Sydney Girls High School

2011

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

Extension 2

This is a trial paper ONLY. It does not necessarily reflect the format or the contents of the 2011 HSC Examination Paper in this subject.

Name: ____________________

Teacher: ____________________

General Instructions

- Reading Time - 5 minutes
- Working time - 3 hours
- Attempt ALL questions
- ALL questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used.
- Diagrams are not to scale
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

Question 1 (15 marks) — Marks

(a) Find $\int_0^1 x(5x^2-2)^4\,dx$ — 3

(b) Find $\int \cot x\,dx$ — 2

(c) Find $\int \frac{1}{x(x^2-1)}dx$ — 3

(d) Using the substitution $t=\tan\frac{x}{2}$, or otherwise, evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{1-\cos x}$ — 4

(e) Evaluate $\int_e^{e^2} \log_e x\,dx$ — 3

Question 2 (15 marks) Start a new page **Marks**

(a) Let $z = 3 + 2i$

(i) Find $\overline{z}$ 1

(ii) Find $\frac{1}{z}$ in the form $x + iy$ 2

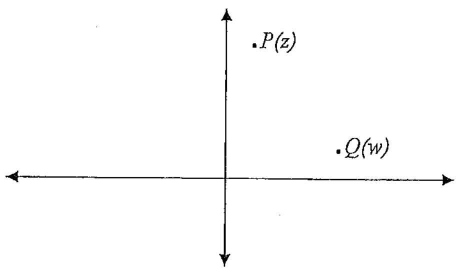
(iii) Find z^{-2} in the form $x + iy$ 1

(b) (i) Express $1-\sqrt{3}i$ in modulus-argument form. 2

(ii) Find $(1-\sqrt{3}i)^5$ in the form $x + iy$ 2

(c) Sketch the region in the complex plane where the inequalities $z \leq 2$ and $\left|\arg z\right| \leq \frac{\pi}{4}$ hold simultaneously. 2

(d) The points P and Q on the Argand diagram represent the complex numbers z and w respectively

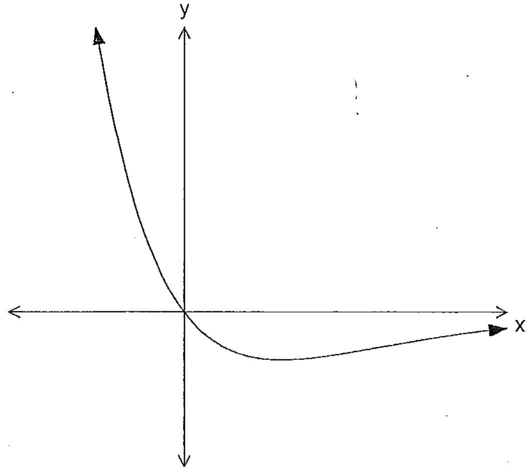


Copy the diagram and mark on it the following points:

(i) The point A representing $-z$ 1
(ii) The point B representing $2w$ 1
(iii) The point S representing $\overline{z}$ 1
(iv) The point T representing iw 1
(v) The point U representing $z + w$ 1

Question 3 (15 marks) Start a new page **Marks**

(a) The following diagram shows the graph of $y = f(x)$



Draw separate one-third page sketches of the graphs of the following:

(i) $y = \left|f(x)\right|$ 1

(ii) $y = \frac{1}{f(x)}$ 2

(iii) $y = e^{f(x)}$ 2

(iv) $y = f(\left|x\right|)$ 2

(b) Find the coordinates of the points where the tangent to the curve $x^2 + xy + y^2 = 12$ is horizontal 3

(c) The zeros of $2x^3 - 3x^2 + 4x - 1$ are α, β and γ

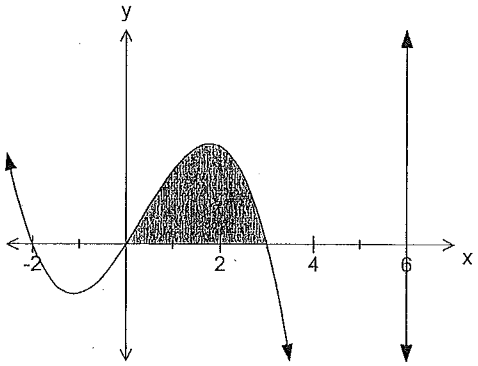
Find a cubic polynomial with integer coefficients whose zeros are

(i) $2\alpha,\ 2\beta$ and 2γ 2

(ii) $\alpha^2,\ \beta^2$ and γ^2 3

Question 4 (15 marks) Start a new page — Marks

(a) The region shaded in the diagram is bounded by the *x-axis* and the curve $y = 6x + x^2 - x^3$ — 4



The shaded region is rotated about the line $x = 6$.
Find the volume generated.

