Sydney Girls High School

2005
TRIAL HIGHER SCHOOL
CERTIFICATE EXAMINATION

Mathematics
Extension 2

This is a trial paper ONLY. It does not necessarily reflect the format or the contents of the 2005 HSC Examination Paper in this subject.

General Instructions

- Reading Time – 5 mins
- Working time – 3 hours
- Attempt ALL questions
- ~~ALL questions are of equal value~~
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used
- Diagrams are not to scale
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

Question One (15 marks) **Marks**

a) Use a substitution to find $\int \frac{\log_e x \, dx}{x}$ 1

b) Use integration by parts to find $\int xe^{-2x}dx$ 2

c) Find $\int \sin^3 x dx$ 2

d) Find real numbers A, B and C such that: 2

i) $\frac{2x-x^2}{(x^2+4)(x+2)} = \frac{Ax+B}{(x^2+4)} - \frac{C}{(x+2)}$

ii) Hence find $\int \frac{2x-x^2}{(x^2+4)(x+2)}dx$ 2

e) Find $\int \frac{dx}{\sqrt{8-2x-x^2}}$ 2

f) Find $\int_0^2 \frac{x^2}{\sqrt{16-x^2}}\, dx$ using the substitution $x = 4\sin\theta$ 4

Question Two (15 marks) **Marks**

a) Given $z = 3+4i$ and $w = a+bi$ where a and b are real.
Find the following:
   i) $|z|$ 1
   ii) zw in the form $x+iy$ 1
   iii) $\dfrac{w}{z}$ in the form $x+iy$ 1

b) On an Argand diagram shade the region containing all points z representing complex numbers such that $|z| \le 2$ and $-1 \le \text{Im}(z) \le 1$ 2

c) Given $z = -\sqrt{3}+i$
   i) Express z in modulus argument form 2
   ii) Hence find $\dfrac{z}{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}$ in modulus argument form 1

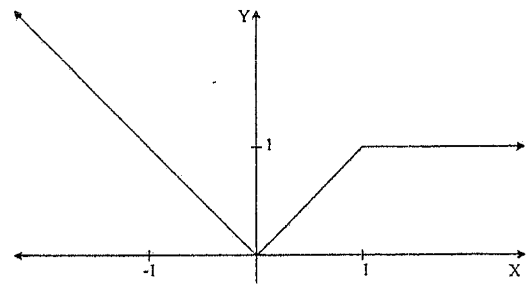
d) Find the complex cube roots of -1 in modulus argument form 2

e) Find the square roots of $15-8i$, giving your answer in the form $x+iy$ 2

f) i) Sketch the locus given by $\text{Re}(z-2+i) = 2$ 2
   ii) Hence find the minimum value of $|z|$ subject to the above condition. 1

Question Three (15 marks) **Marks**

a) The diagram below shows the graph of $y = f(x)$



Draw separate one- third page sketch graphs of:
   i) $y = \dfrac{1}{f(x)}$ 2
   ii) $y = [f(x)]^2$ 2
   iii) $y = \log_e f(x)$ 2
   iv) $y = e^{f(x)}$ 2

b) Sketch(without using calculus) the curve $y = \dfrac{x+1}{x^2+2x}$ showing all asymptotes 3

c) The parametric equations of an ellipse are $x = 3\cos\theta$; $y = 2\sin\theta$
   i) Find the Cartesian equation 1
   ii) Find the eccentricity 1
   iii) Find the co ordinates of the foci 1
   iv) Find the equations of the directrixes 1

Question Four (15 marks) **Marks**

a) The base of a solid is the region in the x-y plane enclosed by the lines $y = x$, $y = -x$ and $y = 4$. Each cross section perpendicular to the y-axis is a semi circle

   i) Find the area of the cross section in terms of y 1

   ii) Find the volume of the solid 2

b) Factorise $x^4 + 3x^2 + 2$ over the complex field 3

c) i) Use mathematical induction to prove De Moivre's Theorem that 3

   $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for n positive

   ii) Show that if $z = (\cos\theta + i\sin\theta)$ then 2

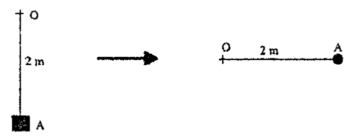
   $z^n + z^{-n} = 2\cos n\theta$

   iii) Hence or otherwise solve $z^4 + 2z^3 + 3z^2 + 2z + 1 = 0$ 4

Question Five (15 marks) **Marks**

a) A 0.5kg mass is attached to a 2m piece of string and hung vertically The string can just sustain the mass.

   i.) Find the tension in the string (use g = 9.8) 1



The mass at A is replaced with a 0.2kg mass. The string is then rotated in a horizontal circle about O. The new tension in the string is 1.6 Newtons.

   ii.) Find the speed of the mass in ms^{-1} 1

   iii.) Find the angular velocity of the mass. 1

   iv.) Find the maximum speed(in ms^{-1}) at which the mass could be rotated without breaking the string. 1

b) The diagram below shows the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent to the hyperbola at $P(x_1, y_1)$ meets the directrix d at the point X. A line is drawn from point X to the focus S.

