 - \frac{7}{9}}}$ ② 1

$I_n = \int_1^2 (\log_e x)^n \, dx$

let $u = (\log_e x)^n$, $v' = 1 \rightarrow u' = \frac{n}{x}(\log_e x)^{n-1}$

$v = x$

$\therefore I_n = \left[x(\log_e x)^n\right]_1^2 - \int_1^2 n(\log_e x)^{n-1}\,dx$

$I_n = \underline{2(\log_e 2)^n - n\,I_{n-1}}$ ③

$I_3 = 2(\log_e 2)^3 - 3I_2$ ②

$= 2(\ln 2)^3 - 3\left[2(\ln 2)^2 - 2I_1\right]$

$= 2(\ln 2)^3 - 6(\ln 2)^2 + 6\left[2\ln 2 - I_0\right]$

$= \underline{2(\ln 2)^3 - 6(\ln 2)^2 + 12\ln 2 - 6}$

Question 2

let $a+bi = \sqrt{-3-4i}$

$a^2+b^2+2abi = -3-4i$

$a^2-b^2 = -3$ ①, $2ab = -4$

$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2$

$= 9 + 16$

$a^2+b^2 = 5$ ② $(a^2+b^2>0)$

①+② $2a^2 = 2$

$a = \pm 1 \qquad b = \mp 2$

$\therefore \sqrt{-3-4i} = \underline{\underline{\pm(1-2i)}}$ ②

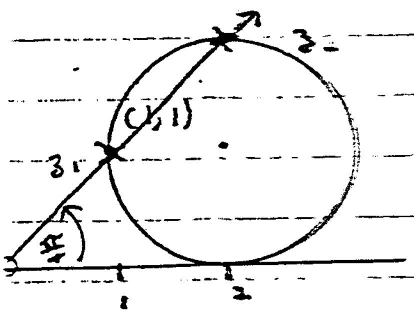
i) $x^2-3x+3+i=0$,

$x = \dfrac{3 \pm \sqrt{9-4(1)(3+i)}}{2}$

$= \dfrac{3+\sqrt{-3-4i}}{2}, \dfrac{3-\sqrt{-3-4i}}{2}$

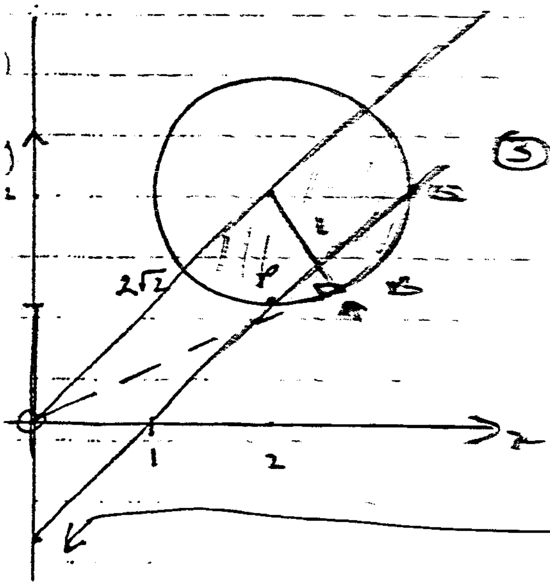
$= \dfrac{3+(1-2i)}{2}, \dfrac{3-1+2i}{2}$

$= \underline{\underline{(2-i), (1+i)}}$ ②



i) From diagram $z_1(1,1)$, $z_2(2,2)$
(or solve algebraically) ①

ii) $|z_1| = \sqrt{1^2+1^2} = \underline{\underline{\sqrt{2}}}$ ① $\qquad |z_2| = \sqrt{2^2+2^2} = \underline{\underline{2\sqrt{2}}}$ ①



③

α)
ii) arg z min at B, $OB = \sqrt{8-1}$ ②
$= \sqrt{7}$

β) $\arg(z-1) = \frac{\pi}{4}$ is the line $y = x-1$ and cuts circle at P, Q

Solving $(x-2)^2+(y-2)^2 = 1$
$y = x-1$ | or from Diagram

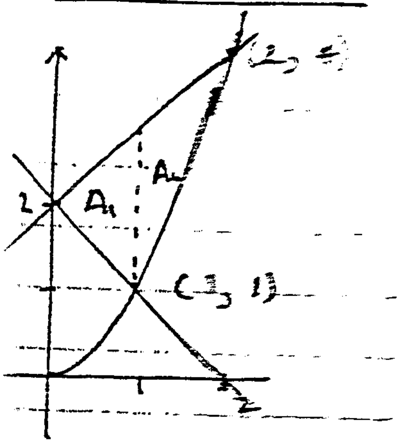
$(x-2)^2+(x-3)^2 = 1$

$x^2-4x+4+x^2-6x+9-1 = 0$ | $(x-3)(x-2) = 0$

$2x^2-10x+12 = 0$ | $x = 2, y = 1$; $x = 3, y = 2$

$x^2-5x \quad = 0$

Question 3



$$A_1 = \int_0^1 [(2+x) - (2-x)]\,dx$$
$$= \int_0^1 2x\,dx$$
$$= [x^2]_0^1$$
$$= 1 \text{ unit}^2$$

$$A_2 = \int_1^2 (2 + x - x^2)$$
$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_1^2$$
$$= \left(4 + 2 - \frac{8}{3}\right) - \left(2 + \frac{1}{2} - \frac{1}{3}\right)$$
$$= 1\frac{1}{6} \text{ units}^2$$

$A = 2\frac{1}{6}$ units^2 (3)

$$V_{shell} = \pi[R^2 - r^2]h \quad \text{(shell 1)}$$
$$= \pi[(x+\Delta x)^2 - x^2][2x]$$
$$= \pi[x^2 + 2x\Delta x + (\Delta x)^2 - x^2]\,2x$$
$$= \pi(4x^2)\Delta x \qquad [(\Delta x)^2 \text{ v. small}]$$
$$\text{solid} = 4\pi \int_0^1 x^2\,dx$$
$$= \frac{4\pi}{3}[x^3]_0^1$$
$$= \frac{4\pi}{3} \text{ units}^3$$

$$V_{shell} = \pi[R^2 - r^2]h \quad \text{(shell 2)}$$
$$= \pi[(x+\Delta x)^2 - x^2](2 + x - x^2)$$
$$= \pi[x^2 + 2x\Delta x + (\Delta x)^2 - x^2](2 + x - x^2)$$
$$= \pi[2x\Delta x](2 + x - x^2)$$
$$= 2\pi[2x + x^2 - x^3]\Delta x$$
$$\text{solid} = 2\pi \int_1^2 (2x + x^2 - x^3)\,dx$$
$$= 2\pi\left[x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right]_1^2$$
$$= 2\pi\left[\left(4 + \frac{8}{3} - \frac{16}{4}\right) - \left(1 + \frac{1}{3} - \frac{1}{4}\right)\right]$$
$$= \frac{19\pi}{6} \text{ units}^3$$

$$\text{Volume} = \frac{4\pi}{3} + \frac{19\pi}{6}$$
$$= \frac{9\pi}{2} \text{ units}^3 \quad (4)$$