(b) (i) Show that the equation of the tangent at the point $(x_1 y_1)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x_1 x}{a^2}+\frac{y_1 y}{b^2}=1$ — 2

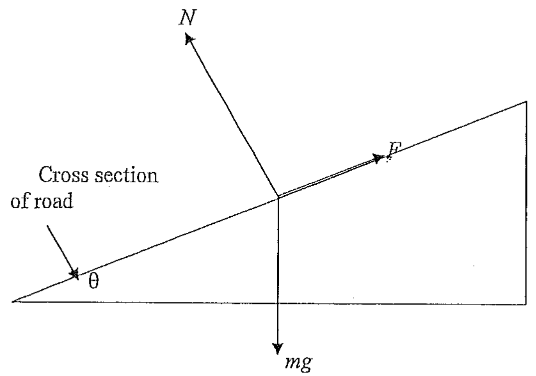
(ii) Find the equation of the tangent that passes through the point $(1,\frac{3\sqrt{3}}{2})$ on the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ — 1

(iii) Find the equation of the tangent parallel to the one in (ii) — 2

(iv) Find the equation of the chord joining the points of contact of the tangents in (ii) and (iii) — 1

Question 4 (continued) — Marks

(c)



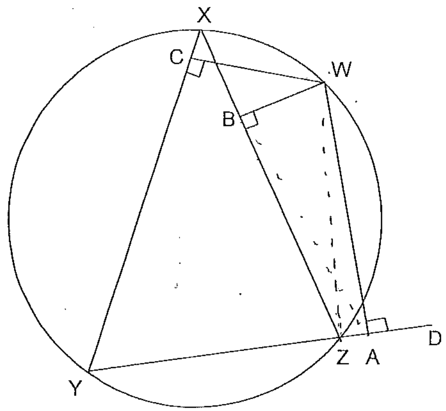
A road contains a bend that is part of a circle radius r. At the bend, the road is banked at an angle θ to the horizontal. A car travels around the bend at constant speed v. Assume that the car is represented by a point of mass m, and that the forces acting on the car are the gravitational force mg, a sideways friction force F (acting up the road as drawn) and a normal reaction N to the road.

(i) By resolving the horizontal and vertical components of force, find expressions for $N\cos\theta$ and $N\sin\theta$ — 3

(ii) Show that $N = \frac{m(v^2 + gr\cot\theta)}{r}\sin\theta$ — 2

Question 5 (15 marks) Start a new page **Marks**

(a)



In the diagram W, X, Y and Z are concyclic, and the points A, B, C are the perpendiculars from W to YZ produced, ZX and XY respectively.

(i) Show that $<WBA = <WZA$ 2

(ii) Show that $<WBC + <WXC = 180^0$ 2

(iii) Deduce that the points A, B and C lie in the same straight line. 2

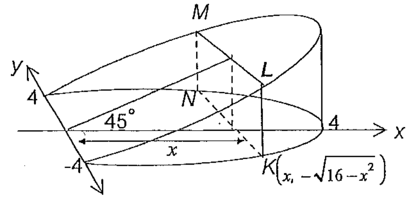
(b) For each integer $n \geq 0$, let $I_n = \int_0^1 x^n e^x dx$

(i) Show that for $n \geq 1, I_n = e - nI_{n-1}$ 2

(ii) Hence, or otherwise, calculate I_4 2

Question 5 (continued) **Marks**

(c) The base of a right cylinder is the circle in the xy-plane with centre 0 and radius 4. A wedge is obtained by cutting this cylinder with the plane through the y-axis at 45^0 to the xy-plane, as shown in the diagram.



A rectangular slice KLMN is taken perpendicular to the base of the wedge at a distance x from the y-axis.

(i) Show that the area of KLMN is given by $x\sqrt{64-4x^2}$ 2

(ii) Find the volume of the wedge. 3

Question 6 (15 marks) Start a new page | **Marks**

(a) Let w be the complex number satisfying $w^3 = 1$ and $Im(w) > 0$

(i) Show that $1 + w + w^2 = 0$ — 2

(ii) Simplify $w^4 + w^6 + w^8$ — 2

(iii) Show that $\frac{1}{w^2}$ is a zero of $P(x) = x^4 + 3x^3 + 2x^2 + x - 1$ — 2

(b) (i) Show that $\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx = \frac{\pi}{2}$ — 2

(ii) By making the substitution $x = \pi - u$ find $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ — 3

(c) (i) Show that the equation of the tangent at the point $P(ct, \frac{c}{t})$ on the hyperbola $xy=c^2$ is $x+t^2y=2ct$ — 2