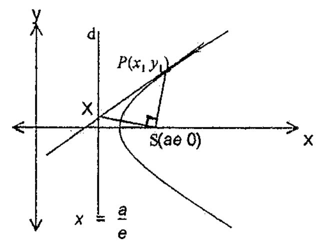


Diagram not to scale

   i.) Show that the equation of the tangent at P is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ 3

   ii.) Find the co ordinates of X 2

   iii.) Show that $\angle PSX = 90°$ 3

c) Find the equation of the tangent to the curve $7y^4 + x^3y + x = 4$ at the point with co ordinates (4, 0) 3

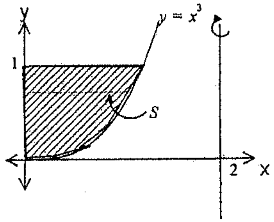
Question Six (15 marks) **Marks**

a) The roots of the polynomial equation $2x^3 - 8x^2 + 3x + 5 = 0$ are α, β, γ.

i) Find the polynomial equation with roots $\alpha^2, \beta^2, \gamma^2$ 2

ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$ 1

b) In the diagram the shaded region is bounded by $y = 1$, $x = 0$ and the curve $y = x^3$

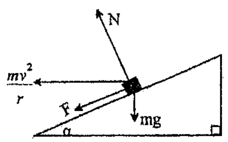


The area is rotated about the line $x = 2$ to form a solid. When the region is rotated, the line segment S at height y sweeps out an annulus.

i) Show that the area of the annulus at height y is equal to $\pi\left(4y^{\frac{1}{3}} - y^{\frac{2}{3}}\right)$ 2

ii) Hence or otherwise find the volume of the solid 2

c) A railway line is banked at an angle of α^0 when it is on a bend, which is part of a circular arc. A train of mass m kg is traveling at v ms^{-1} around the bend. N represents the Normal force and F represents the sideways Frictional force. The circular arc is radius r metres.



i.) By resolving horizontal and vertical components of the forces, find expressions for $F\cos\alpha$ and $F\sin\alpha$ 2

ii.) Find an expression for F independent of N 2

iii.) Give that the radius of the circular arc is 2 km and the train is designed to travel at 180 km/h, find the value of α^0 which will ensure that there is no sideways friction force. (Assume $g = 10ms^{-2}$) 2

d) Sketch (without using calculus) the graph of $y = \frac{\sin x}{x}$ for $0 \le x \le 3\pi$ 2

Question Seven (15 marks) **Mark**

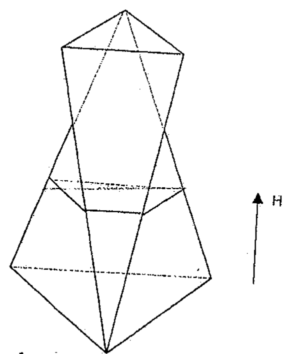
a) i) If α is a multiple root of the polynomial equation $P(x) = 0$, prove that $P'(\alpha) = 0$ 2

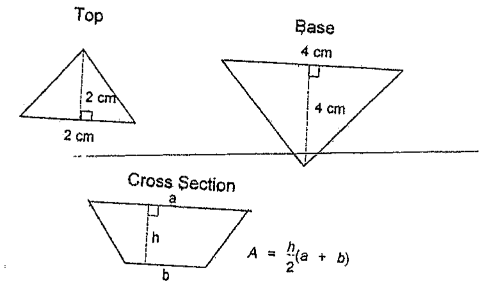
ii) The polynomial equation $Q(x) = x^4 + 2x^3 - ax^2 + bx + 12$ has a double root at $x = -2$. Find the values of a and b 3

iii) Factorise $Q(x)$ over the real field. 2

iv) Factorise $Q(x)$ over the complex field 1

b) A saltshaker has a triangular base and top as shown below. The height of the shaker is 10 cm. All other dimensions are shown below on the right





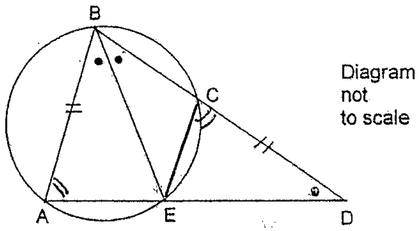
Each cross section parallel to the base is an isosceles trapezium

i) Show that the area of the cross section at height H cm from the base is given by: $\left[8 - \frac{4H}{5} + \frac{H^2}{50}\right]$ 5

ii) Calculate the volume of solid 2

Question Eight (15 marks) **Marks**

a) In the diagram BE bisects $\angle ABD$, and CD = AB



i) Prove that ΔCED is congruent to ΔBAE 2

ii) Prove that $\angle BEA = 2 \times \angle EDB$ 2

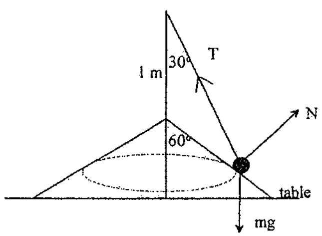
b) Given that the roots of the quadratic equation
$(1+i)z^2 - (\sqrt{3}+i)z + (-1+\sqrt{3}i) = 0$ are z_1 and z_2 find:

i) $|z_1 + z_2|$ 1

ii) $\arg z_1 + \arg z_2$ 1

c) If the line $px + qy + r = 0$ is a tangent to the hyperbola $xy = c^2$ 3
Prove that $4c^2 pq = r^2$

d) An inverted cone of semi-vertex angle 60^0 is placed on a table. From a point 1m above the apex of the cone a 2 kg mass is suspended on a light string. The string makes an angle of 30^0 with the vertical and the mass rotates around the smooth surface of the cone in circular motion.



i.) If the mass is rotating at a constant angular velocity of one radian per second, find the values of T (Tension in the string) and N (Normal reaction force). Take $g = 10ms^{-2}$ 4

ii.) The angular velocity of the rotating mass is increased until it is about to lose contact with the cone. Find the value of this angular velocity when contact with the cone is about to be lost 2

$\alpha\beta\chi\delta\varepsilon\phi\varphi\gamma\eta\iota\kappa\lambda\mu\nu o\pi\varpi\theta\vartheta\rho\sigma\varsigma\tau\upsilon\omega\xi\psi$

S.G.H.S Extn 2 Trial Solutions 2005

Question One

(a) $I = \int \frac{\ln x}{x}\,dx$ let $u = \ln x$, $du = \frac{1}{x}\,dx$.

1

$I = \int u\,du$

$I = \frac{u^2}{2} \Rightarrow \frac{(\ln x)^2}{2} + c$

(b). $I = \int x.e^{-2x}\,dx$. let $u = x$, $du = dx$. $dv = e^{-2x}$, $v = -\frac{1}{2}e^{-2x}$.