-t 3 (continued)

i) $r = 8000\text{ km} = 8\,000\,000\text{ m}$

$$\omega = \frac{1\cancel{2} \times \cancel{2}\pi}{\cancel{24} \times 60 \times 60}\ \text{rad/s}$$

$$= \frac{\pi}{3600}\ \text{rad/s}$$

$$v = r\omega$$

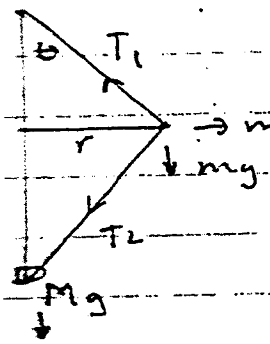
$$= 8000\cancel{000} \times \frac{\pi}{36\cancel{00}}$$

$$= \frac{20000\pi}{9}\ \text{m s}^{-1} \quad ② \quad (\doteqdot 6981 \cdot 3)\ \text{m s}^{-1}$$

ii) $F = m r \omega^2$

$$= 500 \times 8\,000\,000 \times \frac{\pi^2}{(3600)^2}$$

$$\doteqdot \underline{\underline{3046\ \text{N}}} \quad ②$$



At M, $T_2 \cos\theta = Mg \Rightarrow T_2 = \frac{Mg}{\cos\theta}$

at m, $T_1 \cos\theta = T_2 \cos\theta + mg$ (vert)

ie $T_1 \cos\theta = Mg + mg \Rightarrow T_1 = \frac{Mg + m}{\cos\theta}$

$T_1 \sin\theta + T_2 \sin\theta = m r \omega^2$ (hor)

but $r = \sin\theta$

$$T_1 + T_2 = m\omega^2$$

$$\frac{Mg + mg}{\cos\theta} + \frac{Mg}{\cos\theta} = m\omega^2$$

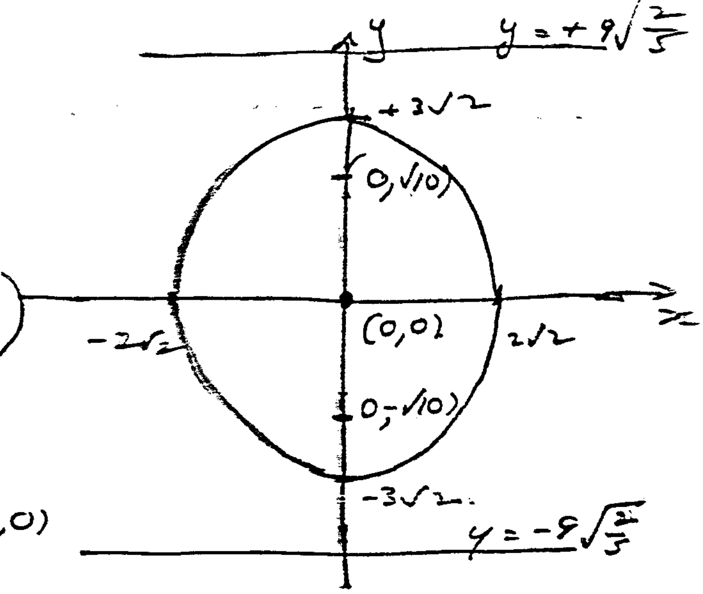
$$\frac{2Mg + mg}{m\omega^2} = \cos\theta$$

but $d = \cos\theta$

$$d = \left(\frac{2M + m}{m\omega^2}\right) g \quad ④$$

Q4

(a) (i)



1

Centre (0,0)

$$\frac{x^2}{4} + \frac{y^2}{9} = 2$$

$$\frac{x^2}{8} + \frac{y^2}{18} = 1.$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore a = 2\sqrt{2}$$

$$b = 3\sqrt{2}$$

Foci $(0, \pm be)$

$$a^2 = b^2(1-e^2)$$

$$e^2 = \frac{b^2 - a^2}{b^2}$$

$$\therefore e = \frac{18-8}{18} = \frac{10}{18} = \frac{5}{9}$$

$$e = \frac{\sqrt{5}}{3}.$$

1 Foci $(0, \pm 3\sqrt{2} \cdot \frac{\sqrt{5}}{3})$

Foci $(0, \pm \sqrt{10})$

1 Directrices $y = \pm \frac{b}{e} = \frac{\pm 3\sqrt{2}}{\frac{\sqrt{5}}{3}} = \frac{\pm 9\sqrt{2}}{\sqrt{5}} = \pm 9\sqrt{\frac{2}{5}}$

$$y = \pm 9\sqrt{\frac{2}{5}}$$

Q4

a)(ii) α)

c Using $P(2\sqrt{2}\cos\theta, 3\sqrt{3}\sin\theta)$

$y = 3\sqrt{3}\sin\theta$. $\quad x = 2\sqrt{2}\cos\theta$

$\frac{dy}{d\theta} = 3\sqrt{3}\cos\theta$. $\quad \frac{dx}{d\theta} = -2\sqrt{2}\sin\theta$.

$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = 3\sqrt{3}\cos\theta \times \frac{-1}{2\sqrt{2}\sin\theta}$.

Grad of tangent: $\frac{3\sqrt{3}\cos\theta}{-2\sqrt{2}\sin\theta}$.

Grad of normal: $\frac{2\sqrt{2}\sin\theta}{3\sqrt{3}\cos\theta}$.

Eqn of normal:

$$y - 3\sqrt{3}\sin\theta = \frac{2\sqrt{2}\sin\theta}{3\sqrt{3}\cos\theta}(x - 2\sqrt{2}\cos\theta)$$

$$y - 3\sqrt{3}\sin\theta = \frac{2\sqrt{2}\sin\theta}{3\sqrt{3}\cos\theta}x - \frac{8\sin\theta}{3\sqrt{3}}$$

$$y = \frac{2\sqrt{2}\sin\theta}{3\sqrt{3}\cos\theta}x + 3\sqrt{3}\sin\theta - \frac{8\sqrt{3}}{9}\sin\theta.$$

$$y = \frac{2\sqrt{2}\sin\theta}{3\sqrt{3}\cos\theta}x + \frac{19\sqrt{3}}{9}\sin\theta$$

β) Passes thru $(-2\sqrt{2}, 0)$.

$$\therefore 0 = \frac{-8\sin\theta}{3\sqrt{3}\cos\theta} + \frac{19\sqrt{3}}{9}\sin\theta.$$

$$\frac{8\cancel{\sin\theta}}{3\sqrt{3}\cos\theta} = \frac{19\sqrt{3}}{9}\cancel{\sin\theta}.$$

$$\Rightarrow 8 = \frac{19\times 9}{9}\cos\theta$$

$$\cos\theta = \frac{8}{19}$$

$$\underline{\theta = 65^\circ}$$

Q4

(a) (ii) α) $P(2\sqrt{2}\cos\theta,\ 3\sqrt{3}\sin\theta)$.