(ii) Find the equation of the locus of the mid point PG if G is the x intercept of the tangent in (i) — 2

Question 7 (15 marks) Start a new page | **Marks**

(a) The curves $y = \sin x$ and $y = \cot x$ intersect at a point A whose x-coordinate is a

(i) Show that $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$ — 1

(ii) Show that the curves intersect at right angles at A — 3

(iii) Show that $\text{cosec}^2 a = \frac{1+\sqrt{5}}{2}$ — 2

(b) The force of attraction between the earth and a communications satellite in circular orbit around it is given by $F = \frac{mgR^2}{x^2}$ where x is the distance of the satellite from the earth's centre, m is the mass of the satellite, g is gravity and R is the radius of the earth. A 300kg satellite is orbiting the earth at 3000m above the surface of the earth.
If $R = 6400$km and $g = 10ms^{-2}$ find

(i) The velocity of the satellite correct to one significant figure — 2

(ii) The period of the satellite correct to the nearest minute — 2

(iii) F — 1

(c) (i) Differentiate $\sin^{-1} x - \sqrt{1-x^2}$ — 2

(ii) Hence show that
$\int_0^a \sqrt{\frac{1+x}{1-x}} dx = \sin^{-1} a + 1 - \sqrt{1-a^2}$ for $0<a<1$ — 2

Question 8 (15 marks) Start a new page **Marks**

(a) (i) Show that $2\cos A\sin B=\sin(A+B)-\sin(A-B)$ **1**

(ii) Show that $\sin\frac{\theta}{2}(1+2\cos\theta+2\cos 2\theta+2\cos 3\theta)=\sin\frac{7\theta}{2}$. **2**

(iii) Show that if $\theta=\frac{2\pi}{7}$, then $1+2\cos\theta+2\cos 2\theta+2\cos 3\theta=0$. **2**

(iv) By writing $1+2\cos\theta+2\cos 2\theta+2\cos 3\theta$ in terms of $\cos\theta$ prove that $\cos\frac{2\pi}{7}$ is a solution of $8x^3+4x^2-4x-1=0$ **2**

(b) Consider the function $f(x)=e^x(1-\frac{x}{8})^8$

(i) Find the turning points of the graph of $y=f(x)$. **2**

(ii) Sketch the curve $y=f(x)$ and label the turning points and any asymptotes **2**

(iii) From your graph deduce that $e^x\le(1-\frac{x}{8})^{-8}$ for $x<8$. **2**

(iv) Using (iii), show that $\left(\frac{9}{8}\right)^8\le e\le\left(\frac{8}{7}\right)^8$ **2**

--- End of Exam ---

(a)

$$\int_0^1 x\left(5x^2-2\right)^4 dx$$

let $u=5x^2-2, \frac{du}{dx}=10x \Rightarrow \frac{du}{10}=xdx$

when $x=0, u=-2$ & $x=1, u=3$

$$\therefore I=\int_{-2}^{3}\frac{u^4}{10}du=\left[\frac{u^5}{50}\right]_{-2}^{3}=\frac{3^5-(-2)^5}{50}$$

$$=\frac{243+32}{50}=\frac{11}{2} \text{ or } 5.5$$

(b) $\int \cot x dx=\int \frac{\cos x}{\sin x}dx=\ln(\sin x)+C$

(c)

$$\int \frac{1}{x\left(x^2-1\right)}dx$$

$$\frac{1}{x\left(x^2-1\right)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1}$$

$$1=A\left(x^2-1\right)+Bx(x-1)+Cx(x+1)$$

$x=1 \Rightarrow 1=2C \qquad C=\frac{1}{2}$

$x=0 \Rightarrow 1=-A \; A=-1$

$x=-1 \Rightarrow 1=2B \qquad B=\frac{1}{2}$

$$I=\int\left(\frac{-1}{x}+\frac{1}{2}\left(\frac{1}{x+1}+\frac{1}{x-1}\right)\right)dx$$

$$\therefore I=-\ln x+\frac{1}{2}\left(\ln(x+1)+\ln(x-1)\right)+C$$

(d)

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{dx}{1-\cos x}$$

$t=\tan\frac{x}{2}, dx=\frac{2dt}{1+t^2}, \cos x=\frac{1-t^2}{1+t^2}$

$$\therefore I=\int_{\tan\left(\frac{\pi}{6}\right)}^{\tan\left(\frac{\pi}{4}\right)}\frac{\frac{2dt}{1+t^2}}{1-\frac{1-t^2}{1+t^2}}=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{2dt}{1+t^2-\left(1-t^2\right)}$$

$$=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{2dt}{2t^2}=\int_{\frac{1}{\sqrt{3}}}^{1}t^{-2}dt=\left[\frac{t^{-1}}{-1}\right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$=\left(-\frac{1}{1}\right)-\left(-\frac{1}{\frac{1}{\sqrt{3}}}\right)=\sqrt{3}-1$$

