Sydney Girls High School

2004
TRIAL HIGHER SCHOOL
CERTIFICATE EXAMINATION

Mathematics

Extension 2

This is a trial paper ONLY. It does not necessarily reflect the format or the contents of the 2004 HSC Examination Paper in this subject.

Candidate Number

General Instructions

- Reading Time - 5 mins
- Working time - 3 hours
- Attempt **ALL** questions
- **ALL** questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used.
- Diagrams are not to scale
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

QUESTION 1

Marks

(a) Find $\int \frac{e^x}{(e^x+1)^2}.dx$ **2**

(b) Find $\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}}.dx$ **3**

(c) Evaluate $I = \int_0^{\frac{\pi}{3}} \sec^4 \theta . \tan\theta . d\theta$ **3**

(d) Evaluate $I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta+\cos\theta}.d\theta$ **3**

(e) (i) Express $\frac{3x+7}{(x+1)(x+2)(x+3)}$ in partial fractions **4**

(ii) Hence, evaluate $\int_0^1 \frac{(3x+7).dx}{(x+1)(x+2)(x+3)}$

Question 2.

Marks

(a) Given $z = 1 + i\sqrt{3}$ and $w = 1 + i$ **6**

(i) Evaluate: $\frac{z}{w}$ in Cartesian Form

(ii) Plot z, w and $\frac{z}{w}$ on the Argand Diagram

(iii) Express z and w in modulus – argument form.

(iv) Show $\frac{z}{w} = \sqrt{2}cis(\frac{\pi}{12})$ and hence find the exact value of $\cos\frac{\pi}{12}$

(b) (i) Find both square roots of $8 + 6i$, in the form $x + iy$ **4**

(ii) Solve the quadratic equation $z^2 + (2 + 4i)z - 11 - 2i = 0$

(c) Given the locus of z is $|z - 2 - 2i| = 1$ **5**

(i) Sketch the locus of z on the Argand Diagram.

(ii) Find the maximum value of $\arg z$

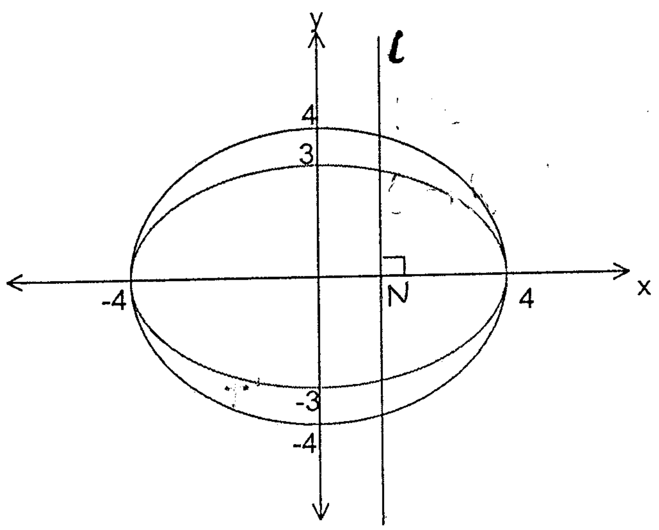
(iii) Find the maximum value of mod z

Question 3. **Marks**

(a) Given $\frac{x^2}{25}-\frac{y^2}{9}=1$ **6**

Find: (a) eccentricity
(b) foci
(c) directrices
(d) asymptotes
(e) the equation of the tangent at $P(5\sec\theta, 3\tan\theta)$

(b) **9**



$x = 4\cos\theta; y = 4\sin\theta$

The diagram shows the ellipse, E, with equation $\frac{x^2}{16}+\frac{y^2}{9}=1$ and its auxiliary circle C
The coordinates of a point P on E are $(4\cos\theta, 3\sin\theta)$.

A straight line, l, parallel to the y axis intersects the x axis at N and the curves E and C at the points P and Q respectively.

(i) Find the eccentricity of E,

(ii) Write down the coordinate of N and Q,

(iii) Find the equations of the tangents at P and Q to the curves E and C respectively,

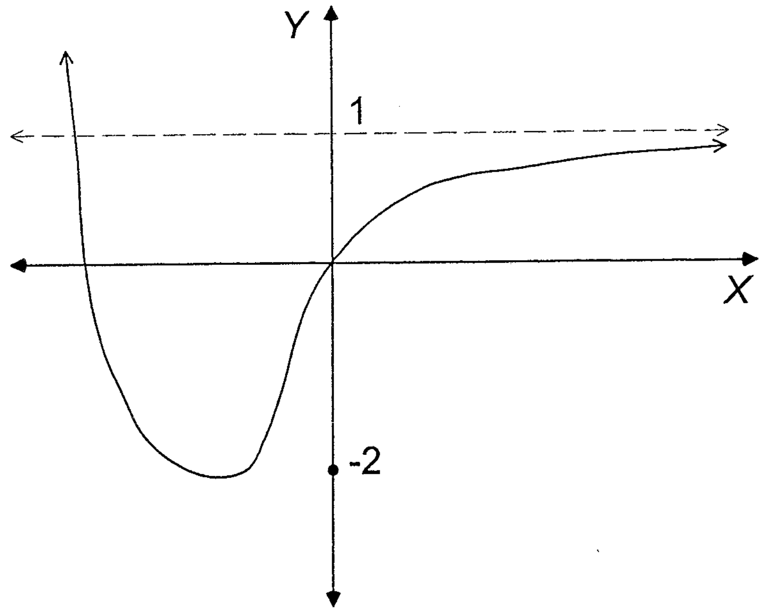
(iv) The tangents at P and Q intersect at a point R. Show that R lies on the x axis,

(v) Prove that $ON.OR$ is independent of the positions P and Q.