A) Using $\frac{x^2}{4} + \frac{y^2}{9} = 2$.

(1) $\frac{2x}{4} + \frac{2y}{9}\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-9x}{4y}$.

At $P(2\sqrt{2}\cos\theta, 3\sqrt{3}\sin\theta)$ $\frac{dy}{dx} = \frac{-18\sqrt{2}\cos\theta}{12\sqrt{3}\sin\theta} = \frac{-3}{2}\frac{\sqrt{2}\cos\theta}{\sqrt{3}\sin\theta}$

(1) Grad of normal: $\frac{+2\sqrt{3}\sin\theta}{3\sqrt{2}\cos\theta}$.

Eqn of normal. $y - 3\sqrt{3}\sin\theta = \frac{2\sqrt{3}\sin\theta}{3\sqrt{2}\cos\theta}(x - 2\sqrt{2}\cos\theta)$

$y - 3\sqrt{3}\sin\theta = \frac{2\sqrt{3}\sin\theta}{3\sqrt{2}\cos\theta}x - \frac{4}{3}\sqrt{3}\sin\theta$.

(2) $\underline{y = \frac{2\sqrt{3}\sin\theta}{3\sqrt{2}\cos\theta}x + \frac{5}{3}\sqrt{3}\sin\theta}$.

B) Passing thru $(-2\sqrt{2}, 0)$

(1) $0 = \frac{2\sqrt{3}\sin\theta}{3\cancel{\sqrt{2}}\cos\theta} \times -2\cancel{\sqrt{2}} + \frac{5}{3}\sqrt{3}\sin\theta$

$+10\sqrt{3}\sin\theta = \frac{2\sqrt{3}\sin\theta}{\cos\theta}$

$5\cos\theta = 1$

(1) $\cos\theta = \frac{1}{5}$

$\underline{\theta = 78°}$

Q4

(ii) α) $\frac{x^2}{4} + \frac{y^2}{9} = 2 \quad \Rightarrow \frac{dy}{dx} = \frac{-9x}{4y}$.

β Using: $P(2\sqrt{2}\cos\theta, 3\sqrt{2}\sin\theta)$.

Gradient: $\frac{dy}{dx} = -\frac{9.2\sqrt{2}\cos\theta}{4.3\sqrt{2}\sin\theta} = \frac{-3\cos\theta}{2\sin\theta}$.

Gradient of Normal.

$$\frac{dy}{dx} = \frac{2\sin\theta}{3\cos\theta}.$$

Equation of normal.

$$y - 3\sqrt{2}\sin\theta = \frac{2\sin\theta}{3\cos\theta}(x - 2\sqrt{2}\cos\theta)$$

$$y - 3\sqrt{2}\sin\theta = \frac{2\sin\theta}{3\cos\theta}x - \frac{4\sqrt{2}}{3}\sin\theta.$$

$$\therefore y = \frac{2\sin\theta}{3\cos\theta}x - \frac{4}{3}\sqrt{2} + 3\sqrt{2}\sin\theta.$$

$$y = \frac{2\sin\theta}{3\cos\theta}x + \frac{5}{3}\sqrt{2}\sin\theta.$$

Passes thru $(-2\sqrt{2}, 0)$

$$0 = \frac{2\sin\theta}{3\cos\theta}.-2\sqrt{2} + \frac{5}{3}\sqrt{2}\sin\theta.$$

$$\therefore \frac{4\cancel{\sqrt{2}}\cancel{\sin\theta}}{\cancel{3}\cos\theta} = \frac{5\cancel{\sqrt{2}}\cancel{\sin\theta}}{\cancel{3}}$$

$$\cos\theta = \frac{4}{5}$$

$$\underline{\theta = 37^\circ}$$

Q4 (b) $P(3p, \frac{3}{p})$ and $Q(3q, \frac{3}{q})$ lie on $xy = 9$.

(i) $y = 9x^{-1}$

$\frac{dy}{dx} = \frac{-9}{x^2}$ but $x = 3p$ $\frac{dy}{dx} = \frac{-9}{9p^2} = \frac{-1}{p^2}$

2

Eqn of tangent $y - \frac{3}{p} = -\frac{1}{p^2}(x - 3p)$

$p^2y - 3p = -x + 3p$

$\underline{x + p^2y = 6p}$ *.

(ii) Tangent at P $x + p^2y = 6p$ — ①

" " Q $x + q^2y = 6q$. — ②.

$y(p^2 - q^2) = 6(p - q)$.

$\therefore y = \frac{6}{p+q}$.

$q^2x + p^2q^2y = 6pq^2$.

$p^2x + p^2q^2y = 6qp^2$.

$x(q^2 - p^2) = 6pq(q - p)$

2

$x = \frac{6pq}{p+q}$, $y = \frac{6}{p+q}$.

$\underline{T\left(\frac{6pq}{p+q}, \frac{6}{p+q}\right)}$

(iii) Chord of contact

1 . $xy_0 + yx_0 = 18$.

$x\left(\frac{6}{p+q}\right) + y\left(\frac{6p}{p+q}\right) = 18$.

passes thru $(0, 2)$.

$2\left(\frac{6pq}{p+q}\right) = 18$

$\frac{6pq}{p+q} = 9$

But $x = \frac{6pq}{p+q}$.

$\therefore$ Locus is

1 $\underline{x = 9}$

Q5 (a) (i) $f(x) = \dfrac{7x}{(x^2+3)(x+2)}$

$$\frac{7x}{(x^2+3)(x+2)} = \frac{Ax+B}{x^2+3} + \frac{C}{x+2}$$

$$\Rightarrow (Ax+B)(x+2) + C(x^2+3) = 7x$$

$x=-2 \Rightarrow 7C = -14 \Rightarrow C = -2.$

Equate $x^2 \Rightarrow A + C = 0 \quad \therefore A = +2.$

$x = 0 \quad 2B + 3C = 0$

$2B - 6 = 0 \Rightarrow B = 3.$

2 $\quad \therefore f(x) = \dfrac{2x+3}{x^2+3} - \dfrac{2}{x+2}.$

(ii) $$I = \int_0^3 \frac{7x \, dx}{(x^2+3)(x+3)}$$

$$I = \int_0^3 \left(\frac{2x+3}{x^2+3}\right) . dx - \int_0^3 \frac{2}{x+2} . dx.$$

$$I = \left[\ln(x^2+3)\right]_0^3 + \left[\frac{3}{\sqrt{3}} . \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_0^3 - \left[2\ln(x+2)\right]$$

$$I = \left[\ln 12 - \ln 3\right] + \sqrt{3}\tan^{-1}\sqrt{3} - 0 - 2\ln 5 + 2\ln 2.$$

$$I = \ln 4 + \sqrt{3} . \frac{\pi}{3} - \ln 25 + \ln 4.$$

4 $$I = 2\ln 4 - \ln 25 + \frac{\pi}{\sqrt{3}}$$

$$I = \ln\frac{16}{25} + \frac{\pi}{\sqrt{3}} \stackrel{\text{OR}}{\equiv} \ln\left(\frac{16}{25}\right) + \frac{\sqrt{3}\pi}{3}$$