(e)

$$\int_e^{e^2}\log_e x\,dx$$

$U=\log_e x, V'=1$

$U'=\frac{1}{x}, V=x$

$I=x\log_e x-\int dx$

$=x\log_e x-x$

$$\int_e^{e^2}\log_e x\,dx=\left[x\log_e x-x\right]_e^{e^2}$$

$$=e^2\log_e e^2-e^2-\left(e\log_e e-e\right)$$

$$=2e^2-e^2-e+e=e^2$$

(a)(i) $z=3+2i \qquad \bar{z}=3-2i$

(a)(ii) $\frac{1}{z}=\frac{\bar{z}}{z\bar{z}}=\frac{3-2i}{9-4i^2}=\frac{3}{13}-\frac{2}{13}i$

(a)(iii) $z^{-2}=\left(\frac{1}{z}\right)^2=\left(\frac{3-2i}{13}\right)^2=\frac{9-12i+4i^2}{169}=\frac{5}{169}-\frac{12}{169}i$

(b)(i) $1-\sqrt{3}i \qquad \theta=-\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=-\frac{\pi}{3} \qquad r^2=1^2+\left(\sqrt{3}\right)^2 \Rightarrow r=2$

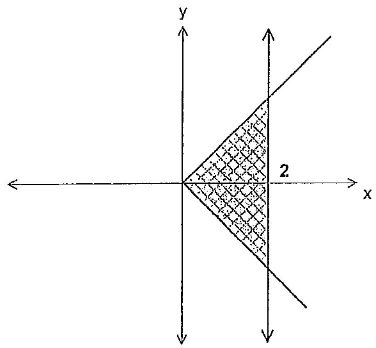
$\therefore 1-\sqrt{3}i=2cis\left(-\frac{\pi}{3}\right)$

(b)(ii) $\left(1-\sqrt{3}i\right)^5=\left[2cis\left(-\frac{\pi}{3}\right)\right]^5=32cis\left(-\frac{5\pi}{3}\right)$

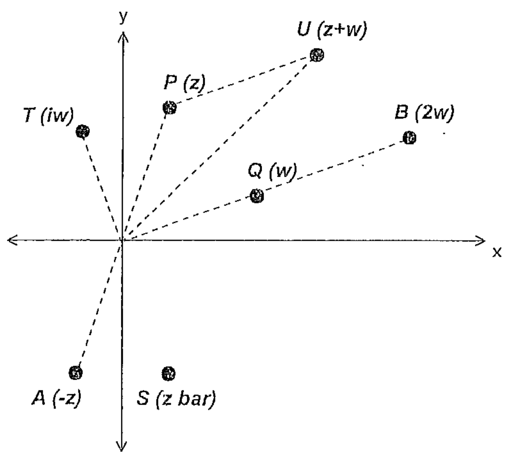
$=32cis\left(\frac{\pi}{3}\right)=32\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=16+16\sqrt{3}i$

(c)

$z\le 2 \qquad -\frac{\pi}{4}\le\arg(z)\le\frac{\pi}{4}$

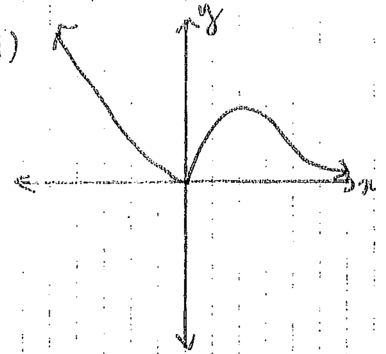


(d)

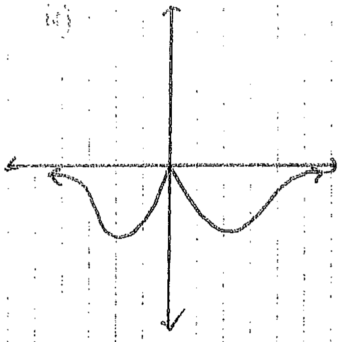


Ext 2 Trial 2011 – Question 3

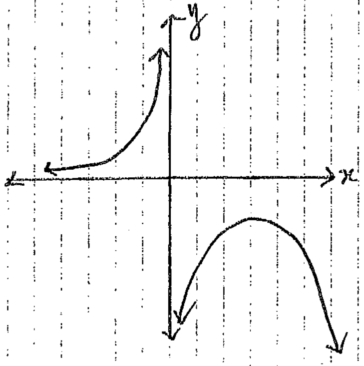
a) i)