Question 4

Marks

(a) 6



Given the above curve is $y = f(x)$

Sketch (i) $y = f(-x)$

(ii) $y = \dfrac{1}{f(x)}$

(iii) $y = e^{f(x)}$

(iv) $y = \ln f(x)$

(v) $y^2 = f(x)$

(b) (i) By expanding $(\cos\theta + i\sin\theta)^4$ find expressions for $\sin 4\theta$ and $\cos 4\theta$. 9

(ii) Prove $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

(iii) Solve Solve $\tan 4\theta = 1$ for $0 \le \theta \le \pi$

(iv) By taking $x = \tan\theta$, find the roots of $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

(v) Using (iv) find the value of: $\tan^2\dfrac{\pi}{16} + \tan^2\dfrac{5\pi}{16} + \tan^2\dfrac{9\pi}{16} + \tan^2\dfrac{13\pi}{16}$

Question 5 **Marks**

(a) Let α, β and δ be the roots of $x^3 - x^2 + 2x - 1 = 0$ **6**

(i) Find the value of $\alpha + \beta + \delta$, hence, or otherwise find the equation with roots $-(\alpha + \beta)$, $-(\beta + \delta)$, and $-(\delta + \alpha)$

(ii) (a) Find the equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\delta}$

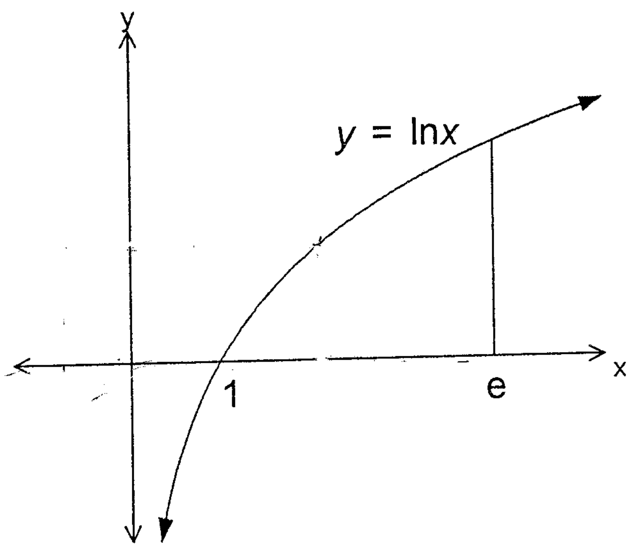
(b) hence, or otherwise, evaluate $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\delta}$

(b) Given the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ **5**

(i) If (x) has zeroes $a + bi$, $a - 2bi$, where a and b are real find the values of a and b

(ii) Hence, express $P(x)$ as the product of two quadratic factors with real Coefficients.

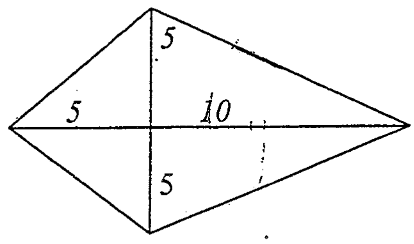
(c) Using the method of cylindrical shells, find the volume generated by revolving the region bounded by $y = \log_e x$ the *x-axis* and $1 \leq x \leq e$, about the *y-axis*. **4**



Question 6

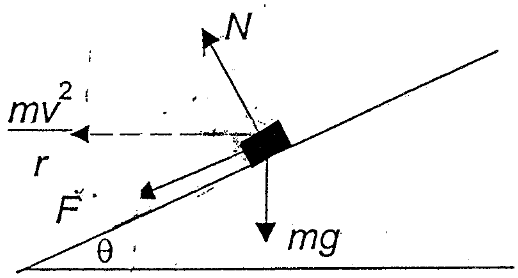
Marks

(a) The base of a solid is the kite shown below, with measurements in cm. 6



Each slice vertical to the base and perpendicular to the major axis of the kite is a semi-circle. By taking the major axis of the kite as the x-axis and the minor axis of the kite as the y-axis find the volume of the solid formed.

(b) An object of mass m kg is travelling around a banked circular track of radius r and angle of banking θ. The mass is travelling at v m/s. By resolving forces vertically and horizontally derive expressions for N (the normal force) and F (the sideways frictional force). 5



Given the radius of the curve is $1km$ and $\tan\theta = \dfrac{1}{100}$ find the velocity in m/s which will ensure no sideways friction (i.e. $F = 0$). Take $g = 10m/s^2$

(c) If $x^3 + 3kx + l = 0$ has a double root, where k and l are real, prove that $l^2 + 4k^3 = 0$ 4

Question 7

Marks

(a) (i) Simplify: $\sin(A-B)+\sin(A+B)$ 3

(ii) Hence find $\int_0^{\frac{\pi}{4}} \sin 5x.\cos 3x.dx.$

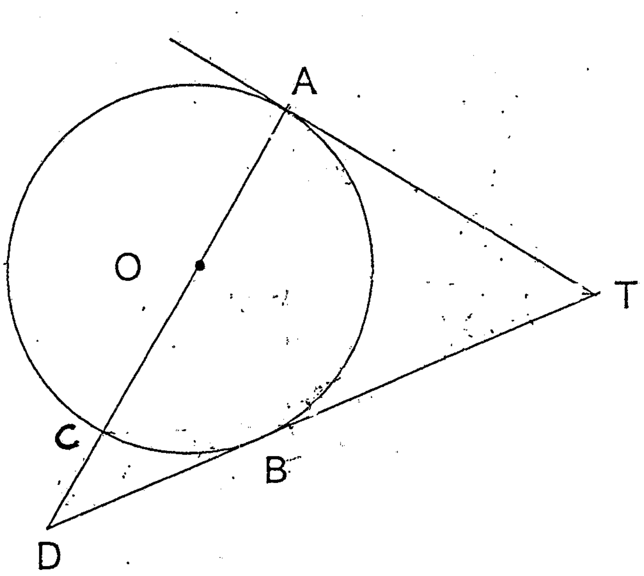
(b) Given $f(x)=\frac{\ln x}{x}$ 4

(i) Find the stationary point and its nature.