Q5(b) (i) $y = \dfrac{x(x-2)}{x-1}$ $\quad x \neq 1 \quad x=0, y=0 \quad x=2 \;\; y=0$

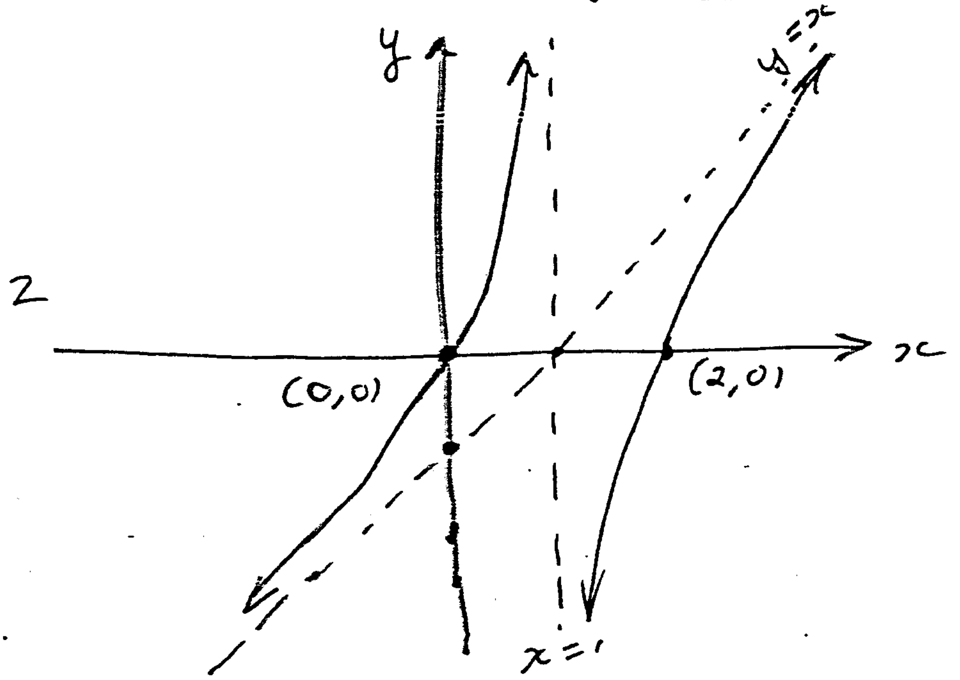
$x \to \infty \quad y \to \infty$

$x \to -\infty \quad y \to -\infty$

$$y = \frac{x^2-2x}{x-1} = \frac{x(x-1)-x}{x-1}$$

$$y = x - \frac{x}{x-1} \quad \text{as } x \to \infty$$

$$y = \underline{\underline{x-1}}$$

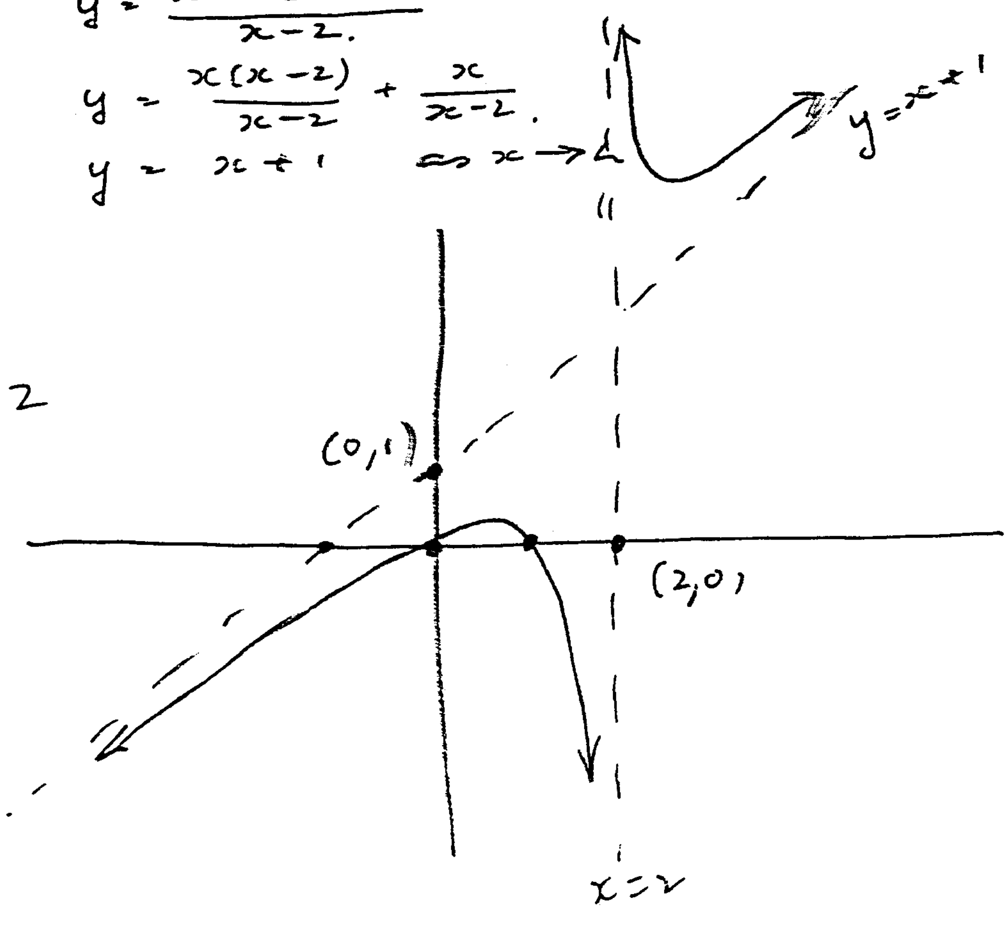


(ii) $y = \dfrac{x(x-1)}{x-2}$ $\quad x=0 \;\; y=0 \quad x=1 \;\; y=0 \quad x \neq 2.$

$$y = \frac{x^2-2x+x}{x-2}$$

$$y = \frac{x(x-2)}{x-2} + \frac{x}{x-2}.$$

$$y = x+1 \quad \text{as } x \to \infty$$



Q5 (c). $y = \dfrac{x^3}{(x-1)^2}$ $\quad x \neq 1.$ $\quad x=0, y=0$

$$\begin{array}{r} x+2 \\ x^2-2x+1 \overline{) \, x^3 \qquad\qquad} \\ \underline{x^3-2x^2+x} \\ 2x^2-x \\ \underline{2x^2-4x+2} \\ 3x-2 \end{array}$$