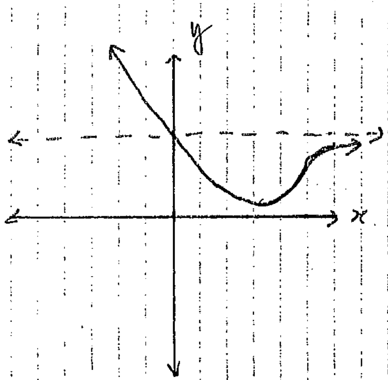
ii)



ii)



iii)



b) $x^2 + xy + y^2 = 12$

$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$2x + y + \frac{dy}{dx}(x + 2y) = 0$

$\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$

when tangent is horizontal

$\frac{dy}{dx} = 0$

$0 = -2x - y$

$y = -2x$ ①

$x^2 + xy + y^2 = 12$ ②

sub ① into ②

$x^2 + x(-2x) + (-2x)^2 = 12$

$x^2 = 4$

$x = \pm 2$

when $x = 2$, $y = -4$

$x = -2$, $y = 4$

$\therefore$ tangent horiz at $(2, -4)$ $(-2, 4)$

c) i) $2x^3 - 3x^2 + 4x - 1$

$2\left(\frac{x}{2}\right)^3 - 3\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right) - 1 = 0$

$x^3 - 3x^2 + 8x - 4 = 0$

ii) $2(\sqrt{x})^3 - 3(\sqrt{x})^2 + 4(\sqrt{x}) - 1 = 0$

$2\sqrt{x}(x + 2) = 3x + 1$

$4x(x + 2)^2 = (3x + 1)^2$

$4x^3 + 7x^2 + 10x - 1 = 0$

Question 4

a) $V_{shell} = \pi(R^2 - r^2)h$

$= \pi\left[(6 - x)^2 - (6 - (x + \delta x))^2\right] y$

$= \pi(12 - 2x - \delta x)\,\delta x\, y$

$= 2\pi y(6 - x)\,\delta x$ (as $\delta x^2 \approx 0$)

$V_{solid} = \int_0^3 2\pi(6 - x)(6x + x^2 - x^3)\,dx$

$= 2\pi\int_0^3 x^4 - 7x^3 + 36x\,dx$

$= 2\pi\left[\frac{x^5}{5} - \frac{7x^4}{4} + 18x^2\right]_0^3$

$= \frac{1377\pi}{10}$ units^3

b) i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

at (x_1, y_1)

$$\frac{dy}{dx} = \frac{-b^2 x_1}{a^2 y_1}$$

$$\therefore \quad y - y_1 = \frac{-b^2 x_1}{a^2 y_1}(x - x_1)$$

$$a^2 y y_1 + b^2 x x_1 = b^2 x_1^2 + a^2 y_1^2$$

$$\frac{y y_1}{b^2} + \frac{x x_1}{a^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

$= 1$ (since (x_1, y_1) lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$)

ii) $\frac{x}{4} + \frac{3\sqrt{3}y}{9 \times 2} = 1$

$$\frac{x}{4} + \frac{\sqrt{3}y}{6} = 1$$

iii) By symmetry of the ellipse, tangent passes through $(-1, -\frac{3\sqrt{3}}{2})$

$$-\frac{x}{4} - \frac{3\sqrt{3}y}{18} = 1$$

$$-\frac{x}{4} - \frac{\sqrt{3}y}{6} = 1$$

iv) Chord passes through the origin:

$$m = \frac{\frac{3\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}}{2}$$

$$= \frac{3\sqrt{3}}{2}$$

$$y = \frac{3\sqrt{3}}{2}x$$

i) Horizontally:

$$N\sin\theta - F\cos\theta = \frac{mv^2}{r}$$

$$N\sin\theta = \frac{mv^2}{r} + F\cos\theta \quad (1)$$

Vertically:

$$F\sin\theta + N\cos\theta = mg$$

$$N\cos\theta = mg - F\sin\theta \quad (2)$$

ii) $N\sin^2\theta = \frac{mv^2}{r}\sin\theta + F\cos\theta\sin\theta \quad (3) \quad [(1) \times \sin\theta]$

$$N\cos^2\theta = mg\cos\theta - F\cos\theta\sin\theta \quad (4) \quad [(2) \times \cos\theta]$$

$$N = \frac{mv^2}{r}\sin\theta + mg\cos\theta \quad ((3) + (4))$$

$$= \frac{mv^2}{r}\sin\theta + \frac{mgr\cot\theta\sin\theta}{r}$$

$$= \frac{m(v^2 + gr\cot\theta)\sin\theta}{r}$$

)(a)(i) $W\hat{B}Z = W\hat{A}D$ (given)