(ii) Show $f(e) > f(\pi)$

(iii) Show $e^{\pi} > \pi^{e}$

(c) From an external point T, two tangents TA and TB are drawn to touch a circle with centre O at A and B respectively. Angle ATB is acute. The diameter AC produced meets TB produced at D. 8



(i) Prove that $\angle CBD = \frac{1}{2}\angle ATB$

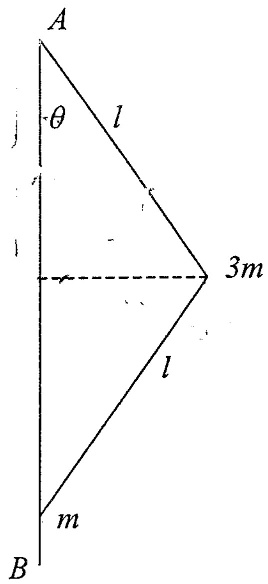
(ii) Prove that ΔABC is similar to ΔTBO

(iii) Deduce that $BC.OT = 2.(OA)^2$

Question 8

Marks

(a) A light string $2l$ metres long is attached to point A. A mass of $3m$ kg is attached to the middle of the string and a second mass of m kg is in the form of a ring and is attached at the end of the string at B. The $3m$ kg mass is rotating in circular motion at w radians/sec and the m kg mass is free to move up or down the smooth rod AB (see diagram). The string makes an angle of θ with the vertical. 7



(i) Find an expression for h in terms of g and w only.

(ii) If the $3m$ kg and m kg masses are interchanged and the speed of the rotating mass is doubled to $2w$ determine if h is increased or decreased. (note $w > 1$)

(b) A magic square is shown below 2

4	3	8
9	5	1
2	7	6

Note that the sum of the diagonals, rows and columns is fifteen
Three different numbers are chosen at random from the square.
Find the probability that the sum of the numbers is 15 if:

(i) A five is chosen first
(ii) A two is chosen first

(c) z is a complex number such that: 3

$$z = k(\cos\theta + i\sin\theta) \quad \text{where } k \text{ is real.}$$

Show $\arg(z + k) = \dfrac{\theta}{2}$

(d) Given $I_n = \dfrac{1}{n!}\displaystyle\int_0^a x^n . e^{-x} . dx.$ 3

Show $\dfrac{a^n}{n!} = e^a (I_{n-1} - I_n)$

Note $n! = n \times (n-1) \times (n-2) \times \ldots\ldots \times 3 \times 2 \times 1$

End of Exam

S. G. H. S Extn 2 Trial 2004

Question One

a) let $u = e^x + 1 \Rightarrow du = e^x\,dx$

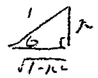
$I = \int u^{-2}\,du$

$= -u^{-1} + c$

$= \dfrac{-1}{e^x + 1} + c$

(2)

b) let $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$



$I = \int \dfrac{\sin^2\theta\cos\theta\,d\theta}{(1-\sin^2\theta)^{\frac{3}{2}}}$

$= \int \dfrac{\sin^2\theta\cos\theta}{\cos^3\theta}\,d\theta$

$= \int \tan^2\theta\,d\theta$

$= \int (\sec^2\theta - 1)\,d\theta$

$= \tan\theta - \theta + c$

$= \dfrac{x}{\sqrt{1-x^2}} - \sin^{-1}x + c$

(3)

c) $I = \int_0^{\frac{\pi}{3}} \sec^4\theta\tan\theta\,d\theta$

$I_1 = \int \sec^2\theta(1 + \tan^2\theta)\tan\theta\,d\theta$

let $u = \tan\theta \Rightarrow du = \sec^2\theta\,d\theta$

$I_1 = \int (1 + u^2)u\,du$

$= \int u + u^3\,du$

$= \dfrac{u^2}{2} + \dfrac{u^4}{4}$

(3)

$I = \left[\frac{1}{2}\tan^2\theta + \frac{1}{4}\tan^4\theta\right]_0^{\frac{\pi}{3}}$

$= \frac{1}{2}(\sqrt{3})^2 + \frac{1}{4}(\sqrt{3})^4$

$= \frac{3}{2} + \frac{9}{4}$

$= 3\frac{3}{4}$ OR

$I = \int_0^{\frac{\pi}{3}} \dfrac{\sin\theta}{\cos^5\theta}\,d\theta = \int_{\frac{\pi}{3}}^{0} \dfrac{-\sin\theta\,d\theta}{\cos^5\theta}$

let $u = \cos\theta$, $du = -\sin\theta\,d\theta$

when $\theta = \frac{\pi}{3} \Rightarrow u = \frac{1}{2}$, $\theta = 0 \Rightarrow u = 1$

$I = \int_{\frac{1}{2}}^{1} u^{-5}\,du$

$= -\frac{1}{4}\left[u^{-4}\right]_{\frac{1}{2}}^{1}$

$= -\frac{1}{4}[1 - 16]$

d) let $t = \tan\frac{\theta}{2}$, $d\theta = \dfrac{2\,dt}{1+t^2}$, $\sin\theta = \dfrac{2t}{1+t^2}$

when $\theta = \frac{\pi}{2}, t = 1$, when $\theta = 0$, $t = 0$ $\cos\theta = \dfrac{1-t^2}{1+t^2}$

$I = \int_0^1 \dfrac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \dfrac{2\,dt}{1+t^2}$

$= \int_0^1 \dfrac{2\,dt}{1 + t^2 + 2t + 1 - t^2}$

$= \int_0^1 \dfrac{2\,dt}{2 + 2t}$

(3)

$= \int_0^1 \dfrac{dt}{1+t}$

$= \left[\log_e(1+t)\right]_0^1$

$= \log_e 2 - \log_e 1$

$= \log_e 2$

e) $\dfrac{3x+7}{(x+1)(x+2)(x+3)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+2)} + \dfrac{C}{(x+3)}$

$3x + 7 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

put $x = -1$, $x = -2$, $x = -3$

$2A = 4 \therefore A = 2$ $B = -1$ $2C = -2 \therefore C = -1$

$\dfrac{3x+7}{(x+1)(x+2)(x+3)} = \dfrac{2}{x+1} - \dfrac{1}{x+2} - \dfrac{1}{x+3}$

(2)

$I = \int_0^1 \left(\dfrac{2}{x+1} - \dfrac{1}{x+2} - \dfrac{1}{x+3}\right)dx$

$= \left[2\log_e(x+1) - \log_e(x+2) - \log_e(x+3)\right]_0^1$

$= \left[\log_e \dfrac{(x+1)^2}{(x+2)(x+3)}\right]_0^1$

$= \log_e \frac{4}{12} - \log_e \frac{1}{6}$

$= \log_e \frac{4}{12} \times \frac{6}{1}$

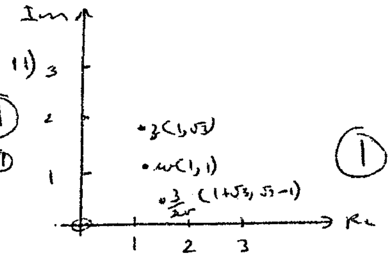
$= \log_e 2$

(2)