$y = x+2 + \dfrac{3x-2}{(x-1)^2}$

$x \to \infty$

$y = x+2.$

(i) Asymptotes $\left. \begin{array}{l} x = 1 \\ y = x+2 \end{array} \right\}.$ $\quad$ 1

(ii) $\dfrac{dy}{dx} = \dfrac{3.(x-1)^2.x^2 - x^3.2(x-1)}{(x-1)^4}$

$= \dfrac{3x^2(x-1) - 2x^3}{(x-1)^3} = \dfrac{3x^3-3x^2-2x^3}{(x-1)^3} = \dfrac{x^3-3x^2}{(x-1)^3}$

Put $\dfrac{dy}{dx} = 0$

$\therefore x^3 - 3x^2 = 0 \quad x^2(x-3) \Rightarrow x=0, x=3$

Test $\dfrac{dy}{dx}$ at $x=0.$

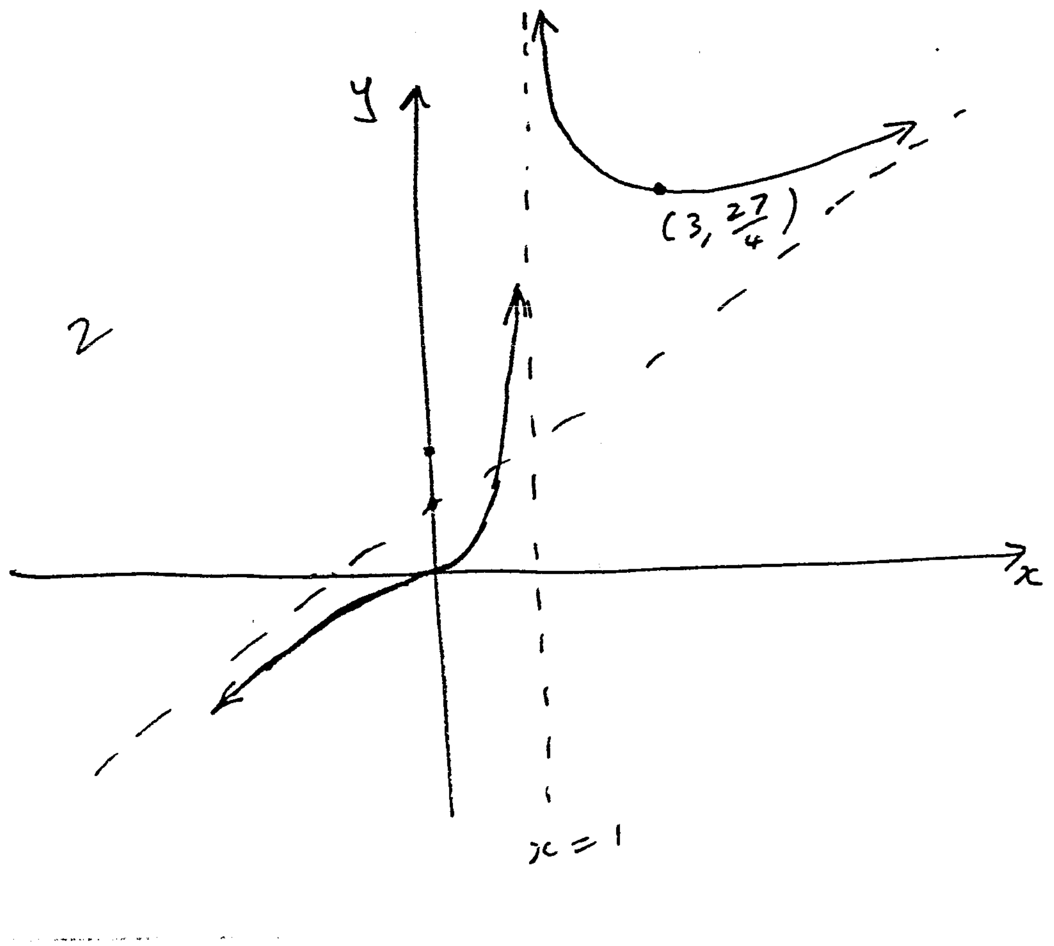
$x<0 \quad \dfrac{dy}{dx} > 0 \qquad x>0 \quad \dfrac{dy}{dx} > 0$

$\therefore$ horizontal inflexion at $(0,0)$

Test at $x=3.$ $\quad$ 2

$x<3 \qquad x>3$

$\dfrac{dy}{dx} < 0 \qquad \dfrac{dy}{dx} > 0 \quad \therefore$ MIN at $x=3$, $y = \dfrac{27}{4}$



2

Q6 (a) (i) $P(x) = (x-b)^m \cdot Q(x)$.

2

$$P'(x) = m(x-b)^{m-1} \cdot Q(x) + (x-b)^m \cdot Q'(x)$$
$$= (x-b)^{m-1} \cdot \{m\,Q(x) + (x-b)\cdot Q'(x)\}$$
$$= (x-b)^{m-1} \cdot S(x).$$

$\therefore$ $x = b$ is a root of multiplicity $m-1$ for $P'(x)$.

(ii) $Q(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$.

$$Q'(x) = 4x^3 - 15x^2 + 8x + 3$$
$$Q'(1) = 0$$
$$Q(1) = 1 - 5 + 4 + 3 + 9 \neq 0.$$

$\therefore$ $x = 1$ is no a zero.

$$Q'(3) = 4\times 3^3 - 15\times 3^2 + 8\times 3 + 3$$
$$= 5\times 27 - 9\times 15 = 0.$$

Try $Q(3) = 3^4 - 5\times 3^3 + 4\times 3^2 + 3\times 3 + 9$
$$= 81 - 135 + 36 + 9 + 9$$
$$=$$

$\therefore$ $x = 3$ is a double root

$\therefore$ $(x-3)^2$ is a factor

$x^2 - 6x + 9$.