∴ WBZA is concyclic (exterior ∠ of quadrilateral equals interior opposite ∠)

$W\hat{B}A = W\hat{Z}A$ (∠s in same segment)

(ii) $X\hat{C}W = X\hat{B}W$ (given)

∴ WXBC is concyclic (WX subtends two equal ∠s)

$W\hat{B}C + W\hat{X}C = 180°$ (opposite ∠s of a cyclic quadrilateral)

(iii) $W\hat{Z}A = W\hat{X}C$ (exterior ∠ of cyclic quadrilateral equals interior opposite ∠s)

∴ $W\hat{B}C + W\hat{Z}A = 180°$

∴ $W\hat{B}C + W\hat{B}A = 180°$

(b)(i) let $u = x^n$ $\quad v' = e^x$

$u' = nx^{n-1}$ $\quad v = e^x$

$I_n = \left[x^n e^x\right]_0^1 - n\int_0^1 x^{n-1} e^x\,dx$

$= e^1 - nI_{n-1}$

(ii) $I_4 = e - 4I_3$

$I_3 = e - 3I_2$

$I_2 = e - 2I_1$

$I_1 = e - I_0$

$I_0 = \int_0^1 e^x\,dx$

$= \left[e^x\right]_0^1$

$= e - 1$

$I_1 = e - e + 1 = 1$

$I_2 = e - 2$

$I_3 = e - 3e + 6 = 6 - 2e$

$I_4 = e - 24 + 8e = 9e - 24$

(c)(i) $\tan 45° = \frac{kL}{x}$

∴ $x° = kL$

$A = x \times 2y$

$= x \times 2\sqrt{16 - x^2}$

(ii) $V = \lim_{\delta x \to 0} \sum_{x=0}^{4} x\sqrt{64 - 4x^2}\,\delta x$

$= \int_0^4 x\sqrt{64 - 4x^2}\,dx$

$= -\frac{1}{8}\int_0^4 y^{\frac{1}{2}}\,dy$ $\quad$ let $y = 64 - 4x^2$, $\frac{dy}{dx} = -8x$, $-\frac{dy}{8x} = dx$

$= -\frac{1}{8}\left[\frac{2y^{\frac{3}{2}}}{3}\right]$

$= \frac{1}{12}\left[y^{\frac{3}{2}}\right]_0^{64}$

$= \frac{1}{12} \times 64^{\frac{3}{2}}$

$= \frac{128}{3}$

6 (a)(i) $w^3 + w + w^2$

$= w(w^2 + 1 + w)$

$w \neq 0$ or 1

∴ $w^2 + 1 + w = 0$

(ii) $w^4(1 + w^2 + w^4)$

$= w^4(1 + w^2 + w)$

$= 0$

(iii) $p\left(\frac{1}{w^2}\right) = \frac{1}{w^8} + \frac{3}{w^6} + \frac{2}{w^4} + \frac{1}{w^2} - 1$

$= w + 3 + 2w^2 + w - 1$

$= 2 + 2w + 2w^2$

$= 0$

(b)(i) let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$-\frac{du}{\sin x} = dx$

$\int_1^{-1} -\frac{du}{\sqrt{1+u^2}}$

$= \left[\tan^{-1} u\right]_{-1}^{1}$

$= \frac{\pi}{4} - -\frac{\pi}{4}$

$= \frac{\pi}{2}$

(ii) $\frac{dx}{du} = -1$

$-dx = du$

$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$

$= -\int_\pi^0 \frac{(\pi - u)\sin(\pi - u)}{1 + \cos^2(\pi - u)}\,du$

$= \int_0^\pi \frac{\pi \sin u - u\sin u}{1 + \cos^2 u}\,du$

$= \int_0^\pi \frac{\pi \sin x}{1+\cos^2 x}\,dx - \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$

∴ $\int_0^\pi \frac{x\sin x}{1+\cos^2 x} = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$

$= \frac{\pi}{2} \times \frac{\pi}{2}$

$= \frac{\pi^2}{4}$

(c)(i) $x\frac{dy}{dx} + y \times 1 = 0$

$\frac{dy}{dx} = -\frac{y}{x}$

$m_A = -\frac{\frac{c}{t}}{ct}$

$= -\frac{1}{t^2}$

eqn of tangent

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$t^2 y - ct = -x + ct$

$x + t^2 y = 2ct$

(ii) when $y = 0$, $x = 2ct$

$m_P = \left(\frac{ct + 2ct}{2}, \frac{\frac{c}{t} + 0}{2}\right)$

$= \left(\frac{3ct}{2}, \frac{c}{2t}\right)$

$x = \frac{3ct}{2}$ $\quad y = \frac{c}{2t}$

$xy = \frac{3c^2}{4}$

Question 7:

a) i.