Question Two.

a) i) $\frac{z}{w} = \frac{1+i\sqrt{3}}{1+i} \times \frac{1-i}{1-i}$

$= \frac{1+\sqrt{3} + i(\sqrt{3}-1)}{2}$ (1) (1)

ii) 

(1)

iii) $z = 1 + i\sqrt{3}$

$= 2 \text{ cis } \frac{\pi}{3}$ (1)

$w = 1 + i$

$= \sqrt{2} \text{ cis } \frac{\pi}{4}$ (1)

iv) $\frac{z}{w} = \frac{2}{\sqrt{2}} \text{ cis } \left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$ (2)

equate real from (1) and (2)

$\sqrt{2}\cos\frac{\pi}{12} = \frac{1}{2}(1+\sqrt{3})$

$\cos\frac{\pi}{12} = \frac{1}{2\sqrt{2}}(1+\sqrt{3})$ (2)

b) i) $a + ib = \sqrt{8+6i}$

$a^2 - b^2 + 2abi = 8 + 6i$

$a^2 - b^2 = 8$ (1)

$2ab = 6$

$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2$

$= 64 + 36$

$a^2 + b^2 = 10$ (2)

(1)+(2) $2a^2 = 18$

$a = \pm 3$, $b = \pm 1$

ie $(3+i)$, $(-3-i)$ (2)

ii) $z^2 + (2+4i)z - 11 - 2i = 0$

$z = \frac{-(2+4i) \pm\sqrt{(2+4i)^2 - 4(1)(-11-2i)}}{2}$

$= \frac{-(2+4i) \pm\sqrt{4 - 16 + 16i + 44 + 8i}}{2}$

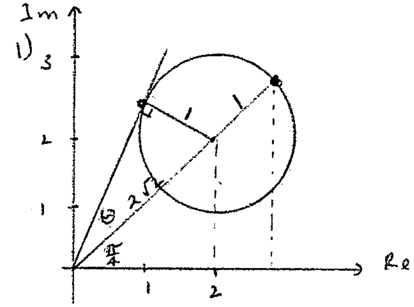
$= \frac{-2-4i \pm\sqrt{32+24i}}{2}$

$z = \frac{-2-4i \pm\sqrt{32+24i}}{2}$

$= \frac{-2-4i \pm 2\sqrt{8+6i}}{2}$

$= -1 - 2i \pm (3+i)$

$= (2-i), (-4-3i)$ (2)

c) i) 

(1)

ii) Max $\arg(z) = \frac{\pi}{4} + \sin^{-1}\frac{1}{2\sqrt{2}}$ (2) ($\doteqdot 45° + 20.8° = 65.8°$)

iii) Max $\text{mod}(z) = 2\sqrt{2} + 1$ (2)

Question Three

a) $\frac{x^2}{25} - \frac{y^2}{9} = 1$ $\quad a = 5, b = 3$

a) $b^2 = a^2(e^2 - 1)$

$9 = 25e^2 - 25$

$e^2 = \frac{34}{25}$

$e = \frac{\sqrt{34}}{5}$ (1)

b) foci $(\pm ae, 0)$

$(\pm\sqrt{34}, 0)$ (1)

c) directrices

$x = \pm \frac{a}{e}$

$x = \pm \frac{25}{\sqrt{34}}$ (1)

d) asymptotes

$y = \pm \frac{b}{a}x$

$= \pm \frac{3x}{5}$ (1)

e) $P(x = 5\sec\theta, y = 3\tan\theta)$

$\frac{dx}{d\theta} = 5\sec\theta\tan\theta$, $\quad \frac{dy}{d\theta} = 3\sec^2\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$

$= \frac{3\sec^2\theta}{5\sec\theta\tan\theta}$

$= \frac{3\sec\theta}{5\tan\theta}$ (2)

Eqn of tangent

$y - 3\tan\theta = \frac{3\sec\theta}{5\tan\theta}(x - 5\sec\theta)$

$5y\tan\theta - 15\tan^2\theta = 3x\sec\theta - 15\sec^2\theta$

$3x\sec\theta - 5y\tan\theta = 15\sec^2\theta - 15\tan^2\theta$

$3x\sec\theta - 5y\tan\theta = 15$

$\frac{x\sec\theta}{5} - \frac{y\tan\theta}{3} = 1$

b) $\frac{x^2}{16} + \frac{y^2}{9} = 1$ $\quad a = 5, b = 3$

i) $b^2 = a^2(1 - e^2)$

$9 = 16 - 16e^2$

$e^2 = \frac{7}{16}$

$e = \frac{\sqrt{7}}{4}$ (1)

ii) $N(4\cos\theta, 0)$, $\quad Q(4\cos\theta, 4\sin\theta)$ (2)

iii) at P $\quad x = 4\cos\theta$, $y = 3\sin\theta$

$\frac{dx}{d\theta} = -4\sin\theta$, $\quad \frac{dy}{d\theta} = 3\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$

$= -\frac{3\cos\theta}{4\sin\theta}$

$y - 3\sin\theta = -\frac{3\cos\theta}{4\sin\theta}(x - 4\cos\theta)$

$-4y\sin\theta + 12\sin^2\theta = 3x\cos\theta - 12\cos^2\theta$

$3x\cos\theta + 4y\sin\theta = 12(\sin^2\theta + \cos^2\theta)$

$\frac{x\cos\theta}{4} + \frac{y\sin\theta}{3} = 1$ ② (2)

at $Q(4\cos\theta, 4\sin\theta)$

$\frac{dx}{d\theta} = -4\sin\theta$, $\quad \frac{dy}{d\theta} = 4\cos\theta$

$\frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta}$

$y - 4\sin\theta = -\frac{\cos\theta}{\sin\theta}(x - 4\cos\theta)$

$-y\sin\theta + 4\sin^2\theta = x\cos\theta - 4\cos^2\theta$

$x\cos\theta + y\sin\theta = 4(\sin^2\theta + \cos^2\theta)$

$\frac{x\cos\theta}{4} + \frac{y\sin\theta}{4} = 1$ ① (2)

iv) ① × 16 $\quad 4x\cos\theta + 4y\sin\theta = 16$ ③

② × 12 $\quad 3x\cos\theta + 4y\sin\theta = 12$ ④

③ − ④ $\quad x\cos\theta = 4 \Rightarrow x = \frac{4}{\cos\theta}$

sub in ③ $\quad 16 + 4y\sin\theta = 16$ (1)