$I = [-\frac{1}{2}.x.e^{-2x}] - \int -\frac{1}{2}.e^{-2x}\,dx$.

2

$I = -\frac{1}{2}x.e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$

$I = -\frac{1}{2}x.e^{-2x} - \frac{1}{4}.e^{-2x} + c$

(c) $I = \int \sin^3 x\,dx$.

$I = \int \sin x(1 - \cos^2 x).dx$.

$I = \int \sin x.\,dx - \int \cos^2 x.\sin x.dx$

2

$I = -\cos x. - I_2$ $\quad I_2 = \int \cos^2 x.\sin x.dx$

let $u = \cos x$

$I = -\cos x + \frac{\cos^3 x}{3} + c$

$du = -\sin x.dx$

$I_2 = -\int u^2.du = -\frac{u^3}{3} + c.$

(d) (i) $\frac{2x - x^2}{(x^2+4)(x+2)} = \frac{Ax + B}{x^2+4} - \frac{C}{x+2}$.

$2x - x^2 = (Ax+B)(x+2) - C(x^2+4)$

$x = -2$.

$-4 - 4 = 0 - 8C \Rightarrow C = 1$.

Equate x^2 $\quad -1 = A - 1 \Rightarrow A = 0$.

Equate x $\quad B = 2$.

2

$\therefore \frac{2x - x^2}{(x^2+4)(x+2)} = \frac{2}{x^2+4} - \frac{1}{x+2}$

(ii) $\int \frac{2x - x^2}{(x^2+4)(x+2)}.dx = \int\left(\frac{2}{x^2+4} - \frac{1}{x+2}\right).dx$

$= \frac{2.1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \ln|x+2| + c$

2

$= \tan^{-1}\left(\frac{x}{2}\right) - \ln|x+2| + C$

(e) $\int \frac{dx}{\sqrt{8 - 2x - x^2}} = \int \frac{dx}{\sqrt{8 - (x^2 + 2x + 1) + 1}}$

2

$= \int \frac{dx}{\sqrt{9 - (x+1)^2}} = \int \frac{dx}{\sqrt{3^2 - (x+1)^2}}$

$I = \sin^{-1}\left(\frac{x+1}{3}\right) + c$

(f) $I = \int_0^1 \frac{x^2}{\sqrt{16 - x^2}}\,dx$. let $x = 4\sin\theta$. $x = 2$, $2 = 4\sin\theta$

1 $\quad dx = 4\cos\theta.d\theta$ $\quad \sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}$

1 $\quad I = \int \frac{16\sin^2\theta.4.\cos\theta.d\theta}{\sqrt{16 - 16\sin^2\theta}} = \int 16\sin^2\theta\,d\theta$.

4

$= \int \frac{1}{2}.16.(1 - \cos 2\theta).d\theta$.

$= 8\int(1 - \cos 2\theta).d\theta$.

1 $\quad = 8\theta - 4\sin 2\theta$.

$= 8\times\frac{\pi}{6} - 4.\sin\frac{\pi}{3}$.

1 $\quad = \frac{4\pi}{3} - 4\times\frac{\sqrt{3}}{2} = \frac{4\pi}{3} - 2\sqrt{3}$

Question Two

(a) $z = 3+4i \quad w = a+ib.$

1 (i) $|z| = \sqrt{3^2+4^2} = 5.$

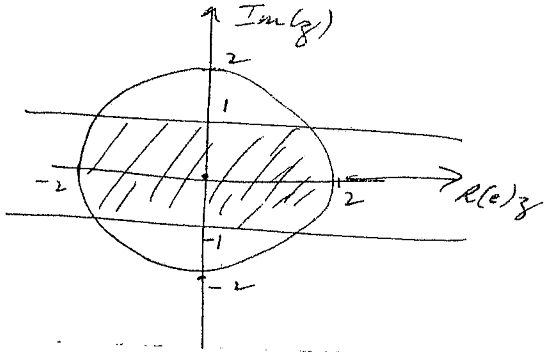
1 (ii) $zw = (3+4i)(a+ib) = 3a-4b+i(4a+3b)$

1 (iii) $\dfrac{w}{\bar{z}} = \dfrac{a+ib}{3-4i} = \dfrac{a+ib}{3-4i} \times \dfrac{3+4i}{3+4i}$

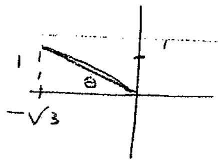
$= \dfrac{3a-4b+i(4a+3b)}{25}$

(b)

2



(c) $z = -\sqrt{3}+i$



$\tan\theta = \dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6}.$

$|z| = \sqrt{3+1} = 2.$

2 (i) $z = 2\,\text{cis}\,\dfrac{5\pi}{6}.$

(ii) $\dfrac{z}{\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})} = \dfrac{2\,\text{cis}\,\frac{5\pi}{6}}{\sqrt{2}\,\text{cis}(\frac{\pi}{4})}$

1 $= \sqrt{2}\,\text{cis}\left(\dfrac{5\pi}{6}-\dfrac{\pi}{4}\right) = \sqrt{2}\,\text{cis}\,\dfrac{20\pi-6\pi}{24}$

$= \sqrt{2}\,\text{cis}\,\dfrac{7\pi}{12}$

(d) $z^3 = -1.$ Let $z = \cos\theta + i\sin\theta, = \text{cis}\,\theta.$

$\therefore \text{cis}\,3\theta = \text{cis}(\pi+2k\pi)$ By De Moivre's Thm.