4

$$\begin{array}{r|l} & x^2 + x + 1 \\ \hline x^2 - 6x + 9 & x^4 - 5x^3 + 4x^2 + 3x + 9 \\ - & x^4 - 6x^3 + 9x^2 \\ \hline & x^3 - 5x^2 + 3x \\ & x^3 - 6x^2 + 9x \\ \hline & x^2 - 6x + 9 \\ & x^2 - 6x + 9 \\ \hline \end{array}$$

$x^2 + x + 1$

$$x = \frac{-1 \pm \sqrt{-3}}{2} \qquad x = \frac{-1 \pm i\sqrt{3}}{2}$$

Over the Real Field

$\therefore$ $x = 3, 3, \frac{-1 \pm i\sqrt{3}}{2}$

(6) (b) $\alpha, \beta, \gamma.$ $3x^3+4x^2+5x+1=0$

$$\frac{1}{\alpha^2\beta^2}+\frac{1}{\beta^2\gamma^2}+\frac{1}{\gamma^2\alpha^2}$$

$$=\frac{\gamma^2+\alpha^2+\beta^2}{\alpha^2\beta^2\gamma^2}$$

$$\alpha+\beta+\gamma=-\frac{4}{3}$$
$$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{5}{3}$$
$$\alpha\beta\gamma=-\frac{1}{3} \Rightarrow (\alpha\beta\gamma)^2=\frac{1}{9}$$

$$(\alpha+\beta+\gamma)^2=(\alpha+\beta+\gamma)(\alpha+\beta+\gamma)$$
$$=\alpha^2+\alpha\beta+\alpha\gamma$$
$$+\beta\alpha+\beta^2+\beta\gamma+\gamma\alpha+\beta\gamma+\gamma^2.$$

4

$$\therefore \alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$$
$$=\left(-\frac{4}{3}\right)^2-2\left(\frac{5}{3}\right)$$
$$=\frac{16}{9}-\frac{10}{3}$$
$$=\frac{16}{9}-\frac{30}{9}=\frac{-14}{9}.$$

$$\therefore \frac{\gamma^2+\alpha^2+\beta^2}{\alpha^2\beta^2\gamma^2}=\frac{-\frac{14}{9}}{\frac{1}{9}}=\underline{\underline{-14}}$$

(c)(i) $x^2 - 3x + 4 = 0 \implies 9x^2 = 9(3x-4)$

$x^2 = 3x - 4$

$x^4 = (3x-4)^2 = 9x^2 - 24x + 16$

$\alpha + \beta = 3$

$\alpha\beta = 4$

2

$= 9(3x-4) - 24x + 16$

$= 27x - 36 - 24x + 16$

$\underline{\underline{x^4 = 3x - 20}}$

(ii) Roots of $x^2 - 3x + 4 = 0$

α and β.

$\alpha^4 = 3\alpha - 20$

$\beta^4 = 3\beta - 20$

$\alpha^4 + \beta^4 = 3(\alpha+\beta) - 40$

$\therefore \alpha^4 + \beta^4 = \underline{-31}$

$\alpha + \beta = 3$

$\alpha\beta = 4$

$\alpha^4\beta^4 = (\alpha\beta)^4$

$= (4)^4 = \underline{\underline{256}}$

$\cancel{\alpha^4 + \beta^4} = (\alpha+\beta)^4$

$= \alpha^4 + 4\alpha^3\beta + 6\alpha^2\beta^2 + 4\alpha\beta^3 + \beta^4$

$(\alpha\beta)^4 = \alpha^4\beta^4$

$= 4^4$

$= \underline{\underline{256}}$

$\therefore \alpha^4 + \beta^4 = (\alpha+\beta)^4 - 4\alpha\beta(\alpha^2+\beta^2)$

$- 6(\alpha^2\beta^2)$

$= (3)^4 - 4.4(\alpha^2+\beta^2) - 6(4)^2$

3

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= 3^2 - 2.4$

$= 1$

$= 3^4 - 16(1) - 6.(4)^2$

$= 81 - 16 - 96 = 81 - 112 = \underline{\underline{-31}}$

$\therefore \underline{X^2 + 31X + 256 = 0}$

Q6 (e) (ii)

$x^2 - 3x + 4 = 0$ Roots α and β

$\alpha + \beta = 3$

$\alpha\beta = 4.$

$x^4 = 3x - 20.$

$\therefore \alpha^4 = 3\alpha - 20$ $\alpha^4 + \beta^4 = 3(\alpha+\beta) - 2\times 20$

$\beta^4 = 3\beta - 20.$ $= 9 - 40 = -31.$

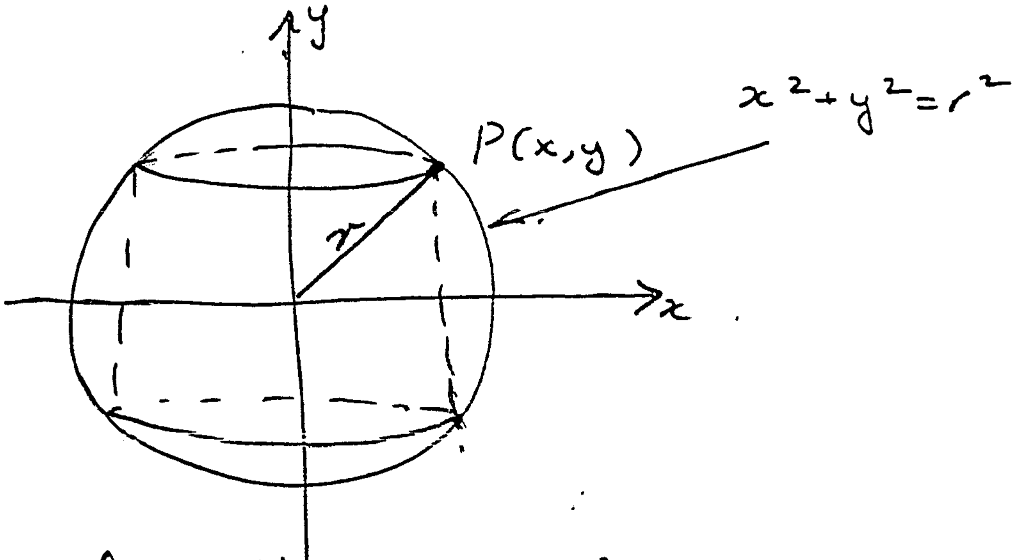
3

$\therefore$ sum of roots $= -31.$

$\alpha^4\beta^4 = (\alpha\beta)^4 = 4^4 = 256.$

$\therefore$ Eqn $\underline{X^2 + 31X + 256 = 0}$

Q7 (a)



Volume of a Cylinder $V = \pi R^2 H$.

from diagram $V = \pi(x^2) \times 2y$

$= 2\pi x^2 y$

But $x^2 = r^2 - y^2 \Rightarrow V = 2\pi(r^2 - y^2)y$

$= 2\pi y r^2 - 2\pi y^3$

For max volume require $\frac{dV}{dy}$

$\frac{dV}{dy} = 2\pi r^2 - 6\pi y^2$.

Put $\frac{dV}{dy} = 0 \Rightarrow \therefore r^2 = 3y^2 \Rightarrow y = \pm\frac{r}{\sqrt{3}}$

But y is a distance $\therefore y > 0$

$\therefore y = \frac{r}{\sqrt{3}}$.