$4y\sin\theta = 0 \Rightarrow y = 0$ $\quad R\left(\frac{4}{\cos\theta}, 0\right)$

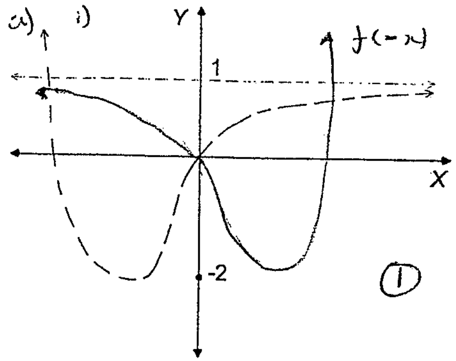
v) $ON = 4\cos\theta$, $\quad OR = \frac{4}{\cos\theta}$

$ON \times OR = 4\cos\theta \times \frac{4}{\cos\theta}$

$= 16$ (1)

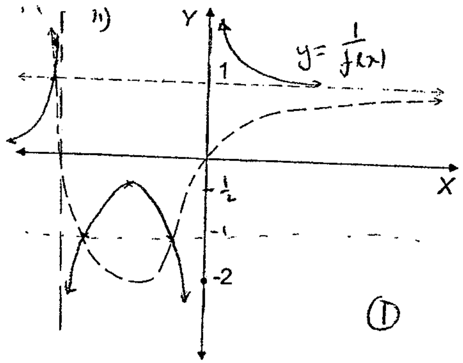
Question Four

a) i)



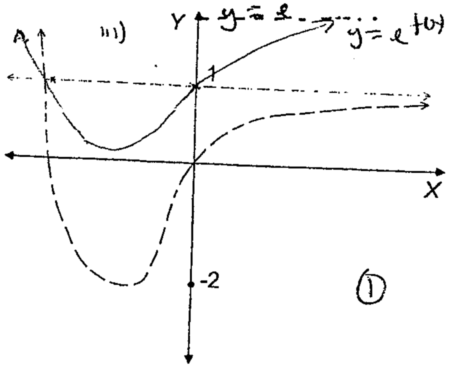
(1)

ii)



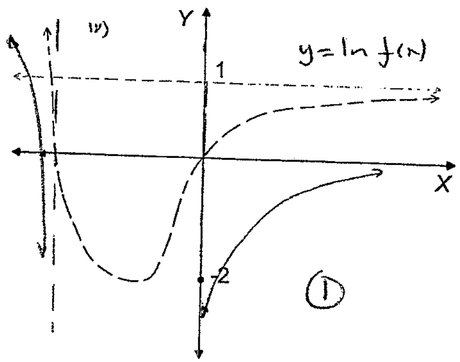
(1)

iii)



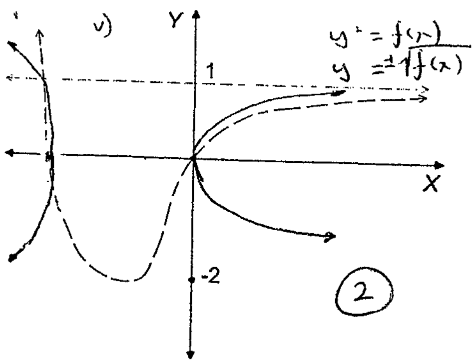
(1)

iv)



(1)

v)



(2)

b) i) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ [De Moivre's Th]

$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

Equate imaginary

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ (3)

Equate real

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

ii) $\dfrac{\sin 4\theta}{\cos 4\theta} = \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \begin{matrix}\div\cos^4\theta \\ \div\cos^4\theta\end{matrix}$

$\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ (1)

iii) $\tan 4\theta = 1$ for $0 \le \theta \le \pi$

$4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \ldots$ (1)

$\theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$

iv) put $\tan 4\theta = 1 = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

let $x = \tan\theta$ then $1 = \dfrac{4x - 4x^3}{1 - 6x^2 + x^4}$ (2)

$x^4 - 6x^2 + 1 = 4x - 4x^3$

$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

then the roots are $\tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$

v) let $\alpha = \tan\frac{\pi}{16}$, $\beta = \tan\frac{5\pi}{16}$, $\gamma = \tan\frac{9\pi}{16}$, $\delta = \tan\frac{13\pi}{16}$

require $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

and $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\sum\alpha\beta)$

$= (-4)^2 - 2(-6)$

$= 28$ (2)

Question Five

(a) (i) $x^3 - x^2 + 2x - 1 = 0$

1 $\alpha + \beta + \gamma = 1$ ∴

$\alpha + \beta = 1 - \gamma$

$-(\alpha + \beta) = \gamma - 1$

∴ Eqn with roots. $Y = -(\alpha + \beta)$

$Y = x - 1$

$x = Y + 1$.

∴ Eqn. $(Y+1)^3 - (Y+1)^2 + 2(Y+1) - 1 = 0$.

$Y^3 + 3Y^2 + 3Y + 1 - Y^2 - 2Y - 1 + 2Y + 2 - 1 = 0$.

$Y^3 + 2Y^2 + 3Y + 1 = 0$.

2 Eqn $x^3 + 2x^2 + 3x + 1 = 0$.

(ii) (a) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ ∴ $Y = \frac{1}{x} \Rightarrow x = \frac{1}{Y}$

$\left(\frac{1}{Y}\right)^3 - \left(\frac{1}{Y}\right)^2 + 2\left(\frac{1}{Y}\right) - 1 = 0$

$1 - Y + 2Y^2 - Y^3 = 0$

∴ $Y^3 - 2Y^2 + Y - 1 = 0$

2 Eqn. $x^3 - 2x^2 + x - 1 = 0$.

1 (b) Now $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = --2$ $\underline{\underline{= 2}}$

(b) $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$.