$$y = \cot x$$
$$= \tan\left(\frac{\pi}{2} - x\right)$$
$$y' = -\sec^2\left(\frac{\pi}{2} - x\right)$$
$$= -\text{cosec}^2 x$$

ii. At $A(x = a)$:

$$\sin a = \cot a$$
$$\sin a = \frac{\cos a}{\sin a}$$
$$\sin^2 a = \cos a \text{ ---(1)}$$

$y = \sin x$	$y = \cot x$
$y' = \cos x$	$y' = -\text{cosec}^2 x$
At $A\ (x = a)$: $m_1 = \cos a$	At $A\ (x = a)$: $m_2 = -\text{cosec}^2 a$

$$m_1 \times m_2 = \cos a \times -\text{cosec}^2 a$$
$$= \cos a \times -\frac{1}{\sin^2 a}$$
$$= \cos a \times -\frac{1}{\cos a} \quad [\text{from } (1)]$$
$$= -1$$

$\therefore$ curves intersect at right angles at A

iii. From (1):

$$\sin^2 a = \cos a$$
$$1 - \cos^2 a = \cos a$$
$$\cos^2 a + \cos a - 1 = 0$$
$$\cos a = \frac{-1 \pm \sqrt{1 - 4(-1)}}{2}$$
$$= \frac{-1 \pm \sqrt{5}}{2}$$
$$\sin^2 a = \frac{-1 + \sqrt{5}}{2} \quad [\text{from } (1)]$$

$$\text{cosec}^2 a = \frac{2}{-1+\sqrt{5}} \times \frac{-1-\sqrt{5}}{-1-\sqrt{5}}$$
$$= \frac{2(-1-\sqrt{5})}{1-5}$$
$$= \frac{-2(1+\sqrt{5})}{-4}$$
$$\text{cosec}^2 a = \frac{1+\sqrt{5}}{2}$$

b) i.

$$F = \frac{mgR^2}{x^2}$$
$$\frac{mv^2}{x} = \frac{mgR^2}{x^2}$$
$$v^2 = \frac{gR^2}{x}$$
$$= \frac{10 \times (6.4 \times 10^6)^2}{6.403 \times 10^6}$$
$$v \cong 8000 \text{ ms}^{-1}$$

ii.

$$v = rw$$
$$v = xw$$
$$8000 = (6.403 \times 10^6) w$$
$$w = 1.249 \times 10^{-3} \text{ rad/s}$$
$$T = \frac{2\pi}{w}$$
$$= 5030 \text{ s}$$
$$= 1 \text{ h } 24 \text{ min}$$

iii.

$$F = \frac{mgR^2}{x^2}$$
$$= \frac{300 \times 10 \times (6.4 \times 10^6)^2}{(6.403 \times 10^6)^2}$$
$$\cong 2997 N$$

c) i. $y=\sin^{-1}x-\sqrt{1-x^2}$

$=\sin^{-1}x-\left(1-x^2\right)^{\frac{1}{2}}$

$y'=\frac{1}{\sqrt{1-x^2}}-\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}\cdot-2x$

$=\frac{1}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}$

$=\frac{1+x}{\sqrt{1-x^2}}$

---

ii. $\frac{1+x}{\sqrt{1-x^2}}=\frac{1+x}{\sqrt{(1+x)(1-x)}}$

$=\sqrt{\frac{1+x}{1-x}}$

$\int_0^a\sqrt{\frac{1+x}{1-x}}dx=\left[\sin^{-1}x-\sqrt{1-x^2}\right]_0^a$

$=\left(\sin^{-1}a-\sqrt{1-a^2}\right)-\left(\sin^{-1}0-\sqrt{1-0^2}\right)$

$=\sin^{-1}a+1-\sqrt{1-a^2}$

Question 8:

a) i. $\sin(A+B)-\sin(A-B)=\sin A\cos B+\cos A\sin B-(\sin A\cos B-\cos A\sin B)$

$=2\cos A\sin B$

---

ii. $LHS=\sin\frac{\theta}{2}(1+2\cos\theta+2\cos2\theta+2\cos3\theta)$

$=\sin\frac{\theta}{2}+2\cos\theta\sin\frac{\theta}{2}+2\cos2\theta\sin\frac{\theta}{2}+2\cos3\theta\sin\frac{\theta}{2}$