$\frac{d^2V}{dy^2} = -12\pi y < 0$ for $y > 0$

$\therefore$ Max Volume for $y = \frac{r}{\sqrt{3}}$

Max Volume $V = 2\pi y(r^2 - y^2) = 2\pi\frac{r}{\sqrt{3}}\left(r^2 - \frac{r^2}{3}\right)$

$= \frac{2\pi r}{\sqrt{3}}\left(\frac{3r^2 - r^2}{3}\right)$

(4)

$V = \frac{4\pi r^3}{3\sqrt{3}}$ c.u.

Q7(b)(i) $z^5 = 1$ Let $z = cis\,\theta$ $|z| = r = 1$

then $(cis\,\theta)^5 = cis\,(0 + 2k\pi)$ k an integer.

$\therefore$ $cis\,5\theta = cis\,(0 + 2k\pi)$ $k = 0, 1, 2, 3, 4$

By De Moivre's Thm

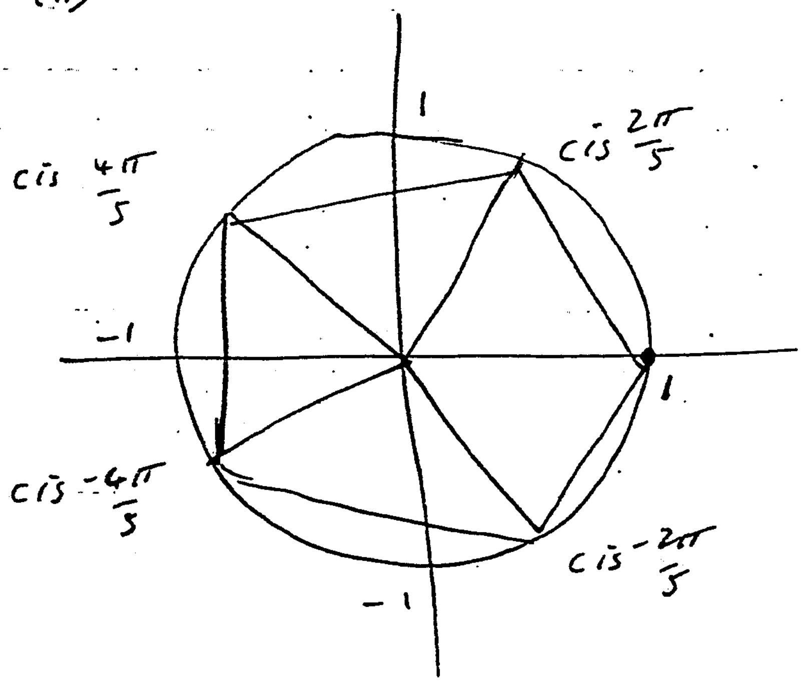
$\therefore 5\theta = 0 + 2k\pi$

$\theta = \frac{2k\pi}{5}$ where $k = 0, 1, 2, 3, 4.$

3 $\therefore \theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}\left(-\frac{4\pi}{5}\right), \frac{8\pi}{5}\left(-\frac{2\pi}{5}\right)$

3 $\therefore$ 5 roots are $cis\,0 = 1$, $cis \pm \frac{2\pi}{5}$, $cis \pm \frac{4\pi}{5}$

(ii)



Area of $\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}.1.1.\sin\frac{2\pi}{5}.$

Area of Pentagon $= 5 \times \frac{1}{2}\sin\frac{2\pi}{5}$

2 $= \frac{5}{2}\sin\frac{2\pi}{5}$ s.u.

2

Q7(b)(iii) $z^5 - 1 = (z-1)(z-z_1)(z-z_2)(z-z_3)(z-z_4)$

But $z_1 = \text{cis}\frac{2\pi}{5}$ $\quad z_2 = \text{cis}\frac{4\pi}{5}$

(2) $\quad z_4 = \text{cis}-\frac{2\pi}{5}$ $\quad z_3 = \text{cis}-\frac{4\pi}{5}$

with $z_1 = \overline{z_4}$ and $z_2 = \overline{z_3}$

2 $\quad \therefore z^5 - 1 = (z-1)(z^2 - 2\cos\frac{2\pi}{5}z + 1)(z^2 - 2\cos\frac{4\pi}{5}z + 1)$

Roots in conjugate pairs.

(iv) Now $(z^5 - 1) = (z-1)(z^4 + z^3 + z^2 + z + 1)$.

$\therefore z^4 + z^3 + z^2 + z + 1 = (z^2 - 2\cos\frac{2\pi}{5}z + 1)(z^2 - 2\cos\frac{4\pi}{5}z + 1)$.

Equating coefficients of z^3

$1 = -2\cos\frac{2\pi}{5} - 2\cos\frac{4\pi}{5}z$.

But. $\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow 2\cos\frac{4\pi}{5} = 2\{2\cos^2\frac{2\pi}{5} - 1\}$

$\therefore 1 = -2\cos\frac{2\pi}{5} - 2\{2\cos^2\frac{2\pi}{5} - 1\}$.

$1 = -2\cos\frac{2\pi}{5} - 4\cos^2\frac{2\pi}{5} + 2$.

$\Rightarrow 4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} = 1$

Above eqn of form $4x^2 + 2x - 1 = 0$.

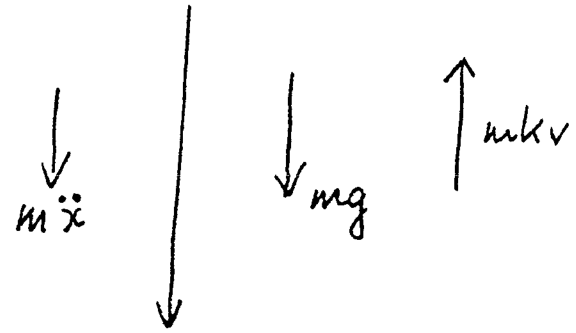
4 $\quad$ Exact Value: $x = \dfrac{-2 \pm \sqrt{4+16}}{8} = \dfrac{-2 \pm \sqrt{20}}{8}$

$x = \dfrac{-2 \pm 2\sqrt{5}}{8} = \dfrac{-1 \pm \sqrt{5}}{4}$

But $\frac{2\pi}{5} < \frac{\pi}{2} \Rightarrow \therefore \cos\frac{2\pi}{5} > 0$

(4) $\quad \therefore \cos\frac{2\pi}{5} = \dfrac{-1+\sqrt{5}}{4}$

Q8 (a)



(i) $m\ddot{x} = mg - mkv$

$\therefore \ \ddot{x} = g - kv \qquad\qquad \ddot{x} = \dfrac{v\,dv}{dx}$

(1)

$\therefore \ v.\dfrac{dv}{dx} = g - kv. \qquad\qquad$ at $t=0 \quad v=0 \quad =0$