(i) Zeroes $a + bi$, $a - 2bi$ then $a - bi$ and $a + 2bi$ also zeros by property of conjugates.

Now $a + bi + a - bi + a + 2bi + a - 2bi = --4$

2 ∴ $4a = 4 \Rightarrow a = 1$.

Product $(a+bi)(a-bi)(a+2bi)(a-2bi) = 10$

$(a^2 + b^2)(a^2 + 4b^2) = 10$.

$(1 + b^2)(1 + 4b^2) = 10$. Put $B = b^2$

∴ $1 + 5B + 4B^2 = 10$

$4B^2 + 5B - 9 = 0 \Rightarrow (4B + 9)(B - 1) = 0$.

∴ $4B = -9$ $\quad B = 1$

2 no soln $\quad$ ∴ $b^2 = 1$ $\quad \underline{\underline{b = \pm 1}}$

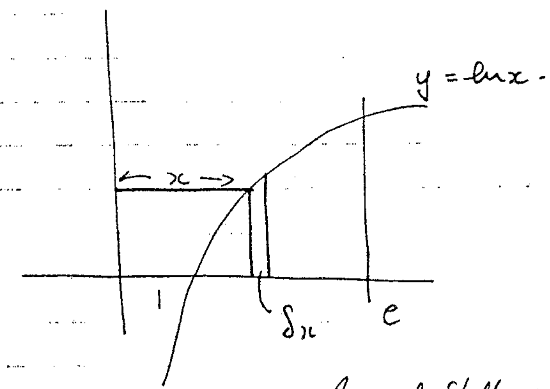
∴ Roots $a \pm bi$, $a \pm 2bi$

are $\underline{1 \pm i}$, $\underline{1 \pm 2i}$

(ii) Quadratic Factors

1 $P(x) = (x^2 - 2x + 2)(x^2 - 2x + 5)$

(c)



Area of Shell $= \pi\{(x + \delta x)^2 - x^2\}$

$= \pi\{x^2 + 2x\,\delta x + (\delta x)^2 - x^2\}$

$\doteq \pi\{2x\,\delta x\}$ as $\delta x \to$ is small

1 $(\delta x)^2$ is negligible.

∴ Volume of a shell.

$\delta V \doteq \pi\{2x \cdot \delta x\} \cdot y = \pi 2x \cdot y \cdot \delta x$.

∴ $V \doteq \sum_a^b \pi 2xy \cdot \delta x$

1 Now $\lim_{\delta x \to 0} V = \int_1^e 2\pi xy \, dx$ where $y = \ln x$.

∴ $V = 2\pi \int_1^e x \ln x \, dx$.

$V = 2\pi \left[\ln x \cdot \frac{x^2}{2}\right]_1^e - 2\pi \int_1^e \frac{1}{x} \cdot \frac{x^2}{2} \cdot dx$.

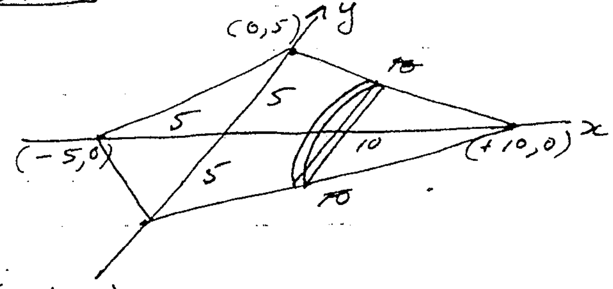
1 $V = 2\pi \left[\frac{x^2}{2} \cdot \ln x\right]_1^e - 2\pi \left[\frac{x^2}{4}\right]_1^e$

$V = 2\pi \left[\frac{e^2}{2} \cdot 1\right] - [0] - 2\pi \left[\frac{e^2}{4} - \frac{1}{4}\right]$

1 $\underline{\underline{V = \frac{\pi e^2}{2} + \frac{\pi}{2}}}$ c.u.

Question Six

(a).



For V_1, $x>0$

Area of a slice $=\frac{1}{2}\pi r^2$ $\quad r=y$

1 $=\frac{1}{2}\pi y^2$

Now $y-0=\frac{5-0}{0-10}(x-10)$

$y=-\frac{1}{2}(x-10)$

$2y=-x+10$

1 $y=\frac{10-x}{2} \Rightarrow y^2=\left(\frac{10-x}{2}\right)^2$

$\therefore A=\frac{1}{2}\pi\left(\frac{10-x}{2}\right)^2$

from 0 to 10.

1 $V_1=\int_0^{10}\frac{\pi}{2}\left(\frac{10-x}{2}\right)^2 dx = \frac{\pi}{8}\int_0^{10}(10-x)^2 dx = \frac{\pi}{24}\left[(10-x)\right]_0^{10}$

$=\frac{\pi}{24}\times 10^3$.

For V_2 $x<0$. area of slice $=\frac{1}{2}\pi R^2$. $R=y$.

$=\frac{1}{2}\pi y^2$.

Now $y-0=\frac{5-0}{0--5}(x--5)$

1 $y=1(x+5)$

$y^2=(x+5)^2$

$\therefore V_2=\int_{-5}^{0}\frac{1}{2}\pi(x+5)^2 dx = \frac{\pi}{6}\left[(x+5)^3\right]_{-5}^{0}$

1 $=\frac{\pi}{6}(5)^3-0$.

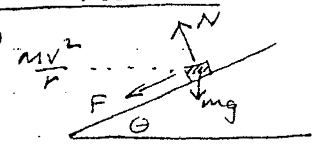
$\therefore V=V_1+V_2=\frac{\pi}{24}\times 10^3+\frac{\pi}{6}\times 5^3$

6 $=\frac{\pi}{24}\{1000+500\}=\frac{1500\pi}{24}$

$=\frac{500\pi}{8}$ c.u. $=\frac{125\pi}{2}$ c.u.