$\therefore 3\theta = \pi+2k\pi \quad k = 0,1,2.$

$\theta = \dfrac{\pi+2k\pi}{3} \quad k=0\ \theta = \dfrac{\pi}{3} \quad k=1\ \theta = \pi$

$k=2\ \theta = \dfrac{5\pi}{3}.$

2 $\therefore$ roots are $z = \text{cis}\,\dfrac{\pi}{3},\ \text{cis}\,\pi = -1,\ \text{cis}\,\dfrac{5\pi}{3}.$

(e) Let $x+iy = \sqrt{15-8i}$

$\therefore x^2-y^2+2xyi = 15-8i$

2 $x^2-y^2 = 15 \qquad xy = -4.$

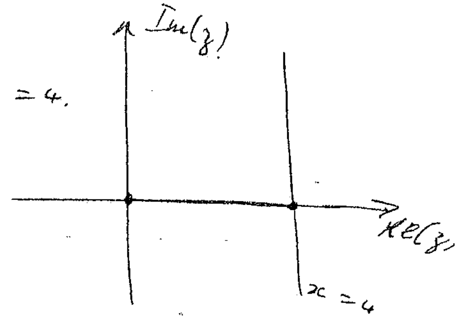
$x=4, y=-1 \qquad x=-4, y=1.$

$\therefore$ roots are $\pm(4-i)$

(f) (i) $\text{Re}(z-2+i) = 2.$

2 $\therefore \text{Re}(x-2+i(y+1)) = 2.$

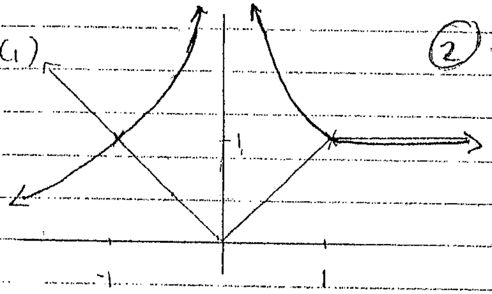
$\therefore x-2 = 2 \Rightarrow x = 4.$



1 (ii) $\min |z| = 4$

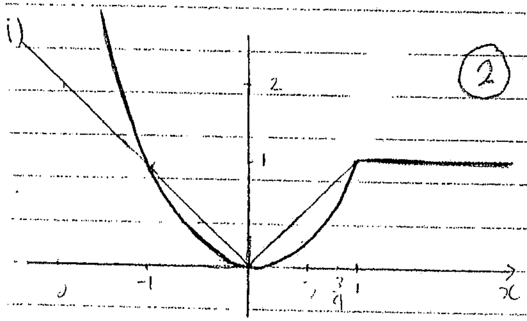
QUESTION 3 (15 marks)

a)(i) 2



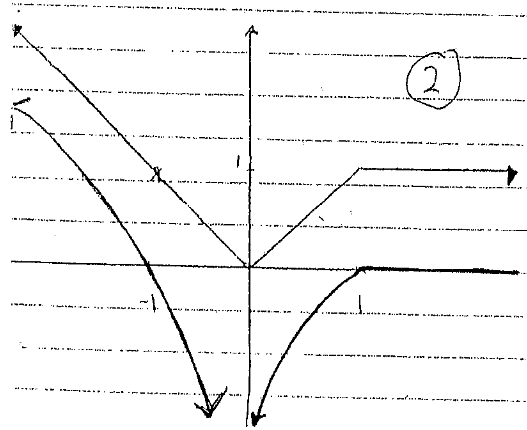
$y = \frac{1}{f(x)}$

i) 2

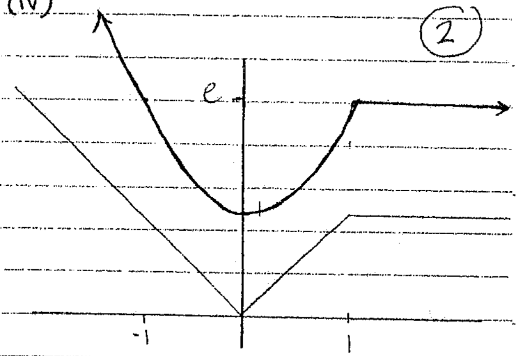


$y = [f(x)]^2$

ii) $y = \log_e f(x)$ 2



(iv) 2



(b) $y = \frac{x+1}{x^2+2x}$

$= \frac{x+1}{x(x+2)}$

Asymptotes: V: $x = 0, -2$

H: $y = 0$

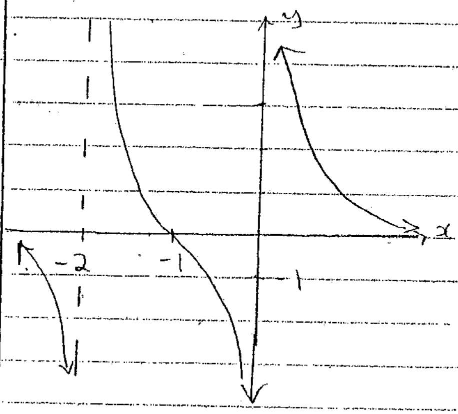
(since as $x \to \pm\infty$ $y = \frac{0}{1} = 0$)

x-intercepts (let $y = 0$)

$0 = \frac{x+1}{x(x+2)}$

$-1 = x$

3



C)(i) $x = 3\cos\theta$

$y = 2\sin\theta$

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$ 1

ii) $e = \sqrt{1 - \frac{b^2}{a^2}}$

$= \sqrt{1 - \frac{4}{9}}$

$= \sqrt{\frac{5}{9}}$ 1

$= \frac{\sqrt{5}}{3}$

iii) foci, $S = (\pm ae, 0)$

$= (\pm 3 \cdot \frac{\sqrt{5}}{3}, 0)$

$= (\pm\sqrt{5}, 0)$ 1

iv) directrices

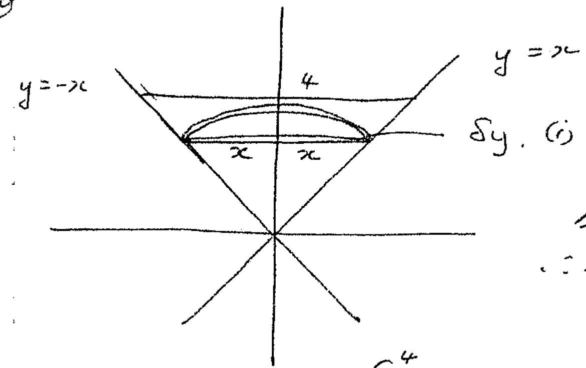
$x = \pm\frac{a}{e}$

$= \pm(3 \div \frac{\sqrt{5}}{3})$ 1

$= \pm\frac{9}{\sqrt{5}}$

Question Four

(a)



δy. (i) Area of semi-circle $= \frac{1}{2}\pi x^2$

But $y = x$

$\therefore$ Area of cross section $= \frac{1}{2}\pi y^2$.