$v=V \quad x=h.$

(ii) $\dfrac{dx}{v.dv} = \dfrac{1}{g-kv}.$

$$\int_0^h dx = \int_0^V \frac{v.dv}{g-kv}$$

$$h = -\frac{1}{k}\int_0^V \frac{g-kv}{g-kv}.dv + \frac{1}{k}\int_0^V \frac{g}{g-kv}.dv$$

$$h = -\frac{V}{k} + -\frac{g}{k^2}\int_0^V \frac{-k.dv}{g-kv}$$

$$h = \frac{-V}{k} - \frac{g}{k^2}\ln\left[g-kv\right]_0^V$$

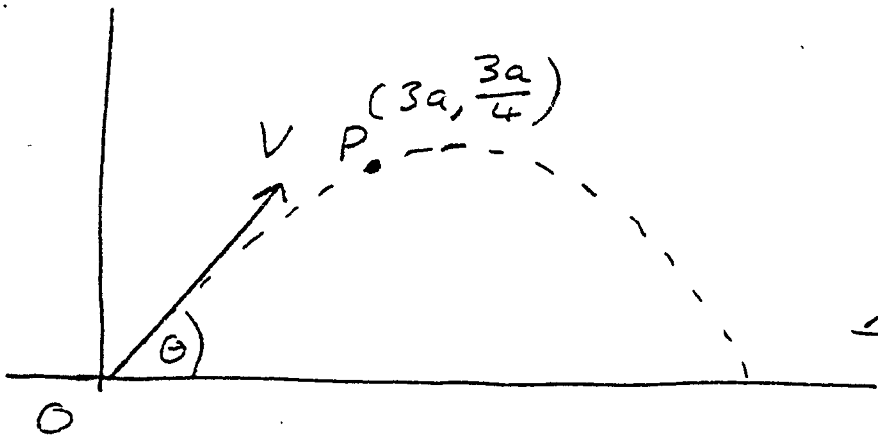
$$h = \frac{-V}{k} - \frac{g}{k^2}\ln\left[\frac{g-kV}{g}\right]$$

$$k^2h = -Vk - g\ln\left[1-\frac{kV}{g}\right]$$

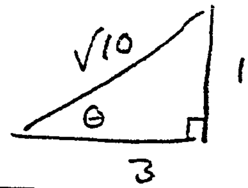
(4) $\therefore \ g\ln\left[1-\dfrac{kV}{g}\right] + Vk + k^2h = 0$

$$\ln\left[1-\frac{kV}{g}\right] + \frac{kV}{g} + \frac{k^2h}{g} = 0$$

Q8 (b)(i)



$\tan\theta = \frac{1}{3}$



$\sin\theta = \frac{1}{\sqrt{10}}$

$\cos\theta = \frac{3}{\sqrt{10}}$

Vertically:

$\ddot{y} = -g.$

$\dot{y} = -gt + c_1$

at $t=0$ $\dot{y} = \frac{V}{\sqrt{10}}$

$\therefore \dot{y} = -gt + \frac{V}{\sqrt{10}}$

$y = -\frac{1}{2}gt^2 + \frac{Vt}{\sqrt{10}} + c_2.$

at $t=0$ $y=0$

$\therefore y = -\frac{1}{2}gt^2 + \frac{Vt}{\sqrt{10}}$ — ①

Horizontally

$\ddot{x} = 0$

$\dot{x} = c_3$

at $t=0$ $\dot{x} = \frac{3V}{\sqrt{10}}$

$\therefore \dot{x} = \frac{3V}{\sqrt{10}}$

$x = \frac{3Vt}{\sqrt{10}} + c_4$

at $t=0$ $x=0$

$\therefore x = \frac{3Vt}{\sqrt{10}}$ — ②

Now passes thro $(3a, \frac{3a}{4})$.

In ② $3a = \frac{3Vt}{\sqrt{10}} \Rightarrow t = \frac{a\sqrt{10}}{V}$

In ① $\frac{3a}{4} = -\frac{1}{2}gt^2 + \frac{Vt}{\sqrt{10}}$

$\frac{3a}{4} = -\frac{1}{2}\cdot g\left(\frac{a\sqrt{10}}{V}\right)^2 + \frac{V}{\sqrt{10}}\cdot\frac{a\sqrt{10}}{V}$

$\frac{3a}{4} = -\frac{1}{2}g.\frac{a^2 10}{V^2} + a.$

5

$\therefore \frac{1}{2}g.\frac{a^2 10}{V^2} = \frac{a}{4}.$

$2ag.10 = V^2$

$\therefore V^2 = 20ga$

Q8(b)(ii)

For max height for Particle P $\dot{y} = 0$

$$\therefore \frac{V}{\sqrt{10}} = gt \Rightarrow t = \frac{V}{g\sqrt{10}}$$

For time of flight for P $y = 0 \Rightarrow \frac{1}{2}gt^2 = \frac{Vt}{\sqrt{10}}$ $t = ($

$$t = \frac{2V}{g\sqrt{10}}.$$

Range of flight $x = 3\frac{Vt}{\sqrt{10}}$

$$x = \frac{3V}{\sqrt{10}} \cdot \frac{2V}{g\sqrt{10}} = \frac{3V^2}{5g}.$$

For Q let u = velocity and angle α.

Then $y = -\frac{1}{2}gt^2 + ut\sin\alpha$ and $x = ut\cos\alpha$ (from (i))

When $t = \frac{V}{g\sqrt{10}}$ $y = 0$ and $x = \frac{3V^2}{5g}$

$\therefore \frac{1}{2}gt = u\sin\alpha$. $\qquad \frac{3V^2}{5g} = u \cdot \frac{V}{g\sqrt{10}}\cos\alpha$.

$u\sin\alpha = \frac{V}{2\sqrt{10}}$ — (3) $\qquad u\cos\alpha = \frac{3\sqrt{10}V}{5}$ — (4).

$(3)^2 \Rightarrow u^2\sin^2\alpha = \frac{V^2}{40}$ $\qquad (4)^2$ $\frac{90}{25}V^2 = u^2\cos^2\alpha$.

$\therefore u^2(\sin^2\alpha + \cos^2\alpha) = \frac{V^2}{40} + \frac{90}{25}V^2 \Rightarrow \frac{29V^2}{8}$

4

But $V^2 = 20ga$

$\therefore u^2 = \frac{29}{8} \times 20ga \Rightarrow u^2 = \frac{145ga}{2}$

$$u = \sqrt{\frac{145ga}{2}} \quad *$$

Now $V^2 = 40u^2\sin^2\alpha$

$20ag = 40 \cdot \frac{145ag}{2} \cdot \sin^2\alpha$.

$\therefore \sin^2\alpha = \frac{1}{145} \Rightarrow \sin\alpha = \frac{1}{\sqrt{145}}$

$\therefore \tan\alpha = \frac{1}{12}$

1

$\alpha = \tan^{-1}\left(\frac{1}{12}\right)$