Question Six

(b)



Vertically: $N\cos\theta = F\sin\theta + mg$ —(1)

Horizontally: $N\sin\theta + F\cos\theta = \frac{mv^2}{r}$ —(2)

(1)× by $\sin\theta$ (2) by $\cos\theta$

$N\cos\theta\sin\theta = F\sin^2\theta + mg\sin\theta$ —(3)

$N\sin\theta\cos\theta + F\cos^2\theta = m\frac{v^2}{r}\cos\theta$, —(4)

$\therefore F\sin^2\theta + mg\sin\theta + F\cos^2\theta = m\frac{v^2}{r}\cos\theta$

2 * $F = m\frac{v^2}{r}\cos\theta - mg\sin\theta = m\left(\frac{v^2}{r}\cos\theta - g\sin\theta\right)$

(1) ×$\cos\theta$ (2) by ×$\sin\theta$

$N\cos^2\theta = F\cos\theta\sin\theta + mg\cos\theta$. —(5)

$N\sin^2\theta + F\cos\theta\sin\theta = m\frac{v^2}{r}\sin\theta$ —(6).

(5) +(6).

2 $\therefore N = m\frac{v^2}{r}\sin\theta + mg\cos\theta = m\left[\frac{v^2}{r}\sin\theta + g\cos\theta\right]$

Given $r=1000$ and $\tan\theta=\frac{1}{100}$ find v when $F=0$

$\therefore \frac{v^2}{r}\cos\theta = g.\sin\theta$.

$v^2 = rg.\tan\theta$. $\quad g=10$

$\therefore v^2 = 1000\times 10\times\frac{1}{100}$

$v^2=100$

1 $\therefore v = 10$ m/s

(c) $x^3+3kx+l=0$.

R.T.P. $l^2+4k^3=0$.

4 Let $x=\alpha$ be the double root

$\therefore \alpha^3+3k\alpha+l=0$

differentiate: $3x^2+3k=0$

$\therefore 3\alpha^2+3k=0 \Rightarrow \alpha^2+k=0$

$\therefore -k\alpha+3k\alpha+l=0$. $\quad \alpha^2=-k$

$2k\alpha = l$. $\quad \alpha^3=-k\alpha$.

$4k^2\alpha^2=l^2$ and $\alpha^2=-k$.

$\therefore 4k^2\times -k = l^2$

$4k^3+l^2=0$

Question 7

(a) (i) $\sin(A-B) + \sin(A+B)$

$= \sin A\cos B - \cos A\sin B + \sin A\cos B + \cos A\sin B.$

1

$= 2\sin A\cos B.$

Now $A = 5x$ $B = 3x$.

(ii)

$$I = \int_0^{\frac{\pi}{4}} \sin 5x \cdot \cos 3x = \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sin(2x) + \sin 8x)\, dx$$

$$I = \left[-\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x\right]_0^{\frac{\pi}{4}}$$

2

$$I = \left[-\frac{1}{4}\cos\frac{\pi}{2} - \frac{1}{16}\cos 2\pi\right] - \left[-\frac{1}{4}\cos 0 - \frac{1}{16}\cos 0\right]$$

$$I = \frac{1}{4} \cdot -\frac{1}{16} + \frac{1}{16} = \frac{1}{4}$$

(b) $f(x) = \frac{\ln x}{x}$

(i) $f'(x) = \frac{x \times \frac{1}{x} - 1 \times \ln x}{x^2} = \frac{1 - \ln x}{x^2}.$

$f'(x) = 0$ for st-pt. $\therefore \ln x = 1$ $\therefore x = e.$

1

Point $(e, \frac{1}{e})$.

For $x < e$ $f'(x) > 0$ for $x > e$ $f'(x) < 0$.

$\therefore$ MAXIMUM at $(e, \frac{1}{e})$.

(ii) Now max at $\{e, f(e)\}$

$\therefore f(e) > f(\pi)$. or. $f'(\pi) = \frac{1 - \ln\pi}{\pi^2}$

$= -\cdot 0147$

1

$\therefore$ Curve is decreasing at $x = \pi$

$\therefore f(e) > f(\pi)$

(iii)

from $f(e) > f(\pi)$ $\frac{\ln e}{e} > \frac{\ln\pi}{\pi}$

$\Rightarrow \frac{1}{e} > \frac{\ln\pi}{\pi}$

2

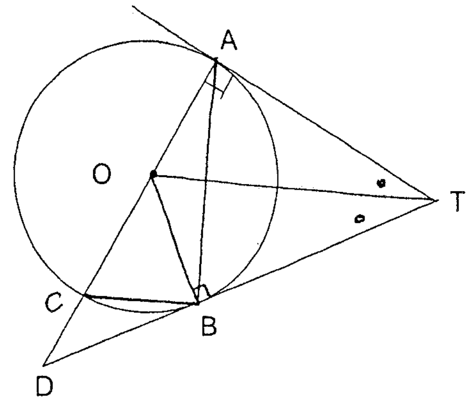
$\therefore \pi > e\ln\pi$

$\pi > \ln\pi^e$

$e^\pi > e^{\ln\pi^e}$

$\therefore e^\pi > \pi^e$

Question 7 (c)



Join BO, OT, CB, AB

4

(i) Proof:

OATB is a cyclic quad. $\angle A + \angle B = 180°$.

$\angle OTA = \angle OTB$

(equal chords support equal angles).

$\angle CBD = \angle DAB$ ($\angle$ in alt. segment)

$\angle OTB = \angle DAB$ ($\angle$s in same segment.

$\therefore \angle CBD = \angle OTB$

i.e. $\angle CBD = \frac{1}{2}\angle ATB$.

(ii). R.T.P. $\triangle ABC \,|||\, \triangle TBO$.