1

(ii) Volume $V = \int_0^4 \frac{1}{2}\pi y^2 \cdot dy = \frac{1}{2}\pi \int_0^4 y^2 \cdot dy$

2 $= \left[\frac{1}{2}\pi \frac{y^3}{3}\right]_0^4 = \frac{32\pi}{3}$ c.u.

(b) $x^4 + 3x^2 + 2$

$= (x^2+2)(x^2+1)$ 1

3 $= (x+\sqrt{2}i)(x-\sqrt{2}i)(x+i)(x-i)$

(c) (i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Step 1: show true for $n=1$

L.H.S: $\cos\theta + i\sin\theta$, R.H.S. $= \cos\theta + i\sin\theta$.

Step 2: Assume true for $n=k$ (where k is a +ve integer

$\therefore (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$.

Step 3 Prove true for $n = k+1$.

L.H.S. $= (\cos\theta + i\sin\theta)^{k+1}$ R.H.S. $= \cos(k+1)\theta + i\sin(k+1)\theta$.

$= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$ from the assumption

$= \cos k\theta.\cos\theta - \sin k\theta.\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$

$= \cos(k+1)\theta. + i\sin(k+1)\theta$.

$=$ R.H.S.

Step 4 It is true for $n=k+1$ if true for $n=k$.

3 It is true for $n=1$ so true for $n=2$, and true for $n=3$ and so on for all +ve integer values of n.

(ii) $z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$.

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$z^{-n} = \cos n\theta - i\sin n\theta$ (cos is even, sine is odd)

2 $\therefore z^n + z^{-n} = 2\cos n\theta$

(iii) $z^4 + 2z^3 + 3z^2 + 2z + 1 = 0$.

$z^2(z^2 + 2z + 3 + 2z^{-1} + z^{-2}) = 0$ $z^2 = 0$ $z = 0$

$z^2 + z^{-2} = 2\cos 2\theta$.

$2(z + z^{-1}) = 2\times 2\cos\theta$.

$\therefore 2\cos 2\theta + 4\cos\theta + 3 = 0$.

$2\{2\cos^2\theta - 1\} + 4\cos\theta + 3 = 0$

$4\cos^2\theta + 4\cos\theta + 1 = 0$.

$(2\cos\theta + 1)^2 = 0$.

$2\cos\theta = -1$

$\cos\theta = -\frac{1}{2}$. $\therefore \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$.

$\therefore z = \text{cis}\,\frac{2\pi}{3}$ $z = \text{cis}\,\frac{4\pi}{3}$

4 $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Question Five (1)

a) i) $T = mg$

$= 9.8 \times 0.5$

$= \underline{4.9}$ ①

ii) $T = \frac{mv^2}{r}$ $m = 0.2$

$1.6 = \frac{0.2v^2}{2}$

$3.2 = 0.2v^2$

$v^2 = 16$

$v = \underline{4ms^{-1}}$ ①

iii) $F = mrw^2$

$1.6 = 0.2 \times 2w^2$

$1.6 = 0.4w^2$

$\underline{w = 2 \text{ rad/sec.}}$ ①

iv) $4.9 = \frac{0.2v^2}{2}$

$9.8 = 0.2v^2$

$v^2 = 49$

$v = \underline{7 ms^{-1}}$ ①

b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{b^2x_1}{a^2y_1}$ ①

$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$

$\frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{x_1^2}{a^2}$ ①

$\frac{xx_1}{} - \frac{yy_1}{} = \frac{x_1^2}{} - \frac{y_1^2}{}$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Question Five (2)

ii) On directrix $x = \frac{a}{e}$

(2)

$\frac{x_1}{ae} - \frac{yy_1}{b^2} = 1$ ①

$\frac{x_1}{ae} - 1 = \frac{yy_1}{b^2}$

$\frac{x_1 - ae}{ae} = \frac{yy_1}{b^2}$

$y = \frac{x_1 - ae}{ae} \times \frac{b^2}{y_1}$ ①

ie $X\left(\frac{a}{e}, \frac{x_1 - ae}{ae}\cdot\frac{b^2}{y_1}\right)$

iii) Gradient PS $\frac{y_1}{x_1 - ae}$ ① $S(ae, 0)$

(3)

Gradient SX $\frac{\frac{x_1 - ae}{ae}\frac{b^2}{y_1} - 0}{\frac{a}{e} - ae}$

$\frac{\frac{(x_1 - ae)}{ae}\frac{a^2(e^2-1)}{y_1}}{\frac{a}{e} - ae}$ ① $[b^2 = a^2(e^2-1)]$

$\frac{\frac{(x_1 - ae)a(e^2-1)}{ey_1}}{\frac{a - ae^2}{e}}$

$= \frac{(x_1 - ae)\cancel{a}(e^2-1)}{\cancel{e}y_1} \cdot \frac{\cancel{e}}{\cancel{a}(1-e^2)}$