$=\sin\frac{\theta}{2}+\left\{\sin\left(\theta+\frac{\theta}{2}\right)-\sin\left(\theta-\frac{\theta}{2}\right)\right\}+\left\{\sin\left(2\theta+\frac{\theta}{2}\right)-\sin\left(2\theta-\frac{\theta}{2}\right)\right\}$

$+\left\{\sin\left(3\theta+\frac{\theta}{2}\right)-\sin\left(3\theta-\frac{\theta}{2}\right)\right\}$

$=\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}-\sin\frac{\theta}{2}+\sin\frac{5\theta}{2}-\sin\frac{3\theta}{2}+\sin\frac{7\theta}{2}-\sin\frac{5\theta}{2}$

$=\sin\frac{7\theta}{2}$

$=RHS$

---

iii. $\sin\frac{\theta}{2}(1+2\cos\theta+2\cos2\theta+2\cos3\theta)=\sin\frac{7\theta}{2}$

When $\theta=\frac{2\pi}{7}$:

$RHS=\sin\frac{7\left(\frac{2\pi}{7}\right)}{2}$

$=\sin\pi$

$=0$

$\sin\frac{\theta}{2}(1+2\cos\theta+2\cos2\theta+2\cos3\theta)=0$

$\sin\frac{\theta}{2}=0$

But when $\theta=\frac{2\pi}{7}$:

$\sin\frac{\pi}{7}\neq0$

$\therefore 1+2\cos\theta+2\cos2\theta+2\cos3\theta=0$

---

b) i. $f(x)=e^x\left(1-\frac{x}{8}\right)^8$

$u=e^x$

$u'=e^x$

$v=\left(1-\frac{x}{8}\right)^8$

$v'=8\left(1-\frac{x}{8}\right)^7\cdot\left(-\frac{1}{8}\right)$

$=-\left(1-\frac{x}{8}\right)^7$

$f'(x)=-e^x\left(1-\frac{x}{8}\right)^7+e^x\left(1-\frac{x}{8}\right)^8$

$=-e^x\left(1-\frac{x}{8}\right)^7\left[1-\left(1-\frac{x}{8}\right)\right]$

$=-e^x\left(1-\frac{x}{8}\right)^7\left(\frac{x}{8}\right)$

$=\frac{-xe^x}{8}\left(1-\frac{x}{8}\right)^7$

Stat. points occur when $f'(x)=0$:

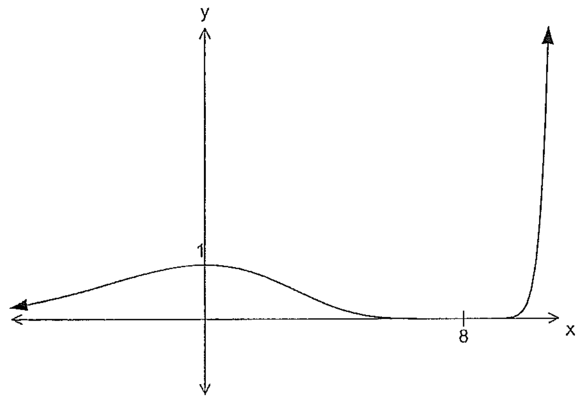
$$\frac{-xe^{x}}{8}=0 \qquad \left(1-\frac{x}{8}\right)^{7}=0$$

$$\left.\begin{array}{l} x=0 \\ y=1 \end{array}\right\} \qquad 1-\frac{x}{8}=0$$

$$\left.\begin{array}{l} x=8 \\ y=0 \end{array}\right\}$$

Stat. points at $(0,1)$ and $(8,0)$

ii.



iii. When $x<8$:

$e^{x}\left(1-\frac{x}{8}\right)^{8}\leq 1$ from graph

$e^{x}\leq\frac{1}{\left(1-\frac{x}{8}\right)^{8}}$ Note: $\left(1-\frac{x}{8}\right)^{8}>0$ when $x<8$

$e^{x}\leq\left(1-\frac{x}{8}\right)^{-8}$

iv. When $x=1$:

$$e\leq\left(1-\frac{1}{8}\right)^{-8}$$

$$e\leq\left(\frac{7}{8}\right)^{-8}$$

$$e\leq\left(\frac{8}{7}\right)^{8}$$

When $x=-1$:

$$e^{-1}\leq\left(1+\frac{1}{8}\right)^{-8}$$

$$\frac{1}{e}\leq\left(\frac{9}{8}\right)^{-8}$$

$$\frac{1}{e}\leq\left(\frac{8}{9}\right)^{8}$$

$$e\geq\left(\frac{9}{8}\right)^{8}$$

$$\therefore\left(\frac{9}{8}\right)^{8}\leq e\leq\left(\frac{8}{7}\right)^{8}$$