2

1. $\angle ABC = \angle OBT = 90°$ ($\angle$ in semi-circle is $90°$ radius meets tangent at $90°$)

2. $\angle BAC = \angle OTB$ (shown in (i))

3. $\angle ACB = \angle TOB$ ($\angle$ sum of $\triangle$)

$\therefore$ $\triangle ABC \,|||\, \triangle TBO$ (equiangular)

(iii)

Now $\frac{BC}{OB} = \frac{AC}{OT}$ (corr sides of similar $\triangle$s are in ratio)

2

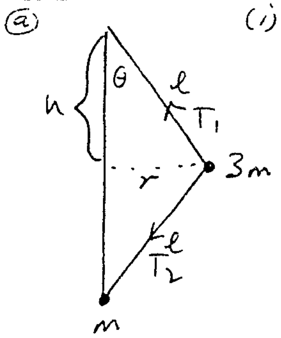
$\therefore BC.OT = OA.AC$ But $AC = 2OA$

$\therefore BC.OT = OA.2OA$

$BC.OT = 2(OA)^2$

Question 8

(a)



(i) For the 3m mass.

Vertically:

$T_1 \cos\theta = T_2 \cos\theta + 3mg$ (1).

Horizontally

$T_1 \sin\theta + T_2 \sin\theta = 3mr\omega^2$ (2).

For the m mass

Vertically

$T_2 \cos\theta = mg$ (3).

$T_2 = \frac{mg}{\cos\theta}$

Put (1) and (3) together.

$\therefore T_1 \cos\theta = 4mg$

$T_1 = \frac{4mg}{\cos\theta}$

Now in (2). $r = \ell\sin\theta$.

$\therefore T_1 \sin\theta + T_2 \sin\theta = 3m\ell.\sin\theta.\omega^2$

$T_1 + T_2 = 3m\ell\omega^2 \Rightarrow \frac{4mg}{\cos\theta} + \frac{mg}{\cos\theta} = 3m\ell\omega^2$

But $\cos\theta = \frac{h}{\ell}$

$h = \ell\cos\theta$.

$\Rightarrow \therefore 5mg = 3m\ell\omega^2\cos\theta$.

$5mg = 3m\omega^2 h$.

4 $\Rightarrow h_1 = \frac{5g}{3\omega^2}$.

(ii) Masses are reversed. and rotating mass speed increased to 2ω.

Rotating Mass

Vertically.

$T_1 \cos\theta = T_2 \cos\theta + mg$. —(1)

Horizontally

$T_1 \sin\theta + T_2 \sin\theta = mr(2\omega)^2$. (2)

Fixed Mass

$T_2 \cos\theta = 3mg$. (3)

$r = \ell\sin\theta$.

(1) and (3).

$\therefore T_1 \cos\theta = 4mg. \Rightarrow T_1 = \frac{4mg}{\cos\theta}$

$T_2 \cos\theta = 3mg \Rightarrow T_2 = \frac{4mg}{\cos\theta}$,

$\therefore$ in (2)

$T_1 + T_2 = 4m\ell\omega^2$.

$\frac{7mg}{\cos\theta} = 4m\ell\omega^2$.

$\ell\cos\theta = h$

$7mg = 4m\omega^2 h$.

2 $\therefore h_2 = \frac{7g}{4\omega^2}$.

Now: $h_1 = \frac{5}{3}\frac{g}{\omega^2} \Rightarrow h_2 = \frac{7g}{4\omega^2} \Rightarrow h_1 : h_2 = \frac{5}{3} : \frac{7}{4}$

1 $\therefore h_2 > h_1$

Question 8

(b) (i) Choose 5 then choose any then choose match

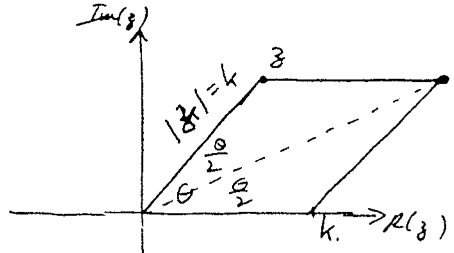
$P(T=15) = \frac{1}{1} \times \frac{8}{8} \times \frac{1}{7} = \frac{1}{7}$

2 (ii) Choose 2 then choose any match then choose match

$P(T=15) = \frac{1}{1} \times \frac{6}{8} \times \frac{1}{7} = \frac{3}{4} \times \frac{1}{7} = \frac{3}{28}$.

(c) $z = k(\cos\theta + i\sin\theta)$. show $\arg(z+k) = \frac{\theta}{2}$.

Method 1



3 Sketch z.

then add k.

Now have a rhombus.

$\therefore$ diagonal from O to z+k. bisects vertex angle. $\therefore$

$\therefore \arg(z+k) = \frac{\theta}{2}$.

Method 2

$z = k\,\text{cis}\,\theta$

$z + k = k + k\,\text{cis}\,\theta$

$= k(1 + \text{cis}\,\theta)$

$= k(1 + \cos\theta + i\sin\theta)$

$= k(1 + 2\cos^2\frac{\theta}{2} - 1 + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2})$

$= k(2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2})$

$= 2k\cos\frac{\theta}{2}\{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\}$.

$\therefore$ Modulus of $z+k$

$|z+k| = 2k\cos\frac{\theta}{2}$.

$\arg(z+k) = \frac{\theta}{2}$.

(d) Given $I_n = \frac{1}{n!}\int_0^a x^n.e^{-x}.dx$ show $\frac{a^n}{n!} = e^a(I_{n-1} - I_n)$.

Let $u = x^n$ $\quad \frac{dv}{dx} = e^{-x}$

$du = nx^{n-1}$ $\quad v = -e^{-x}$

Now $I_n = \frac{1}{n!}\left[x^n.-e^{-x}\right]_0^a - \frac{1}{n!}\int_0^a -e^{-x}.n.x^{n-1}.dx$.

$I_n = \frac{1}{n!}\left[-x^n.e^{-x}\right]_0^a + \frac{n}{n!}\int_0^a x^{n-1}.e^{-x}.dx$.

3 $I_n = \frac{1}{n!}\left[-x^n.e^{-x}\right]_0^a + \frac{1}{(n-1)!}\int_0^a x^{n-1}.e^{-x}.dx$.

$I_n = \frac{1}{n!}\left[-x^n.e^{-x}\right]_0^a + I_{n-1} = \frac{1}{n!}\left[-a^n.e^{-a}\right] - \frac{1}{n!}[0] + I_{n-1}$

$\therefore \frac{1}{n!}\left[a^n e^{-a}\right] = I_{n-1} - I_n$.

$\frac{a^n}{n!} = e^a(I_{n-1} - I_n)$.