$= \frac{ae - x_1}{y_1}$ ①

$PS_m \times SX_m = \frac{y_1}{x_1 - ae} \times \frac{ae - x_1}{y_1}$

$= -1$

Question Five (3)

c) $7y^4 + x^3 y + x = 4$

$$28y^3 \frac{dy}{dx} + x^3 \frac{dy}{dx} + y\,3x^2 + 1 = 0$$

$$\frac{dy}{dx} = \frac{-3y x^2 - 1}{28y^3 + x^3} \quad ①$$

when $x = 4, \quad y = 0$

$$m = -\frac{1}{64} \quad ①$$

$$y - y_1 = m(x - x_1)$$

$$y = -\frac{1}{64}(x - 4)$$

$$64y = -x + 4$$

$$x + 64y - 4 = 0 \quad ①$$

Question Six (1)

a) $2x^3 - 8x^2 + 3x + 5 = 0$

i) $x = y^2$ since $\alpha, \beta, \gamma = x$

2

$$y = x^{\frac{1}{2}}$$

$$2x^{\frac{3}{2}} - 8x + 3x^{\frac{1}{2}} + 5 = 0 \quad ①$$

$$x^{\frac{1}{2}}(2x + 3) = (8x - 5)$$

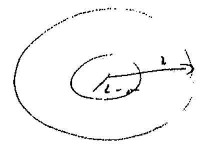
$$x[4x^2 + 12x + 9] = 64x^2 - 80x + 25$$

$$4x^3 + 12x^2 + 9x = 64x^2 - 80x + 25$$

$$\underline{4x^3 - 52x^2 + 89x - 25 = 0} \quad ①$$

ii) $$\alpha^2 + \beta^2 + \gamma^2 = -\frac{b}{a} = \frac{52}{4} = 13 \quad ①$$

b) i) $A = \pi[2^2 - (2 - x)^2]$



2

$$= \pi[4 - (4 - 4x + x^2)]$$

$$= \pi[4 - 4 + 4x - x^2]$$

$$= \pi[4x - x^2] \quad ①$$

Now $y = x^3$

$$x = y^{\frac{1}{3}}$$

$$A = \pi[4y^{\frac{1}{3}} - y^{\frac{2}{3}}] \quad ①$$

ii) $$V = \pi\int_0^1 4y^{\frac{1}{3}} - y^{\frac{2}{3}}\,dy$$

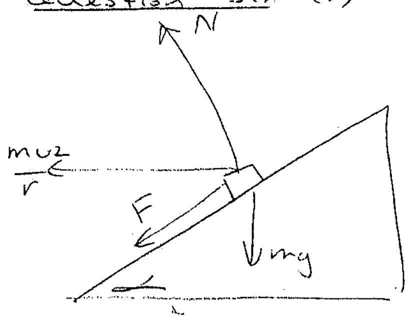
2

$$= \pi\left[4 \times \frac{3}{4} y^{\frac{4}{3}} - \frac{3}{5} y^{\frac{5}{3}}\right]_0^1 \quad ①$$

$$= \left[3 - \frac{3}{5}\right]$$

$$= \frac{12\pi}{5} \text{ units}^3 \quad ①$$

Question six (2)

c) 

Horizontally $N\sin\alpha + F\cos\alpha = \frac{mv^2}{r}$ ①

Vertically $N\cos\alpha - F\sin\alpha = mg$ ②

i) from ② $F\sin\alpha = N\cos\alpha - mg$ ③ [1 mark each]

① $F\cos\alpha = \frac{mv^2}{r} - N\sin\alpha$ ④

ii) ③ × $\sin\alpha$ 1 mark $F\sin^2\alpha = N\sin\alpha\cos\alpha - mg\sin\alpha$ ⑤

④ × $\cos\alpha$ $F\cos^2\alpha = \frac{mv^2}{r}\cos\alpha - N\sin\alpha\cos\alpha$ ⑥

⑤+⑥ $F = \frac{mv^2}{r}\cos\alpha - mg\sin\alpha$ 1 mark

1 mark $F = m(v^2 - gr\tan\alpha)\frac{\cos\alpha}{r}$

iii) $F = 0$

$0 = m(v^2 - gr\tan\alpha)\frac{\cos\alpha}{r}$

$0 = v^2 - gr\tan\alpha$

180 km/h $= \frac{180}{3.6} = 50\,ms^{-1}$ $r = 2000,\ g = 10$

$0 = (50^2) - (2000\times 10)\tan\alpha$ [1 mark correct subst]

$\frac{2500}{20000} = \tan\alpha$

$\tan\alpha = 0.125$

$\alpha = 7°$ [1 mark]

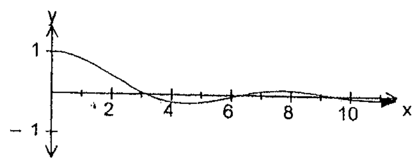
d) $y = \frac{\sin x}{x}$ Note • $\lim_{x\to 0}\frac{\sin x}{x} = 1$

• On x axis $y = 0$

$\frac{\sin x}{x} = 0 \quad (x \neq 0)$

$\sin x = 0$

$x = \pi, 2\pi, 3\pi$



1 mark for graph

• $-1 \le \sin x \le 1$

Question Seven (1)

c) i) Let $P(x) = (x-\alpha)^n Q(x)$

$P'(x) = n(x-\alpha)^{n-1}Q(x) + (x-\alpha)^n Q'(x)$ 1 mark

2 $P'(\alpha) = n(\alpha-\alpha)^{n-1}Q(x) + (\alpha-\alpha)^n Q'(x)$ ← 1 mark

$P'(\alpha) = 0 + 0$

ii) $Q(x) = x^4 + 2x^3 - ax^2 + bx + 12$

$Q'(x) = 4x^3 + 6x^2 - 2ax + b$

$Q(-2) = \cancel{16} - \cancel{16} - 4a - 2b + 12 = 0$

$-4a - 2b + 12 = 0$ ①

3 $Q'(-2) = -32 + 24 + 4a + b = 0$ ← 1 mark

$4a + b - 8 = 0$ ②

① + ② $-b - 4 = 0$

$b = 4$ 1 mark

$4a + 4 - 8 = 0$

$4a = +4$

$a = +1$ 1 mark

iii)

$$
\begin{array}{r|l}
 & x^2 - 2x + 3 \\
x^2+4x+4 & x^4 + 2x^3 - x^2 + 4x + 12 \\
 & x^4 + 4x^3 + 4x^2 \\
 & -2x^3 - 5x^2 + 4x \\
 & -2x^3 - 8x^2 - 8x \\
 & 3x^2 + 12x + 12
\end{array}
$$

1 mark

② Now $b^2 - 4ac = 4 - 4(1)(3)$

< 0

$\therefore P(x) = (x+2)(x+2)(x^2 - 2x + 3)$ 1 mark

iv) $x^2 - 2x + 3 = (x^2 - 2x + 1) + 2$

$= (x-1)^2 - (\sqrt{2}i)^2$

$= (x - 1 + \sqrt{2}i)(x - 1 - \sqrt{2}i)$ 1 mark

$\therefore P(x) = (x+2)(x+2)(x - 1 + \sqrt{2}i)(x - 1 - \sqrt{2}i)$

Quest 7. (2)

b) 1) $a = mH + c$

when $H = 0$ $a = 4 = c$

$a = mH + 4$

when $H = 10$, $a = 0$

$0 = 10m + 4$

$m = -\frac{2}{5}$

$a = \frac{-2H}{5} + 4$ 1 mark

$b = mH + c$

when $H = 0$, $b = 0$, $c = 0$

$b = mH$

when $H = 10$, $b = 2$

$2 = 10m$

$m = \frac{1}{5}$

$b = \frac{H}{5}$ 1 mark

a

$a = \frac{2(10-H)}{5}$

$h = mH + c$

when $H = 0$, $h = 4 = c$

$h = mH + 4$

when $H = 10$, $h = 2$

$2 = 10m + 4$

$-2 = 10m$

$m = -\frac{1}{5}$

$h = -\frac{H}{5} + 4$ 1 mark

$$A = \frac{h}{2}(a+b)$$

$$= \left(2 - \frac{H}{10}\right)\left(\frac{H}{5} - \frac{2H}{5} + 4\right) \quad 1 \text{ mark}$$

$$= \left(2 - \frac{H}{10}\right)\left(4 - \frac{H}{5}\right)$$

$$= 8 - \frac{2H}{5} - \frac{4H}{10} + \frac{H^2}{50}$$

$$= 8 - \frac{4H}{5} + \frac{H^2}{50} \quad 1 \text{ mark}$$

ii) $V = \int_0^{10} 8 - \frac{4H}{5} + \frac{H^2}{50} \, dH$

$$= 8H - \frac{2H^2}{5} + \frac{H^3}{150} \quad 1 \text{ mark}$$

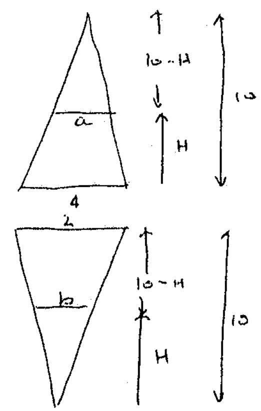
$$= 80 - \frac{200}{5} + \frac{1000}{150}$$

$$= 80 - 40 + \frac{100}{15}$$

$$= 46\frac{2}{3} \text{ units}^3 \quad 1 \text{ mark}$$

Question Seven (3)

b) Bottom to Top Similar Triangles



$$\frac{10-H}{10} = \frac{a}{4}$$

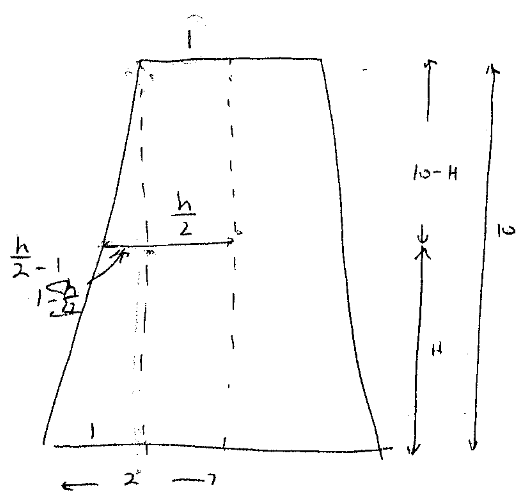
$$a = \frac{40 - 4H}{10}$$

$$= 4 - \frac{2H}{5}$$

$$= \frac{20 - 2H}{5} \quad (1)$$

$$\frac{H}{10} = \frac{b}{2}$$

$$b = \frac{H}{5}. \quad (1)$$



$$\frac{10-H}{10} = \frac{h}{2} - 1$$

$$10 - H = 5h - 10$$

$$20 - H = 5h$$

$$4 - \frac{H}{5} = h \quad (1)$$

$$A = \frac{h}{2}[a+b]$$

$$= 2 - \frac{H}{10}\left[4 - \frac{2H}{5} + \frac{H}{5}\right] \quad (1)$$

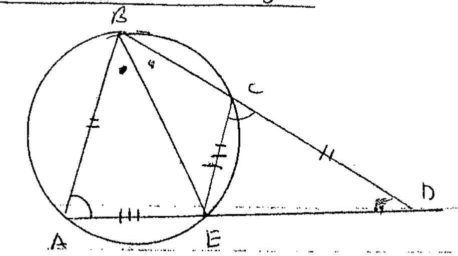
$$= 2 - \frac{H}{10}\left[4 - \frac{H}{5}\right]$$

$$= 8 - \frac{2H}{5} - \frac{4H}{10} + \frac{H^2}{50}$$

$$= 8 - \frac{4H}{5} + \frac{H^2}{50} \quad (1)$$

Question Eight (1)

a)



i) Prove $\Delta CED \equiv \Delta BAE$

$\angle ABC = \angle EBC$ (given)

$AE = EC$ (equal chords subtend equal $\angle$'s) ①

$\angle BAE = \angle ECD$ (int $\angle$ = opp ext $\angle$ cyclic quad) ①

$BA = CD$ given

$\therefore \Delta CED \equiv \Delta BAE$ SAS

ii) $\angle BDE = \bullet$ (corresp $\angle$'s cong Δ's) ①

$\therefore \angle BEA = 2\times\bullet$ (ext $\angle$ = sum int opp $\angle$'s) ①

b) i) $|z_1 + z_2| = \frac{-b}{a} = \left|\frac{\sqrt{3}+i}{1+i} \times \frac{1-i}{1-i}\right|$

$= \left|\frac{\sqrt{3} - i\sqrt{3} + i + 1}{2}\right|$

$= \left|\frac{\sqrt{3}+1}{2} + i\frac{(1-\sqrt{3})}{2}\right|$

$= \sqrt{\frac{3+2\sqrt{3}+1+1-2\sqrt{3}+3}{4}}$

$= \sqrt{\frac{8}{4}}$

$= \sqrt{2}$

$z_1 z_2 = \frac{c}{a} = \frac{-1+\sqrt{3}i}{1+i} \times \frac{1-i}{1-i}$

$= \frac{-1+i+\sqrt{3}i+\sqrt{3}}{2}$

$= \frac{-1+\sqrt{3} + i(1+\sqrt{3})}{2}$

$= \frac{\sqrt{3}-1}{2} + \frac{i(\sqrt{3}+1)}{2}$

$\arg(z_1 z_2) = \tan^{-1} \frac{\sqrt{3}+1}{2} \times \frac{2}{\sqrt{3}-1}$

$= \tan^{-1} \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \tan^{-1} \frac{4+2\sqrt{3}}{2}$

$= \tan^{-1}(2+\sqrt{3})$

$(75°)$

c) $y = \frac{c^2}{x}$

$\frac{dy}{dx} = \frac{-c^2}{x^2}$

Eqn of tangent

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$t^2 y - ct = -x + ct$

Q8 (2)

c) cont

$px + qy + r = 0$

$px + qy = -r$ ①, $x + t^2 y = 2ct$ ②

since both represent the same eqn

$\frac{1}{p} = \frac{t^2}{q} = \frac{2ct}{-r}$

$t^2 = \frac{q}{p}$, ③ $\quad -rt^2 = 2ctq$

$-rt = 2cq$

$t = \frac{-2cq}{r}$ ②

$\therefore \left(\frac{-2cq}{r}\right)^2 = \frac{q}{p}$

$\frac{4c^2q^2}{r^2} = \frac{q}{p}$

$4c^2q^2 \times \frac{p}{q} = r^2$

$4c^2pq = r^2$

OR $px + qy + r = 0$, $xy = c^2$ ①

then $x = \frac{-qy - r}{p}$ sub in ①

$\left(\frac{-qy-r}{p}\right)y = c^2$

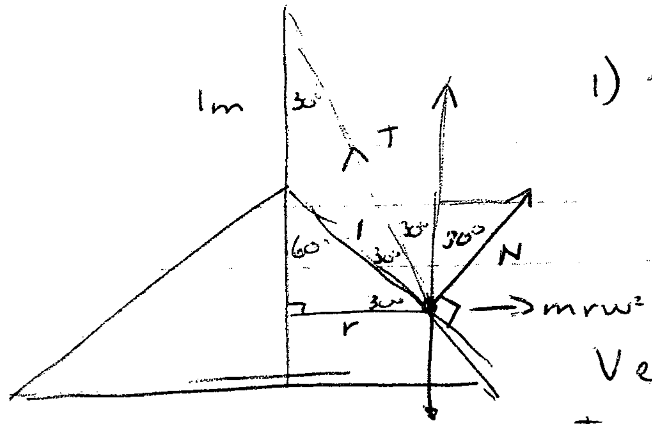
$-qy^2 - ry = c^2p$

$qy^2 + ry + c^2p = 0$ $\quad$ for equal roots $\Delta = 0$

$\Delta = r^2 - 4(q)c^2p$

$= r^2 - 4c^2pq$

$r^2 = 4c^2pq$



i) $\omega = 1$ rad/sec

$\cos 30° = \frac{\sqrt{3}}{2}$

$\frac{r}{1} = \sin 60°$

$r = \frac{\sqrt{3}}{2}$

Vertically

$T\cos 30° + N\cos 30° = mg$

$\frac{\sqrt{3}T}{2} + \frac{\sqrt{3}N}{2} = 20$ 1 mark

$\sqrt{3}T + \sqrt{3}N = 40$

$T + N = \frac{40}{\sqrt{3}}$ ①

Horizontally

$T\sin 30° - N\sin 30° = mr\omega^2$

$\frac{T}{2} - \frac{N}{2} = 2\left(\frac{\sqrt{3}}{2}\right)(1)$ 1 mark

$T - N = 2\sqrt{3}$ ②

① + ②

$2T = 2\sqrt{3} + \frac{40}{\sqrt{3}}$

$T = \frac{\sqrt{3}}{1} + \frac{20}{\sqrt{3}}$

$= \frac{3}{\sqrt{3}} + \frac{20}{\sqrt{3}}$

$T = \frac{23}{\sqrt{3}}$ N sub in ② 1 mark

(13.3)

$\frac{23}{\sqrt{3}} - N = 2\sqrt{3}$

$N = \frac{23}{\sqrt{3}} - 2\sqrt{3}$

$= \frac{23-6}{\sqrt{3}}$

$N = \frac{17}{\sqrt{3}}$ N $(\doteqdot 9.8)$ 1 mark

ii) Put $N = 0$ then vertically

$\frac{\sqrt{3}T}{2} + 0 = 20$

$T = \frac{40}{\sqrt{3}}$ 1 mark

Horizontally

$\frac{1}{2}T = 2\left(\frac{\sqrt{3}}{2}\right)\omega^2$

$\frac{20}{\sqrt{3}} = \sqrt{3}\,\omega^2$

$\frac{20}{\sqrt{3}\times\sqrt{3}} = \omega^2$